e-ISBN: 978-93-6252-929-9
IIP Series

ANALYZING FRACTIONAL DIFFERENTIAL EQUATIONS USING THE POURREZA TRANSFORM

Abstract

This research investigates specific classes of fractional differential equations using a straightforward fractional calculus technique. The employed methodology yields various fascinating results, including a broader adaptation of the widely recognized classical Frobenius method. The approach outlined in this study primarily relies on fundamental theorems concerning the specific solutions of fractional differential equations, making use of the Pourreza transform and binomial series extension coefficients. Additionally, the study presents advanced techniques for solving fractional differential equations effectively, illustrated through practical examples.

Keywords: Fractional-order differential equation; Riemann-Liouville (RL) fractional integrals; Mittag-Leffler function; gamma function; Pourreza transform of the fractional derivative.

Authors

Prabakaran Raghavendran
Department of Mathematics
Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology
Chennai, Tamil Nadu, India.
rockypraba55@gmail.com

Tharmalingam Gunasekar
Department of Mathematics
Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology
Chennai, Tamil Nadu, India.
tguna84@gmail.com

M. L. Suresh
Department of Mathematics
Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology
Chennai, Tamil Nadu, India.
mukunthusuresh@gmail.com

Murugan Suba
Department of Mathematics
S.A. Engineering College (Autonomous)
Chennai, Tamil Nadu, India.
suba.hari87@gmail.com

Shyam Sundar Santra
Department of Mathematics
JIS College of Engineering, Kalyani
West Bengal, India.
shyam01.math@gmail.com

I. INTRODUCTION

In recent years, fractional differential equations have garnered substantial interest because of their capacity to capture complex phenomena in diverse fields of science and engineering. These equations go beyond traditional integer-order differentiation, enabling a more precise description of processes that display memory and non-local behaviors. This research delves into the realm of fractional differential equations, aiming to advance the understanding and solution techniques for specific classes of these equations. The foundation of our investigation draws inspiration from a rich history of mathematical contributions, as evidenced by the works of renowned scholars in this field. Caputo's seminal work [1] on elasticity and anelastic dissipation laid the groundwork for understanding the fundamentals of fractional calculus. Podlubny's comprehensive treatise [2] on fractional differential equations has served as a cornerstone reference for researchers and practitioners alike. Our research also draws upon innovative methods and techniques that have emerged in recent years. Notably, Zhang's Sumudu-based algorithm [3] provides a valuable computational tool for solving differential equations, while Aboodh's transformative work [4] introduced the Aboodh transform, offering a novel approach to tackling fractional differential equations. The study of Laplace transforms in the context of fractional differential equations has been a focus of research, as demonstrated by the contributions of Lin and Lu [5]. Mohamed's Elzaki transformation [6] and Kashuri, Fundo, and Liko's new integral transform [7] represent additional methodologies that have expanded the arsenal of techniques available for solving these equations. Furthermore, the New Integral Transform Mohand Transform [8], as proposed by Abdelrahim Mahgoub, has brought new perspectives to the field. Silva, Moreira, and Moret's work on conformable Laplace transforms [9] adds to the evolving landscape of fractional calculus techniques. Lastly, the Aboodh transform continues to be explored, as evidenced by Aruldoss and Anusuya Devi [10], who have employed it for solving fractional differential equations. Additionally, the study by Raghavendran et al. [11] explores the use of Aboodh transform for fractional integro-differential equations, showcasing its versatility. Burqan, Saadeh, Qazza, and Momani's 2023 [13] paper introduces the ARA-residual power series method, a novel approach for solving partial fractional differential equations. This method offers a valuable contribution to mathematical techniques for addressing complex mathematical problems in engineering and science.

In this investigation, we utilize the Pourreza transform of fractional derivatives and the coefficients from binomial series extensions to address multiple fractional differential equations. Moreover, we unveil various properties that are relevant to our main focus. To illustrate our findings, we present practical examples.

II. PRELIMINARIES

In this section, we are listing some preliminaries that are useful throughout the paper [11].

1. The definition of the RL fractional integral with order $\zeta > 0$ for a function $y(t)$ can be expressed as follows:

$$I^{\zeta}{}_{t}y(t) = \frac{1}{\Gamma(\zeta)}\int_{0}^{t}(t-\eta)^{\zeta-1}\ y(\eta)d\eta$$

2. The Caputo fractional derivative of the function $y(t)$ is defined as follows:

$$D^{\zeta}{}_{t}y(t) = \begin{cases} y^{i}(t) \quad ; \quad if\ \vartheta = i \in \mathbb{N} \\ \dfrac{1}{\Gamma(i-\zeta)}\displaystyle\int_{0}^{\zeta} \frac{y^{i}(t)}{(t-x)^{\zeta-i+1}}dt \quad ; if\ i-1 < \zeta < i \end{cases}$$

