SOME PROPERTIES OF MOBIUS TRANSFORMATION

Abstract

In conformal mapping, we highlight mostly the topic of Möbius transformation. We identify how different areas and curves are transformed by this transformation. There are some elementary mappings which will be used frequently to explain the various concepts of conformal mapping. Here we describe ‘‘Möbius transformation’’ and its related properties.

Keywords: Mobius Transformation, Conformal Transformation, Critical Points, Fixed Points.

Authors

Anamika Rai
Department of Mathematics
K. S. Saket P. G. College
Ayodhya, Uttar Pradesh, India
anamikarai2538@gmail.com

S. K. Tiwari
Department of Mathematics
K. S. Saket P. G. College
Ayodhya, Uttar Pradesh, India
sktiwarisaket@yahoo.com

I. INTRODUCTION

The function w = f(z) = u + iv of complex variables is a rule that assigns a complex number w of w- plane to each complex number z of z- plane . If a point z_0 of $z -$ plane maps into the point w_0 of w- plane then w_0 is known as the image of z_0. If the points in z- plane describes curve C in z- plane then points of w- plane describes a curve C' because corresponding every point (x, y) in z- plane the function w = f(z) defines a corresponding point (u,v) in w- plane .Therefore the relation w = f (z) defines a mapping or transformation of z- plane into w- plane .

II. CONFORMAL TRANSFORMATION

Let two smooth curves curves Υ_1 and Υ_2 intersecting at point z_0 in z- plane and their image curves Υ_1' and Υ_2' in w- plane under the map w = f(z) intersect at the point $w_0 = f(z_0)$. If the angle between the curves Υ_1 and Υ_2 at z_0 is same as the angle between the image curves Υ_1' and Υ_2' at point w_0 both in magnitude and sense, then the mapping w = f(z) is called conformal transformation at z_0 .

1. **Necessary condition for transformation w = f(z) to be a conformal mapping:** If a mapping w = f(z) is conformal at a point z_0 , then f(z) is analytic at z_0.

2. **Sufficient condition for transformation w = f(z) to be a conformal mapping:** If f(z) is analytic function of z of domain D of z- plane and $f'(z) \neq 0$ inside D then the transformation w = f (z) is conformal at all points of D.

3. **Critical Points:** The points at which $f'(z) = \frac{dw}{dz} = 0$ or ∞ are called critical points. In other words, we can say that at critical points the conformal property does not hold good.

4. **Coefficient of Magnification:** Coefficient of magnification for the conformal mapping w = f(z) at $z = \alpha + i\beta$ is $|f'(\alpha + i\beta)|$.

5. **Angle of Rotation:** Angle of rotation for the conformal mapping w = f (z) at $z = \alpha + i\beta$ is arg $[f'(\alpha + i\beta)]$.

III. MÖBIUS TRANSFORMATION

Möbius transformation is composition of the following four types of general transformations:

1. **Some General Transformations**

   - **Translation:** The map $w = z + \beta$ corresponds to a translation. By this transformation the figure in w – plane is same as figure in z – plane with different origin .
   - **Magnification or Dilation:** Consider the map w = a z, where a is real number. If two figures in z- plane and w- plane are similar and similarly situated about their respective origins but the figure in w- plane is ‘a’ times the figure in z- plane, then such map is called magnification.

- **Rotation:** By the transformation $w = e^{i\Theta}z$, $\Theta \in R$ figures in z- plane are rotated through an angle Θ. If $\Theta > 0$, the rotation is anti clockwise. If $\Theta < 0$ then rotation is clockwise.
- **Inversion:** By means of a transformation $w = \frac{1}{z}$, figures in z- plane are mapped upon the reciprocal figures in w- plane.

2. Definition of Möbius Transformation: Möbius transformation is of the form

$$w = T(z) = \left(\frac{az+b)}{cz+d}\right) \tag{3.1}$$

where a, b, c, d are complex constants such that $ad - bc \neq 0$. This transformation is called "Bilinear Transformation" or "Linear Fractional Transformation". From equation (3.1), we have

$$cwz + dw - az - b = 0 \ . \tag{3.2}$$

Above equation is linear in z and w so it is called bilinear transformation. Further from (3.1), we obtained

$$z = T^{-1}(w) = \frac{-dw+b}{cw-a}, \tag{3.3}$$

where $(-d)(-a) - bc = ad - bc \neq 0$. The transformation (3.3) is the inverse of (3.1). Thus the inverse of bilinear transformation is also bilinear transformation with the same determinant.

Note: All the four general transformations mentioned above are bilinear transformation. Further the transformations T and T^{-1} can be discussed in extended complex plane. We defined

$$T(\infty) = \lim_{z\to\infty} T(z) = \lim_{z\to\infty} \frac{a+\frac{b}{z}}{c+\frac{d}{z}} = \frac{a}{c}$$

and it's inverse is $T^{-1}(\frac{a}{c}) = \infty$.

Furthermore, the value of $T^{-1}(\infty)$ is given as

$$T^{-1}(\infty) = \lim_{w\to\infty} T^{-1}(w) = -\frac{d}{c}$$

and it's inverse is $T(-\frac{d}{c}) = \infty$.

Thus, we conclude that transformation $w = T(z)$ is a one to one mapping of the extended complex z – plane onto the extended complex w- plane .

3. Properties of Möbius Transformation: Möbius transformations have a lot of important properties. These properties are valid for whole extended complex z- plane. In this section, we discussed the most significant properties of Möbius transformations.

(P_1) Bilinear transformations are conformal mapping of the extended z- plane.
Proof: A bilinear transformation is given by

$$w = \frac{az+b}{cz+d}, \ ad - bc \neq 0 .$$

Then,

$$\frac{dw}{dz} = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad - bc}{(cz+d)^2} \neq 0.$$

So, w (z) is a conformal mapping .

(P_2) The product of two bilinear transformation is again a bilinear transformation.

Proof: Suppose T and S are two bilinear transformations defined by

$$T(z) = \frac{az+b}{cz+d} ; \ ad - bc \neq 0 ,$$

$$\text{And } S(z) = \frac{a'z+b'}{c'z+d'} ; \ a'd' - b'c' \neq 0.$$

Then the product ToS is given by

$$(ToS)(z) = T(S(z)) = \frac{a\left(\frac{a'z+b'}{c'z+d'}\right) + b}{c\left(\frac{a'z+b'}{c'z+d'}\right) + d} = \frac{(aa'+bc')z + ab' + bd'}{(ca'+dc')z+cb'+dd'} = \frac{Az+B}{Cz+D},$$

where $A = aa' + bc'$, $B = ab' + bd'$, $C = ca' + dc'$, $D = cb' + dd'$.

Also

$$AD - BC = ad\,(a'd' - b'c') - bc\,(a'd' - b'c')$$
$$= (ad - bc)(a'd' - b'c')$$
$$\neq 0 \ (\text{since } ad - bc \neq 0 \text{ and } a'd' - b'c' \neq 0) .$$

Thus ToS is a bilinear transformation.

(P_3) Every bilinear transformation is a product of translation, inversion , and dilation.

Proof: Let us consider a bilinear transformation

$$T(z) = \frac{az+b}{cz+d}, \text{ where } a, b, c, d \in C \text{ and } ad - bc \neq 0 .$$

Case -I : If $c = 0$ then $ad \neq 0$ i.e., $a \neq 0$, $d \neq 0$ and

$$T(z) = \frac{az+b}{d} = \frac{a}{d} z + \frac{b}{d}$$

$$= T_1(z) + \frac{b}{d} \ (\text{ where } T_1(z) = \frac{a}{d} z \text{ and } \frac{a}{d} \neq 0 \text{ i.e., } T_1 \text{ is a dilation })$$

$$= T_2(T_1(z)) \ (\text{ where } T_2(T_1(z) = T_1(z) + \frac{b}{d} \text{ is a translation }) .$$

Hence $T(z) = T_2 o\, T_1$.