The Euler gamma function, denoted as $\Gamma(.)$, is defined as follows:

$$\Gamma(\psi) = \int_{0}^{\infty} t^{\psi-1}\, e^{-t}\, dt \qquad (\mathbb{R} > 0).$$

3. The Pourreza transform of a function $\mathrm{y(t)}$, $\mathrm{t} \in (0,\infty)$ is defined by

$$P[y(t)](\mathcal{E}) = F(\mathcal{E}) = \mathcal{E}\int_{0}^{\infty} e^{-\mathcal{E}^{2}t}\, y(t)\, dt\, ;\ (\mathcal{E} \in \mathbb{C})$$

4. The Mittag-Leffler function is defined by

$$E_{\delta,\gamma}(\psi) = \sum_{i=0}^{\infty} \frac{\psi^{i}}{\Gamma(\delta r + \gamma)} \qquad (\,\delta, \gamma, \psi \in \mathbb{C}, \mathbb{R}(\delta) > 0).$$

5. The Simplest Wright function is defined by

$$\rho(\omega, \psi; \phi) = \sum_{r=0}^{\infty} \frac{1}{\Gamma(\omega r + \psi)} . \frac{\phi^{r}}{r!} \qquad (\phi, \psi, \omega \in \mathbb{C}\,).$$

6. The general Wright function ${}_{\mathrm{i}}\chi_{\mathrm{j}}\,(\varphi)$ is characterized by the following conditions $\varphi \in \mathbb{C}$, $\nu_{1l}, \nu_{2m} \in \mathbb{C}$, and real $\omega_{l}, \phi_{m} \in \mathbb{R}$ $(l = 1, \dots, i, m = 1, \dots, j)$, as determined by the provided series.

$${}_{\mathrm{i}}\chi_{\mathrm{j}}(\nu) = {}_{\mathrm{i}}\chi_{\mathrm{j}}\left(\begin{matrix} (\nu_{1l}, \omega_{l})_{1,i} \\ (\nu_{2m}, \phi_{m})_{1,j} \end{matrix} \mid \varphi \right) = \sum_{r=0}^{\infty} \frac{\prod_{l=1}^{i}\Gamma(\nu_{1l}+\omega_{l}r)}{\prod_{m=1}^{j}\Gamma(\nu_{2m}+\phi_{m}r)} . \frac{\varphi^{r}}{r!}$$

7. The inverse Pourreza transform is defined by

$$P^{-1}\left[\frac{\Gamma(z+1)}{\mathcal{E}^{2z+1}}\right] = t^{z}$$

Remark 2.1

$$P\big[D^{\vartheta} y(t)\big](\mathcal{E}) = \mathcal{E}\int_{0}^{\infty} e^{-\mathcal{E}^{2}t}\ \big[D^{\vartheta}\ y(t)\big]dt$$

$$= \ \mathcal{E}\int_{0}^{\infty} e^{-\mathcal{E}^{2}t}\ \frac{1}{\Gamma(n-\vartheta)}\int_{0}^{t} \frac{y^{(n)}(\zeta)}{(t-\zeta)^{\vartheta-n+1}}\ d\zeta\ dt$$

$$= \frac{\mathcal{E}}{\Gamma(n-\vartheta)} \int_0^\infty \int_\zeta^\infty e^{-\mathcal{E}^2 t} \frac{y^{(n)}(\zeta)}{(t-\zeta)^{\vartheta-n+1}} \, dt \, d\zeta$$

$$= \frac{\mathcal{E}}{\Gamma(n-\vartheta)} \int_0^\infty y^{(n)}(\zeta) \int_0^\infty e^{-\mathcal{E}^2(u+\zeta)} u^{n-\vartheta-1} \, du \, d\zeta$$

$$= \frac{\mathcal{E}}{\Gamma(n-\vartheta)} \int_0^\infty e^{-\mathcal{E}^2 \zeta} y^{(n)}(\zeta) \int_0^\infty e^{-\mathcal{E}^2 u} u^{n-\vartheta-1} \, du \, d\zeta$$

$$= \frac{\mathcal{E}}{\Gamma(n-\vartheta)} \int_0^\infty e^{-\mathcal{E}^2 \zeta} y^{(n)}(\zeta) \frac{\Gamma(\mathrm{n}-\vartheta)}{\mathcal{E}^{n-\vartheta}} d\zeta$$

$$= \mathcal{E}^{\vartheta-n+1} \int_0^\infty e^{-\mathcal{E}^2 \zeta} y^{(n)}(\zeta) \, d\zeta = \mathcal{E}^{\vartheta-n+1} P\left[y^{(n)}(\zeta)\right](\mathcal{E})$$

$$= \mathcal{E}^{\vartheta-n+1} \left[\mathcal{E}^{2n} P[y(t)] - \sum_{m=0}^{n-1} \mathcal{E}^{-2m-1} y^{(n)}(0)\right]$$

$$= \mathcal{E}^{\vartheta+n+1} P[y(t)] - \sum_{m=0}^{n-1} \mathcal{E}^{\vartheta-n-2m} y^{(n)}(0)$$

Note: Fubini's theorem is employed to rearrange the order of integration in the preceding derivative.