Case – II $c \neq 0$

Now,

$$T(z) = \frac{az+b}{cz+d} - \frac{a}{c} + \frac{a}{c}$$

$$= \frac{bc - ad}{c^2} \frac{1}{z+\frac{d}{c}} + \frac{a}{c} \ (\text{here } \frac{bc-ad}{c^2} \neq 0)$$

$$= \frac{bc - ad}{c^2} \frac{1}{T_1(z)} + \frac{a}{c} \ (\text{where } T_1(z) = z + \frac{d}{c}, \text{ is a translation })$$

$= \frac{bc-ad}{c^2} T_2(T_1(z) + \frac{a}{c}$ (where $T_2(T_1(z) = \frac{1}{T_1(z)}$ is the inversion)
$= T_3(T_2(T_1(z)) + \frac{a}{c}$ (where $T_3(T_2(T_1(z)) = \frac{bc-ad}{c^2}(T_2(T_1(z))$ is a dilation)
$= T_4(T_3(T_2(T_1(z)))$ (where $T_4(z) = (T_3(T_2(T_1(z)))) + \frac{a}{c}$ is a translation).

Therefore $T = T_4 o T_3 o T_2 o T_1$.
Hence proved.

(P_4) The inverse of a bilinear transformation is also a bilinear transformation.
Proof: The proof is already done (see section III. 2).
(P_5) The identity mapping w = z is trivially a bilinear transformation.
(P_6) The associative law for composition of bilinear transformation holds.
(P_7) Every bilinear transformation maps circles and straight lines into circles and straight lines (a line is a circle with infinite radius i.e., line is a circle through the point of infinity).
Proof : `Under each of the elementary transformations the family of circles and straight lines are transformed into the family of circles and lines .
Hence the result follows.
Therefore from the above properties (P_2), (P_4), (P_5) and (P_6) we can state the following:

Theorem 3.1: The set of all bilinear transformations form a group with respect to the composition of bilinear transformations.

4. Invariant or Fixed Points

Definition: The point which coinsides with their transformation is called invariant point of the transformation i.e., fixed point of a transformation is obtained by the equation z = f (z).

Proposition 1: Every bilinear transformation (except the identity map) has at most two fixed point.

Proof: If T(z) has a fixed point z, then T(z) = z of

$$\frac{az+b}{cz+d} = z \Leftrightarrow cz^2 + dz = az + b \Leftrightarrow cz^2 - (a\text{-}d)\,z - b = 0.$$

The last equation is quadratic in z and hence can have at most two roots.
For the identity map, I (z) = z, every point of the domain is a fixed point.
This completes the proof.

Proposition 2: If a bilinear transformation w = f(z) has exactly two fixed points z_1 and z_2 , then they satisfy the equation

$$\frac{w-z_1}{w-z_2} = k\,\frac{z-z_1}{z-z_2}\,. \qquad (3.4)$$

where k is non- zero constant .

Further, if T(z) has only one fixed point say z_1 , then it can be written as

$$\frac{1}{w-z_1} = k' + \frac{1}{z-z_1}, \; k' \neq 0 . \quad (3.5)$$

Proof: First Part: Let z_1 and z_2 be the given fixed points of the bilinear transformation $w = \frac{az+b}{cz+d}$ and these are the roots of the equation

$$cz^2 - (a-d)z - b = 0 .$$

This means

$$cz_1^2 - (a-d)z_1 - b = 0 \Leftrightarrow cz_1^2 - a\,z_1 = b - dz_1 , \quad (3.6)$$
$$cz_2^2 - (a-d)z_2 - b = 0 \Leftrightarrow cz_2^2 - a\,z_2 = b - dz_2 . \quad (3.7)$$