III. SOLUTIONS OF THE FRACTIONAL DIFFERENTIAL EQUATIONS

In this section, there are strong indications that the function $k(t)$ alone may be adequate to enable the Pourreza transform $P[k(t)]$ to operate successfully at a certain value of the parameter `$\mathcal{E}'$.

Theorem 3.1. Let $1 < \vartheta < 2$ and σ and $\tau \in \mathbb{R}$. Then the fractional differential equation

$$t) + \sigma\, k^{\vartheta}(t) + \tau\, k(t) = 0 \tag{1}$$

with initial conditions $k(0) = c_0$ and $k^{'}(0) = c_1$ has the unique solution

$$k(t) = c_0 \sum_{m=0}^{\infty} \frac{(-\tau)^m \, t^{2m}}{m!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(m+\aleph+1)\left(-\sigma t^{(2-\vartheta)}\right)^{\aleph}}{\Gamma[(2-\vartheta)\aleph+2m+1]\,\aleph!} + c_1 \sum_{m=0}^{\infty} \frac{(-\tau)^m \, t^{2m+1}}{m!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(m+\aleph+1)\left(-\sigma t^{(2-\vartheta)}\right)^{\aleph}}{\Gamma[(2-\vartheta)\aleph+2m+2]\,\aleph!} \tag{2}$$

$$+\sigma c_1 \sum_{m=0}^{\infty} \frac{(-\tau)^m \, t^{2m-\vartheta+3}}{m!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(m+\aleph+1)\left(-\sigma t^{(2-\vartheta)}\right)^{\aleph}}{\Gamma[(2-\vartheta)\aleph+2m-\vartheta+4]\ \aleph!}$$

$$+\sigma c_0 \sum_{m=0}^{\infty} \frac{(-\tau)^m \, t^{2m-\vartheta+2}}{m!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(m+\aleph+1)\left(-\sigma t^{(2-\vartheta)}\right)^{\aleph}}{\Gamma[(2-\vartheta)\aleph+2m-\vartheta+3]\ \aleph!}$$

Proof: Utilizing the Pourreza transform in (1) and taking into consideration, we have

$$\mathcal{E}^4 F(\mathcal{E}) - \mathcal{E}^3 y(0) - \mathcal{E}\, y'(0) + \sigma\left[\mathcal{E}^{2\vartheta} F(s) - \mathcal{E}^{2\vartheta-1} y(0) - \mathcal{E}^{2\vartheta-3} y'(0)\right] + \tau\, F(s) = 0$$

$$\mathcal{E}^4 P[k(t)] - \mathcal{E}^3 k(0) - \mathcal{E} k'(0) + \sigma \mathcal{E}^{2\vartheta} P[k(t)] - \sigma \mathcal{E}^{2\vartheta-1}\, k(0) - \sigma \mathcal{E}^{2\vartheta-3} k'(0) + \tau\, P[k(t)] = 0$$

$$\left(\mathcal{E}^4 + \sigma \mathcal{E}^{2\vartheta} + \tau\right) P[k(t)] = \mathcal{E}^3 c_0 + \mathcal{E}\, c_1 + \sigma\, \mathcal{E}^{2\vartheta-1} c_0 + \sigma\, \mathcal{E}^{2\vartheta-3} c_1$$

$$P[k(t)] = \frac{\mathcal{E}^3 c_0 + \mathcal{E}\, c_1 + \sigma\, \mathcal{E}^{2\vartheta-1} c_0 + \sigma\, \mathcal{E}^{2\vartheta-3} c_1}{(\mathcal{E}^4 + \sigma\, \mathcal{E}^{2\vartheta} + \tau)} \quad (3)$$

Since

$$\frac{1}{(\mathcal{E}^4 + \sigma\, \mathcal{E}^{2\vartheta} + \tau)} = \frac{\mathcal{E}^{-2\vartheta}}{\mathcal{E}^{4-2\vartheta} + \sigma + \tau \mathcal{E}^{-2\vartheta}}$$

$$= \frac{\mathcal{E}^{-2\vartheta}}{(\mathcal{E}^{4-2\vartheta} + \sigma)\left(1 + \frac{\tau\, \mathcal{E}^{-2\vartheta}}{\mathcal{E}^{4-2\vartheta} + \sigma}\right)}$$

$$= \frac{\mathcal{E}^{-2\vartheta}}{\mathcal{E}^{4-2\vartheta} + \sigma} \sum_{m=0}^{\infty} \left(\frac{-\tau\, \mathcal{E}^{-2\vartheta}}{\mathcal{E}^{4-2\vartheta} + \sigma}\right)^m$$

$$= \sum_{m=0}^{\infty} \frac{(-\tau)^m \mathcal{E}^{-2\vartheta m - 2\vartheta}}{(\mathcal{E}^{4-2\vartheta} + \sigma)^{m+1}}$$

$$= \sum_{m=0}^{\infty} \frac{(-\tau)^m \mathcal{E}^{-4m-4}}{(1 + \sigma\, \mathcal{E}^{2\vartheta-4})^{m+1}}$$

$$= \sum_{m=0}^{\infty} (-\tau)^m \mathcal{E}^{-4m-4} \sum_{\aleph=0}^{\infty} \left(-\sigma \mathcal{E}^{2\vartheta-4}\right)^r \binom{m+\aleph}{\aleph}$$

$$= \sum_{m=0}^{\infty} (-\tau)^m \sum_{\aleph=0}^{\infty} \binom{m+\aleph}{\aleph} (-\sigma)^{\aleph}\, \mathcal{E}^{(2\vartheta-4)\aleph-4m-4} \quad (4)$$

Substituting the above equation (4) in (3), we get

$$
\begin{aligned}
P[k(t)] = {} & c_0 \sum_{m=0}^{\infty} (-\tau)^m \sum_{\aleph=0}^{\infty} \binom{m+\aleph}{\aleph} (-\sigma)^{\aleph}\, \varepsilon^{(\vartheta-2)\aleph-2m-1} \\
& + c_1 \sum_{m=0}^{\infty} (-\tau)^m \sum_{\aleph=0}^{\infty} \binom{m+\aleph}{\aleph} (-\sigma)^{\aleph}\, \varepsilon^{(\vartheta-2)\aleph-2m-2} \\
& + \sigma c_0 \sum_{m=0}^{\infty} (-\tau)^m \sum_{\aleph=0}^{\infty} \binom{m+\aleph}{\aleph} (-\sigma)^{\aleph}\, \varepsilon^{(\vartheta-2)\aleph-2m+\vartheta-3} \\
& + \sigma c_1 \sum_{m=0}^{\infty} (-\tau)^m \sum_{\aleph=0}^{\infty} \binom{m+\aleph}{\aleph} (-\sigma)^{\aleph}\, \varepsilon^{(\vartheta-2)\aleph-2m+\vartheta-4}
\end{aligned}
\tag{5}
$$

Thus, the inverse Pourreza transform to equation (5) yields the solution (2)

$$
\begin{aligned}
k(t) = {} & c_0 \sum_{m=0}^{\infty} \frac{(-\tau)^m\, t^{2m}}{m!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(m+\aleph+1)\left(-\sigma t^{(2-\vartheta)}\right)^{\aleph}}{\Gamma[(2-\vartheta)\aleph+2m+1]\,\aleph!} \\
& + c_1 \sum_{m=0}^{\infty} \frac{(-\tau)^m\, t^{2m+1}}{m!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(m+\aleph+1)\left(-\sigma t^{(2-\vartheta)}\right)^{\aleph}}{\Gamma[(2-\vartheta)\aleph+2m+2]\,\aleph!} \\
& + \sigma c_1 \sum_{m=0}^{\infty} \frac{(-\tau)^m\, t^{2m-\vartheta+3}}{m!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(m+\aleph+1)\left(-\sigma t^{(2-\vartheta)}\right)^{\aleph}}{\Gamma[(2-\vartheta)\aleph+2m-\vartheta+4]\,\aleph!} \\
& + \sigma c_0 \sum_{m=0}^{\infty} \frac{(-\tau)^m\, t^{2m-\vartheta+2}}{m!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(m+\aleph+1)\left(-\sigma t^{(2-\vartheta)}\right)^{\aleph}}{\Gamma[(2-\vartheta)\aleph+2m-\vartheta+3]\,\aleph!}
\end{aligned}
$$

which is (2). This completes the proof of the theorem.

Example 3.1 The fractional differential equation is
$k''(t) + \sqrt{7}\, k^{\left(\frac{3}{2}\right)}(t) + 10\, k(t) = 0$

with initial conditions $k(0) = c_0$ and $k'(0) = c_1$ has the unique solution

$$
\begin{aligned}
k(t) = {} & c_0 \sum_{m=0}^{\infty} \frac{(-10)^m\, t^{2m}}{m!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(m+\aleph+1)\left(-\sqrt{7}\, t^{\left(\frac{1}{2}\right)}\right)^{\aleph}}{\Gamma\left[\left(\frac{1}{2}\right)\aleph+2m+1\right]\,\aleph!} \\
& + c_1 \sum_{m=0}^{\infty} \frac{(-10)^m\, t^{2m+1}}{m!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(m+\aleph+1)\left(-\sqrt{7} t^{\left(\frac{1}{2}\right)}\right)^{\aleph}}{\Gamma\left[\left(\frac{1}{2}\right)\aleph+2m+2\right]\,\aleph!}
\end{aligned}
$$