Using (3.6), we get

$$\begin{aligned} w - z_1 &= \frac{az+b}{cz+d} - z_1 \\ &= \frac{az + b - z_1(cz+d)}{cz+d} \\ &= \frac{(a-z_1c)z + b - dz_1}{cz+d} \\ &= \frac{(a-z_1c)z + cz_1^2 - a\,z_1}{cz+d} \\ &= \frac{(a-z_1c)\,(z-z_1)}{cz+d} . \end{aligned}$$

Similarly using equation (3.7), we have

$$w - z_2 = \frac{(a-cz_2)\,(z-z_2)}{cz+d} .$$

Hence, $\frac{w-z_1}{w-z_2} = \frac{a-cz_1}{a-cz_2} \cdot \frac{z-z_1}{z-z_2} = k\,\frac{z-z_1}{z-z_2}$,

where, $k = \frac{a-cz_1}{a-cz_2}$.

Second Part: For the second part , z_1 is the only fixed point . Then the equation $cz^2 - (a\text{-}d)z - b = 0$ has one root z_1 , say . So

$$cz_1^2 - (a-d)z_1 - b = 0 \Leftrightarrow cz_1^2 - az_1 = b - dz_1$$

and z_1(being the repeated root) is given by

$$z_1 = \frac{a-d}{2c} \Leftrightarrow a - cz_1 = d + cz_1 . \quad (3.8)$$

From previous analysis, we obtained

$$\begin{aligned} \frac{1}{w-z_1} &= \frac{cz + a - cz_1 - cz_1}{(a-cz_1)(z-z_1)} \\ &= \frac{c(z-z_1) + a - cz_1}{(a-cz_1)(z-z_1)} \\ &= \frac{c}{a-cz_1} + \frac{1}{z-z_1} . \end{aligned}$$

Therefore, $\frac{1}{w-z_1} = k' + \frac{1}{z-z_1}$, $k' = \frac{c}{a-cz_1} = \frac{2c}{a+d}$.

Hence proved.

Remarks

- Equations (3.4) and (3.5) are known as the normal form or canonical form of a bilinear transformation.
- A Möbius transformation which has a unique fixed point is parabolic.
- If a Möbius transformation has exactly two fixed points, then it is called loxodromic.

5. Cross Ratio: In this section, we introduce the concept of cross ratio to discuss the specific bilinear transformation that maps three distinct points in the extended z- plane onto three distinct points in the extended w- plane.

Definition : If z, z_1, z_2, z_3 are distinct points , then cross ratio of z, z_1, z_2, z_3 is denoted by (z, z_1, z_2, z_3) and defined by

$$(z, z_1, z_2, z_3) = \frac{(z- z_1)(z_2 - z_3)}{(z_1 - z_2)\,(z_3 - z)} . \tag{3.9}$$

Theorem 3.2: A bilinear transformation preserve cross ratio i.e., if z, z_1, z_2, z_3 are transform to w, w_1, w_2 ,w_3 respectively then (z, z_1, z_2, z_3) = (w, w_1, w_2 ,w_3) .

Proof: The bilinear transformation is given by

$$w = T(z)= \frac{az+b}{cz+d} , \; ad – bc = 1 ,$$

such that , $w_k = T(z_k)$, k= 1, 2, 3 then we have to show that

$$(w, w_1, w_2 ,w_3) = (T(z),T(z_1),T(z_2),T(z_3)) =(z, z_1, z_2, z_3) . \tag{3.10}$$

Since z_k corresponds to w_k ,therefore we have

$$w - w_k = \frac{z - z_k}{(cz_k + d)\,(cz + d)} ,$$

where we have used ad- bc = 1.
From above equation, we have

$$w - w_1 = \frac{z- z_1}{(cz_1 +d)\,(cz + d)} , \; w - w_2 = \frac{z- z_2}{(cz_2+d)\,(cz + d)} ,$$
$$w - w_3 = \frac{z- z_3}{(cz_3+d)(cz+d)} . \tag{3.11}$$

Replace w by w_2 , and z by z_2 in equation (3.11) , we get

$$w_2 - w_1 = \frac{z_2- z_1}{(cz_1+d)(cz_2+d)} ,$$
$$w_2 - w_3 = \frac{z_2- z_3}{(cz_3+d)(cz_2+d)}. \tag{3.12}$$

Equations (3.11) and (3.12) yield

$$\frac{w - w_1}{w_1-w_2} \cdot \frac{w_2- w_3}{w_3 - w} = \frac{z - z_1}{z_1- z_2} \cdot \frac{z_2- z_3}{z_3- z} ,$$

or $$(w,w_1, w_2, w_3) = (z, z_1, z_2, z_3) .$$

Hence proved the theorem.

IV. SOLVED PROBLEMS

Problem 1: Prove that the transformation $w = \frac{z+3}{z-2}$ maps the circle $x^2 + y^2 - 2x = 0$ into the straight line 2u+ 3 = 0.

Solution: The given transformation is

$$w = \frac{z+3}{z-2}. \quad (4.1)$$

From equation (4.1), we have

$$z = \frac{2w+3}{w-1}.$$

Therefore

$$\bar{z} = \frac{2\bar{w}+3}{\bar{w}-1}. \quad (4.2)$$

The given equation of circle is

$$x^2 + y^2 - 2x = 0,$$

i.e., $(x+iy)(x-iy) - 2x = 0$,

or, $z\bar{z} - (z + \bar{z}) = 0$ (since $2x = z + \bar{z}$).

With the help of equation (4.2) , above equation can be written as

$$\frac{2w+3}{w-1} = \frac{2\bar{w}+3}{\bar{w}-1} - \left(\frac{2w+3}{w-1} + \frac{2\bar{w}+3}{\bar{w}-1} \right) = 0,$$

or $5(w + \bar{w}) + 15 = 0$,

or $2u + 3 = 0$ (since $w + \bar{w} = 2u$) .

This is the equation of straight line in u – plane .

Problem 2: Find out the fixed point and the normal for the bilinear transformation $w = \frac{3z-4}{z-1}$

Solution: For fixed (invariant) point, put w = z in the given bilinear transformation $w = \frac{3z-4}{z-1}$ for fixed points , we get $(z-2)^2 = 0$.

Thus z = 2 is the only fixed point .

To obtain normal form of the given bilinear transformation , we proceed as follows

$$w - 2 = \frac{3z-4}{z-1} - 2 = \frac{z-2}{z-1},$$

i.e., $\frac{1}{w-2} = \frac{z-1}{z-2} = \frac{z-2+1}{z-2} = 1 + \frac{1}{z-2}$,

which is the required normal form .

Problem 3: Find out the bilinear transformation that maps the points 1, 2, 0 into points 1, 0, i .

Solution: We know that the bilinear transformation that maps the points z_1, z_2, z_3 into points w_1, w_2, w_3 respectively is

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}.$$

Substituting the points in above equation , we obtain

$$\frac{(w-1)(0-i)}{(1-0)(i-w)} = \frac{(z-1)(2-0)}{(1-2)(0-z)},$$

or $\frac{i - iw}{i - w} = \frac{2(z-1)}{z}$,

or (i – iw) z = 2(i – w) (z - 1),

or $w = \frac{iz - 2i}{(2 - i)z - 2}$.

This is the required bilinear transformation.

Problem 4: Find out the bilinear transformation which maps the points i , ∞, 0 into the points 0 , i , ∞ respectively .