$$+\sqrt{7}c_0\sum_{m=0}^{\infty}\frac{(-10)^m\ t^{2m+\frac{1}{2}}}{m!}\sum_{\aleph=0}^{\infty}\frac{\Gamma(m+\aleph+1)\ \left(-\sqrt{7}t^{\left(\frac{1}{2}\right)}\right)^{\aleph}}{\Gamma\left[\left(\frac{1}{2}\right)\aleph+2m+\frac{3}{2}\right]\ \aleph!}$$

$$+\sqrt{7}c_1\sum_{m=0}^{\infty}\frac{(-10)^m\ t^{2m+\frac{3}{2}}}{m!}\sum_{\aleph=0}^{\infty}\frac{\Gamma(m+\aleph+1)\ \left(-\sqrt{7}t^{\left(\frac{1}{2}\right)}\right)^{\aleph}}{\Gamma\left[\left(\frac{1}{2}\right)\aleph+2m+\frac{5}{2}\right]\ \aleph!}$$

Figure 1 illustrates the solution behavior of the fractional differential equation of Example 3.1 at various values of ϑ with the initial conditions $c_0 = 1$ and $c_1 = 1$.

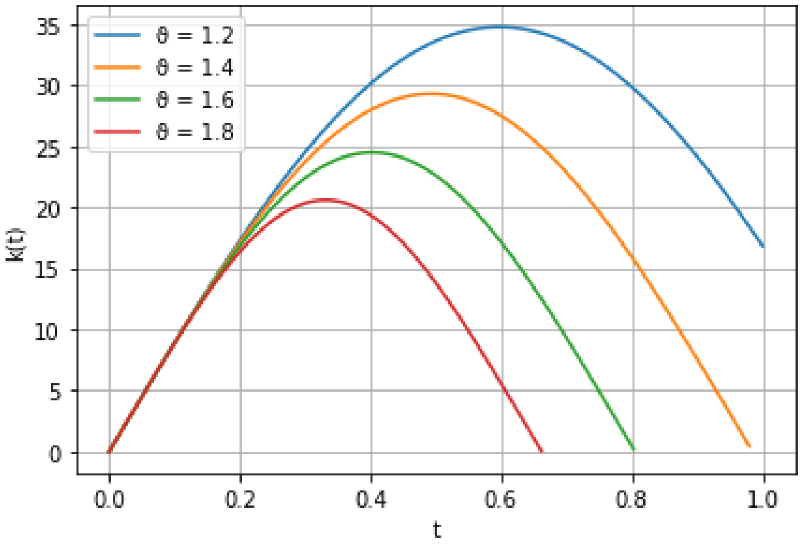

Figure 1: The solution behavior of Example 3.1

Theorem 3.2. Let $1 < \vartheta < 2$ and σ and $\tau \in \mathbb{R}$. Then the fractional differential equation

$$k^{\vartheta}(t) + a\,k'(t) + \tau\,k(t) = 0 \tag{6}$$

with initial conditions $k(0) = c_0$ and $k'(0) = c_1$ has the unique solution

$$k(t) = c_0\sum_{m=0}^{\infty}\frac{(-\tau)^m}{m!}\sum_{\aleph=0}^{\infty}\frac{\Gamma(m+\aleph+1)\ (-\sigma)^{\aleph}t^{(\vartheta-1)\aleph+\vartheta k}}{\Gamma[(\vartheta-1)\aleph+\vartheta m+1]\ \aleph!}$$

$$+c_1\sum_{m=0}^{\infty}\frac{(-\tau)^m}{m!}\sum_{\aleph=0}^{\infty}\frac{\Gamma(m+\aleph+1)\ (-\sigma)^{\aleph}t^{(\vartheta-1)\aleph+\vartheta k+1}}{\Gamma[(\vartheta-1)\aleph+\vartheta m+2]\ \aleph!}$$

$$+\sigma c_0\sum_{m=0}^{\infty}\frac{(-\tau)^m}{m!}\sum_{\aleph=0}^{\infty}\frac{\Gamma(m+\aleph+1)\ (-\sigma)^{\aleph}t^{(\vartheta-1)\aleph+\vartheta k+\vartheta-1}}{\Gamma[(\vartheta-1)\aleph+\vartheta m+\vartheta]\ \aleph!} \tag{7}$$