Solution: We know the bilinear transformation that maps z_1, z_2, z_3 onto w_1, w_2, w_3 respectively is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)},$$

or $$\frac{(w-w_1)(\frac{w_2}{w_3}-1)}{(w_1-w_2)(1-\frac{w}{w_3})} = \frac{(z-z_1)(1-\frac{z_3}{z_2})}{(\frac{z_1}{z_2}-1)(z_3-z)}.$$

Substituting points in above equation , we have

$$\frac{(w-i)(0-1)}{(i-0)(1-0)} = \frac{(z-i)(1-0)}{(0-1)(0-z)},$$

or $\frac{w-i}{i} = -\left(\frac{z-i}{z}\right)$,

or wz – iz = - iz – 1,

or wz = -1,

or $w = -\frac{1}{z}$,

which is the required transformation.

Problem 5: For the conformal mapping $w = z^2$, show that

(a) The coefficient of magnification at z = 3 +i is $2\sqrt{10}$.

(b) The angle of rotation at z = 3 + i is $\tan^{-1}\frac{1}{3}$.

Solution

(a) For conformal mapping w = f(z) , the coefficient of magnification at $z = z_0$ is $|f'(z_0)|$.
Here w = f(z) = z^2,therefore $f'(z) = 2z$.
Hence $f'(3+i) = 2(3+i) = 6+2i$.
Therefore coefficient of magnification at z = 3+i is
$= \quad |f''(3+i)| = |6+2i| = \sqrt{6^2+2^2} = 2\sqrt{10}$.

(b) For conformal mapping w= f(z), angle of rotation at $z = z_0$ is arg $\{f'(z_0)\}$.
Here f(z) = z^2 , therefore $f'(z) = 2z$.
Therefore angle of rotation at z = 3+ i
$= \arg[f'(3+i)] = \arg(6+2i) = \tan^{-1}\frac{2}{6} = \tan^{-1}\frac{1}{3}$.

Problem 6: Find the image of the rectangular region of the z- plane bounded by the lines x= 0, y = 0, x = 2, y =1 under the transformation w = z+(3-i) in the w – plane .

Solution: The given transformation is

$$w = z + (3-i) \quad . \qquad (4.3)$$

Using z= x+ iy and w = u+ iv in above equation, we get

$$u + iv = x + iy + (3 - i) = (x+3) + i(y - 1) .$$

Comparing real and imaginary parts on both sides in above equation, we obtain

$$u = x + 3 \text{ and } v = y - 1. \qquad (4.4)$$

Put
(a) x = 0 in (4.4), we get u = 3 ,
(b) y = 0 in (4.4) , we get v = -1 ,
(c) x = 2 in (4.4), we get u = 5,
(d) y = 1 in (4.4) , we get v = 0 .

Thus , the image of the rectangular region of the z – plane bounded by the lines x = 0 , y = 0, x = 2 and y = 1 under the transformation w = z +(3- i) is the rectangular region bounded by u = 3 , v = -1, u = 5 and v = 0 in the w – plane .

Problem 7: Find out the image of the rectangular region bounded by the lines x = 0, y= 0, x = 1, y= 2 in z- plane under the transformation w = 2z in the w – plane.

Solution: The given transformation is

$$w = 2z. \qquad (4.5)$$

Using z= x +iy and w = u + iv in above equation (4.5), we have

$$u + iv = 2 (x + iy),$$

or $$u + iv = 2x + 2iy .$$

Comparing real and imaginary parts on both sides in above equation, we get

$$u = 2x \quad \text{and } v = 2y. \qquad (4.5)'$$

Put
(a) x = 0 in $(4.5)'$, we get u = 0,
(b) y = 0 in $(4.5)'$, we get v = 0 ,
(c) x = 2 in $(4.5)'$, we get u = 4 ,
(d) y = 3 in $(4.5)'$, we get v = 6 .

Hence, the image of the rectangular region of the z – plane bounded by the lines x = 0, y = 0, x = 2, y = 3 under the transformation w = 2z is the rectangular region bounded by u = 0, v = 0, u =4, v = 6 in w – plane.