Proof: Utilizing the Pourreza transform in (6) and taking into consideration, we have

$$\mathcal{E}^{2\vartheta}F(\mathcal{E})-\mathcal{E}^{2\vartheta-1}y(0)-\mathcal{E}^{2\vartheta-3}y'(0)+\sigma[\,\mathcal{E}^2F(\mathcal{E})-\mathcal{E}\,y(0)]+\tau\,F(\mathcal{E})=0$$
$$\mathcal{E}^{2\vartheta}P[k(t)]-\mathcal{E}^{2\vartheta-1}k(0)-\mathcal{E}^{2\vartheta-3}k'(0)+\sigma\,\mathcal{E}^2\,P[k(t)]-\sigma\,\mathcal{E}\,k(0)+\tau\,P[k(t)]=0$$
$$\mathcal{E}^{2\vartheta}P[k(t)]-\mathcal{E}^{2\vartheta-1}c_0-\mathcal{E}^{2\vartheta-3}c_1+\sigma\,\mathcal{E}^2\,P[k(t)]-\sigma\,\mathcal{E}\,c_0+\tau\,P[k(t)]=0$$
$$P[k(t)]=\frac{\mathcal{E}^{2\vartheta-1}c_0+\mathcal{E}^{2\vartheta-3}c_1+\sigma\,\mathcal{E}\,c_0}{(\mathcal{E}^{2\vartheta}+\sigma\,\mathcal{E}^2+\tau)} \tag{8}$$

Since

$$\frac{1}{(\mathcal{E}^{2\vartheta}+\sigma\,\mathcal{E}^2+\tau)}=\frac{\mathcal{E}^{-2}}{\mathcal{E}^{2\vartheta-2}+\sigma+\tau\,\mathcal{E}^{-2}}$$

$$=\frac{\mathcal{E}^{-2}}{(\mathcal{E}^{2\vartheta-2}+\sigma)\left(1+\frac{\tau\,\mathcal{E}^{-2}}{\mathcal{E}^{2\vartheta-2}+\sigma}\right)}$$

$$=\frac{\mathcal{E}^{-2}}{\mathcal{E}^{2\vartheta-2}+\sigma}\sum_{m=0}^{\infty}\left(\frac{-b\mathcal{E}^{-2}}{\mathcal{E}^{2\vartheta-2}+\sigma}\right)^m$$

$$=\sum_{m=0}^{\infty}\frac{(-\tau)^m\,\mathcal{E}^{-2m-2}}{(\mathcal{E}^{2\vartheta-2}+\sigma)^{m+1}}$$

$$=\sum_{m=0}^{\infty}\frac{(-\tau)^m\,\mathcal{E}^{-2\vartheta m-2\vartheta}}{(1+\sigma\,\mathcal{E}^{2-2\vartheta})^{m+1}}$$

$$=\sum_{m=0}^{\infty}(-\tau)^m\,\mathcal{E}^{-2\vartheta m-2\vartheta}\sum_{\aleph=0}^{\infty}\left(-\sigma\,\mathcal{E}^{2-2\vartheta}\right)^{\aleph}\binom{m+\aleph}{\aleph}$$

$$=\sum_{m=0}^{\infty}(-\tau)^m\sum_{\aleph=0}^{\infty}\binom{m+\aleph}{\aleph}\,(-\sigma)^{\aleph}\,\mathcal{E}^{(2-2\vartheta)\aleph-2\vartheta m-2\vartheta} \tag{9}$$

Substituting the above equation (9) in (8) and taking the inverse, yields the solution (7)

$$k(t)=\;c_0\sum_{m=0}^{\infty}\frac{(-\tau)^m}{m!}\sum_{\aleph=0}^{\infty}\frac{\Gamma(m+\aleph+1)\,(-\sigma)^{\aleph}t^{(\vartheta-1)\aleph+\vartheta k}}{\Gamma[(\vartheta-1)\aleph+\vartheta m+1]\,\aleph!}$$
$$+\,c_1\sum_{m=0}^{\infty}\frac{(-\tau)^m}{m!}\sum_{\aleph=0}^{\infty}\frac{\Gamma(m+\aleph+1)\,(-\sigma)^{\aleph}t^{(\vartheta-1)\aleph+\vartheta k+1}}{\Gamma[(\vartheta-1)\aleph+\vartheta m+2]\,\aleph!}$$
$$+\,\sigma c_0\sum_{m=0}^{\infty}\frac{(-\tau)^m}{m!}\sum_{\aleph=0}^{\infty}\frac{\Gamma(m+\aleph+1)\,(-\sigma)^{\aleph}t^{(\vartheta-1)\aleph+\vartheta k+\vartheta-1}}{\Gamma[(\vartheta-1)\aleph+\vartheta m+\vartheta]\,\aleph!}$$

which is (7). This completes the proof of the theorem. Also, the Wright function can express this solution as

$$k(t) = c_0 \sum_{m=0}^{\infty} \frac{(-\tau)^m\, t^{\vartheta m}}{m!}\, {}_1\lambda_1\left(\begin{matrix} (m+1, 1 \\ (\vartheta m + 1, \vartheta - 1) \end{matrix} \middle| -\sigma\, t^{\vartheta - 1} \right)$$

$$+ c_1 \sum_{m=0}^{\infty} \frac{(-\tau)^m\, t^{\vartheta m + 1}}{m!}\, {}_1\lambda_1\left(\begin{matrix} (m+1, 1 \\ (\vartheta m + 2, \vartheta - 1) \end{matrix} \middle| -\sigma\, t^{\vartheta - 1} \right)$$

$$+ a c_0 \sum_{m=0}^{\infty} \frac{(-\tau)^m\, t^{\vartheta m + \vartheta - 1}}{m!}\, {}_1\lambda_1\left(\begin{matrix} (m+1, 1 \\ (\vartheta m + \vartheta, \vartheta - 1) \end{matrix} \middle| -\sigma\, t^{\vartheta - 1} \right)$$

Example 3.2. The fractional differential equation

$$k^{\frac{3}{2}}(t) + 4k^{'}(t) + 11\, k(t) = 0$$

With initial conditions $k(0) = c_0$ and $k^{'}(0) = c_1$ has the unique solution

$$k(t) = c_0 \sum_{m=0}^{\infty} \frac{(11)^m}{m!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(m + \aleph + 1)\, (4)^{\aleph} t^{\left(\frac{1}{2}\right)\aleph + \frac{3}{2}m}}{\Gamma\left[\left(\frac{1}{2}\right)\aleph + \frac{3}{2}m + 1\right]\, \aleph!}$$

$$+ c_1 \sum_{m=0}^{\infty} \frac{(11)^m}{m!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(m + \aleph + 1)\, (4)^{\aleph} t^{\left(\frac{1}{2}\right)\aleph + \frac{3}{2}m + 1}}{\Gamma\left[\left(\frac{1}{2}\right)\aleph + \frac{3}{2}m + 2\right]\, \aleph!}$$

$$+4 c_0 \sum_{m=0}^{\infty} \frac{(11)^m}{m!} \sum_{r=0}^{\infty} \frac{\Gamma(m + \aleph + 1)\, (4)^{\aleph} t^{\left(\frac{1}{2}\right)\aleph + \frac{3}{2}m + \frac{1}{2}}}{\Gamma\left[\left(\frac{1}{2}\right)\aleph + \frac{3}{2}m + \frac{3}{2}\right]\, \aleph!}$$

Figure 2 illustrates the solution behavior of the fractional differential equation of Example 3.2 at various values of ϑ with the initial conditions $c_0 = 1$ and $c_1 = 1$.

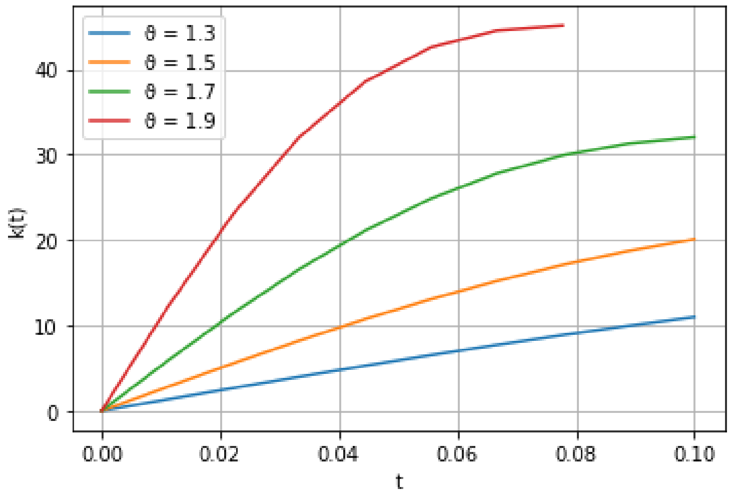

Figure 2: The solution behavior of Example 3.2

Proposition 3.1. Let $1 < \vartheta < 2$ and $\tau \in \mathbb{R}$. Then the fractional differential equation

$$k^{\alpha}(t) - b\,k(t) = 0 \tag{10}$$

with initial conditions $k(0) = c_0$ its proposal is provided by

$$k(t) = c_0 \sum_{m=0}^{\infty} \tau^m \frac{t^{\vartheta m}}{\Gamma(\vartheta m + 1)} = c_0 E_{\vartheta}\left(\tau\, t^{\vartheta}\right) \tag{11}$$

Proof: The proof of this proposition as like as previous theorem.

Remark 3.1. Accordingly, $a = 0$ in (6), then the derivative is

$$k^{\vartheta}(t) + \tau\, k(t) = 0\ ;\ 1 < \vartheta \le 2 \tag{12}$$

with initial conditions $k(0) = c_0$ and $k'(0) = c_1$ its proposal is provided by

$$k(t) = c_0 E_{\vartheta,1}\left(-\tau t^{\vartheta}\right) + c_1 E_{\vartheta,2}\left(-\tau\, t^{\vartheta}\right) \tag{13}$$

Proposition 3.2. A nearly simple harmonic vibration differential equation

$$k^{\vartheta}(t) + z^2 k(t) = 0\,;\ 1 < \vartheta \le 2 \tag{14}$$

with initial conditions $k(0) = c_0$ and $k'(0) = c_1$ its proposal is provided by

$$k(t) = c_0 E_{\vartheta,1}\left(-z^2 t^{\vartheta}\right) + c_1 E_{\vartheta,2}\left(-z^2 t^{\vartheta}\right)$$

Proof: The above proof is accomplished by implanting $\tau = z^2$ in equation(13).