Problem 8: What is the image of triangular region of z – plane bounded by the lines x = 0, y = 0, $\sqrt{3}$ x + y = 1 under the transformation w = $e^{i\pi/3}$z in the w- plane.

Solution: The given transformation is

$$w = e^{i\pi/3}z. \qquad (4.6)$$

Substituting z = x +iy and w = u + iv in above equation (4.6) , we get

$$u + iv = \left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3} \right) (x + iy) \text{ , (since } e^{i\Theta} = \cos\Theta + i\sin\Theta \text{) ,}$$

or $$u + iv = \frac{(1 + i\sqrt{3})}{2} (x + iy) ,$$

or $2(u + iv) = x + iy + i\sqrt{3}\,x - \sqrt{3}\,y$,
or $2u + 2iv = (x - \sqrt{3}\,y) + i(\sqrt{3}\,x + y)$.

Comparing real and imaginary parts on both sides in above equation, we have

$$2u = x - \sqrt{3}\,y \,, \tag{4.7}$$
$$2v = \sqrt{3}\,x + y \,. \tag{4.8}$$

Multiplying both sides of equation (4.8) by $\sqrt{3}$, we have

$$2\sqrt{3}\,v = 3x + \sqrt{3}\,y \,. \tag{4.9}$$

Adding equations (4.7) and (4.9) , we obtain

$2u + 2\sqrt{3}\,v = x + 3x = 4x$,

or $$u + \sqrt{3}\,v = 2x \,. \tag{4.10}$$

Further multiplying both sides of equation (4.7) by $\sqrt{3}$, we get

$$2\sqrt{3}\,u = \sqrt{3}\,x - 3y \,. \tag{4.11}$$

Again subtracting equation (4.8) from equation (4.11), we have

$2\sqrt{3}\,u - 2v = -3y - y$,

or $2(\sqrt{3}\,u - v) = -4y$,

or $\sqrt{3}\,u - v = -2y$,

or $$v - \sqrt{3}\,u = 2y \,. \tag{4.12}$$

Put

(a) x = 0 in (4.10) , we get $v = -\frac{1}{\sqrt{3}}\,u$,

(b) y = 0 in (4.12) , we get $v = \sqrt{3}\,u$.

Further from equation (4.8), we have

$2v = 1$ (since $\sqrt{3}\,x + y = 1$),

or $v = \frac{1}{2}$.

Hence, the image of triangular region of z- plane bounded by the lines x=0, y= 0, $\sqrt{3}\,x + y = 1$ under the transformation $w = e^{i\pi/3}z$ is the triangular region bounded by $v = -\frac{1}{\sqrt{3}}\,u$, $v = \sqrt{3}\,u$ and $v = \frac{1}{2}$ in w – plane .

Problem 9: What is the image of the line $y - x + 1 = 0$ in z- plane under the transformation $w = \frac{1}{z}$ in w – plane.

Solution: The given transformation is

$$w = \frac{1}{z} \,. \tag{4.13}$$

Here, $z = x + iy$ and $w = u + iv$.

Putting above in equation (4.13), we get

$$u + iv = \frac{1}{x + iy},$$

or $$x + iy = \frac{1}{u + iv} \cdot \frac{u - iv}{u - iv} = \frac{u - iv}{u^2 + v^2},$$

or $$x + iy = \frac{u}{u^2 + v^2} - i\frac{v}{u^2 + v^2}.$$

Comparing real and imaginary parts on both sides in above equation, we have

$$x = \frac{u}{u^2 + v^2}, \tag{4.14}$$

and $$y = -\frac{v}{u^2 + v^2}. \tag{4.15}$$

With the help of equations (4.14) and (4.15) , equation y – x + 1 = 0 can be written as

$$-\frac{v}{u^2 + v^2} - \frac{u}{u^2 + v^2} + 1 = 0,$$

or $$-v - u + u^2 + v^2 = 0,$$

or $$u^2 - u + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + v^2 - v + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = 0,$$

or $$\left(u - \frac{1}{2}\right)^2 + \left(v - \frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4},$$

or $$\left(u - \frac{1}{2}\right)^2 + \left(v - \frac{1}{2}\right)^2 = \frac{1}{2}.$$

This is equation of the circle with centre ($\frac{1}{2}, \frac{1}{2}$) and radius $\frac{1}{\sqrt{2}}$.
Thus the image of the line y – x + 1 = 0 in z- plane under the transformation $w = \frac{1}{z}$ is a circle in w – plane.

Problem 10: Show that the transformation $w = \frac{1}{z}$ maps the circle in z – plane to a circle in w – plane or a straight line (if the former passes through the origin).

Solution: The given transformation is

$$w = \frac{1}{z}, \tag{4.16}$$

Here, w = u + iv and z = x +iy.

Putting above in equation (4.16), we get

$$u + iv = \frac{1}{x + iy},$$

or $$x + iy = \frac{1}{u + iv},$$

or $$x + iy = \frac{1}{u + iv} \cdot \frac{u - iv}{u - iv} = \frac{u - iv}{u^2 + v^2},$$

or $$x + iy = \frac{u}{u^2 + v^2} - i\frac{v}{u^2 + v^2}.$$

Comparing real and imaginary parts on both sides, we have

$$x = \frac{u}{u^2 + v^2}, \tag{4.17}$$

and $$y = -\frac{v}{u^2 + v^2}. \tag{4.18}$$

Equation of circle in z- plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0. \tag{4.19}$$

Using equations (4.17) and (4.18) in above , we obtain

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + \frac{2gu}{u^2+v^2} + \frac{2f(-v)}{u^2+v^2} + c = 0,$$

or $\frac{1}{u^2+v^2} + \frac{2gu-2fv}{u^2+v^2} + c = 0$,

or $1 + 2gu - 2fv + c\,(u^2+v^2) = 0$,

or $c(u^2+v^2) + 2gu - 2fv + 1 = 0$. (4.20)

Case (i): If $c \neq 0$, equation (4.20) is a equation of circle .
Case (ii): If c = 0, 2gu – 2fv + 1 = 0, is equation of a straight line.

Problem 11: Show that the transformation $w = \frac{1}{z}$ maps a line in z – plane to a circle or staright line in w – plane.

Solution: Given transformation is

$$w = \frac{1}{z} . \quad (4.21)$$

Putting w = u +iv and z = x + iy in above equation (4.21) , we obtain

$$u + iv = \frac{1}{x+iy} ,$$

or $x + iy = \frac{1}{u+iv}$,

or $x + iy = \frac{1}{u+iv} . \frac{u-iv}{u-iv} = \frac{u-iv}{u^2+v^2}$,

or $x + iy = \frac{u}{u^2+v^2} - i\,\frac{v}{u^2+v^2}$.

Comparing real and imaginary parts on both sides, we have

$$x = \frac{u}{u^2+v^2} , \quad y = -\frac{v}{u^2+v^2} . \quad (4.22)$$

The equation of line in z – plane is

$$ax + by + c = 0 . \quad (4.23)$$

Using equation (4.22) in above equation , we have

$$\frac{au}{u^2+v^2} - \frac{bv}{u^2+v^2} + c = 0 ,$$

or $au - bv + c\,(u^2+v^2) = 0$,

or $c(u^2+v^2) + au - bv = 0$. (4.24)

Case (i): If $c \neq 0$ i.e. , line in z- plane does not pass through origin then equation (4.24) becomes a circle in w – plane .
Case (ii): If c = 0 i.e., line in z – plane passes through origin then equation (4.24) becomes a straight line.

V. CONCLUSIONS

In our book chapter, we basically highlight about the topic "Möbius transformation" and see how it transforms different curves and regions from one complex plane to the other complex plane . We have also discussed some of its remarkable properties like conformality (angle preserving property), circle preserving property and what a Möbius transformation is composed of.

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