IV. CONCLUSION

The article utilized the Pourreza transform to address certain fractional differential equations. The connection between the Pourreza transform and the Laplace transform was explored in greater detail, revealing additional instances of the Pourreza transform's applicability. A unique methodology for tackling fractional differential equations was introduced, involving the application of the Pourreza transform alongside binomial series extension coefficients. The focus also encompassed the examination of various properties and illustrated examples.

REFERENCES

[1] M. Caputo, “Elasticity dissipazione (Elasticity and anelastic dissipation)”, Zanichelli, Bologna, vol. 4, pp. 98, 1969.
[2] I. Podlubny, “Fractional Differential Equation”, Academic Press, San Diego, 1999.
[3] J. A. Zhang, “Sumudu based algorithm for solving differential equations”, Computer Science Journal of Moldova, vol. 15, no. 3, pp. 45–53, 2007.
[4] K. S. Aboodh, “The New Integral transform Aboodh transform”, Global Journal of Pure and Applied Mathematics, vol. 9, no. 1, pp. 35 - 43, 2013.
[5] S. Lin, C. Lu, “Laplace transform for solving some families of fractional differential equations and its applications”, Advances in Difference Equations, pp. 1-9, 2013.
[6] A. E. Mohamed, “Elzaki transformation for Linear Fractional Differential Equations”, Journal of Computational and Theoretical Nanoscience, vol. 12, pp. 2303 - 2315, 2015.
[7] A. Kashuri, A. Fundo, R.Liko, “New Integral transform for solving some fractional differential equations”, International Journal of Pure and Applied Mathematics, vol. 103, no. 4, pp. 675-682, 2015.
[8] M. M. Abdelrahim Mahgoub, “The New Integral Transform Mohand Transform”, Adv Theoret Appl Mathe, vol. 12, no. 2, pp. 113–120, 2017.
[9] F. S. Silva, D. M. Moreira, and M. A. Moret, “Conformable Laplace transform of fractional differential equations. Axioms”, vol. 7, no. 3, pp. 55, 2018.
[10] R. Aruldoss and R. Anusuya Devi, “Aboodh Transform for Solving Fractional Differential Equations”, Global Journal of Pure and Applied Mathematics, vol. 16, no. 2, pp. 145-153, 2020.
[11] P. Raghavendran P, T. Gunasekar, H. Balasundaram, S. S. Santra, D. Majumder, D. Baleanu, “Solving fractional integro-differential equations by Aboodh transform”, Journal of Mathematics and Computer Science, vol. 32, no. 3, pp 229-240, 2024.
[12] S. A. P. Ahmadi, H. Hosseinzadeh, A. Y. Cherati, A New Integral Transform for Solving Higher Order Linear Ordinary Differential Equations.Nonlinear Dyn Syst Theory, vol. 19, no. 2, pp 243-252, 2019.
[13] Burqan, A., Saadeh, R., Qazza, A., & Momani, S. (2023). ARA-residual power series method for solving partial fractional differential equations. Alexandria Engineering Journal, 62, 47-62.
[14] T. Gunasekar, P. Raghavendran, S.S. Santra, D. Majumder, D. Baleanu, & H. Balasundaram, "Application of Laplace transform to solve fractional integro-differential equations," Journal of Mathematics and Computer Science, 33(3) (2024), 225-237.
[15] Th. Gunasekar, P. Raghavendran, Sh. S. Santra, M. Sajid, "Analyzing existence, uniqueness, and stability of neutral fractional Volterra-Fredholm integro-differential equations," Journal of Mathematics and Computer Science, 33(4) (2024), 390-407.
[16] Th. Gunasekar, P. Raghavendran, Sh. S. Santra, M. Sajid, "Existence and controllability results for neutral fractional Volterra-Fredholm integro-differential equations," Journal of Mathematics and Computer Science, 34(4) (2024), 361-380.
[17] T. Gunasekar, and P. Raghavendran, "The Mohand transform approach to fractional integro-differential equations," Journal of Computational Analysis and Applications, 33 (2024), 358-371.
[18] T. Gunasekar, J. Thiravidarani, M. Mahdal, P. Raghavendran, A. Venkatesan, and M. Elangovan, "Study of Non-Linear Impulsive Neutral Fuzzy Delay Differential Equations with Non-Local Conditions," Mathematics, 11(17) (2023), 3734.