CERTAIN REDUCTION FORMULAE FOR SRIVASTAVA–DAOUST TYPE SERIES

Abstract

Many of the Physical, Astrophysical, Statistical, and Mathematical problems can be solved by using the allied special functions. Specifically, various multiple hypergeometric series and reduction formulae can be applied to solve such problems. The aim of the present chapter is to derive certain classes of the reduction formulae for the Srivastava–Daoust type doublehypergeometric series. To prove these reductions, we use one of the extension results on the Bailey transform developed and studied by Joshi and Vyas in 2005.We also obtain some well–known formulae, e.g. the Kampé de Fériet reduction formula, the Euler transformation formula and Whipple's quadratic transformation formula, on particularization of some reduction formulae. 2020 Mathematics Subject Classification. 33C05; 33C15; 33C65

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I. INTRODUCTION

The Gauss hyper geometric series is

$$1+\frac{ab}{c}z+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+\cdots+\frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{c(c+1)\cdots(c+n-1)}\frac{z^n}{n!}+\cdots. \tag{1}$$

This series was studied by the famous German mathematician C.F. Gauss [1].

The part $a(a+1)\cdots(a+n-1)$ in "(1)" in terms of Pochhammer's symbol is denoted by

$$(a)_n = a(a+1)\cdots(a+n-1)\,,\ (1)_n = n!\,,\ n \geq 1 \tag{2}$$

For $n=0$, the values of $(a)_n$ is equal to 1. Similar interpretation is for other Pochhammer's symbol, $(b)_n$ and $(c)_n$, present in "(1)".

Thus, "(1)" in terms of Pochhammer's symbols given by:

$$ {}_2F_1(a,\ b\ ;\ c\ ;\ z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} = {}_2F_1\left[\begin{matrix} a,\ b; \\ c; \end{matrix}\ z\right], c \notin \mathbb{Z}^- \cup \{0\} \tag{3}$$

The series in "(3)" is convergent if $|z|<1$. Eq. "(3)" is convergent if it terminates, i.e., when a or b are negative integer or zero. However, for $|z|=1$, it converges if $\mathrm{Re}(c-a-b)>0$ and diverges if $\mathrm{Re}(c-a-b)\leq 0$. If $0 \geq \mathrm{Re}(c-a-b) > -1,\ z \neq 1$, the series "(3)" converges conditionally, see [2, p. 18].

Also, when $a=-m$ or $b=-m$ where $m=0,1,2,\ldots$ and $c=-k$ where $k=m, m+1,\ldots$ in "(3)" then the series "(3)" terminates otherwise it becomes meaningless.

The generalization of "(3)" is as given below:

$$ {}_rF_s\left[\begin{matrix}\alpha_1,\ldots,\alpha_r; \\ \beta_1,\ldots,\beta_s;\end{matrix}\ z\right] = \sum_{p=0}^{\infty}\frac{(\alpha_1)_p\ldots(\alpha_r)_p}{(\beta_1)_p\ldots(\beta_s)_p}\frac{z^p}{p!} = {}_rF_s\left[\begin{matrix}(\alpha_r); \\ (\beta_s);\end{matrix}\ z\right], \tag{4}$$

which have arbitrary number of numerator (α_r) and denominator (β_s) parameters.

Note that r and s are either zero or positive integers and the argument z may take any real or complex value, provided none of the bottom parameters (β_s) in "(4)" is zero or negative integer. This generalized hypergeometric function is convergent or divergent with the following restrictions:

(i). Converges for all z if $r \leq s$; for all $|z|<1$, if $r=s+1$ and for $|z|=1$ with $r=s+1$, if

$$\mathrm{Re}\left(\sum_{j=0}^{s}\beta_j - \sum_{j=0}^{r}\alpha_j\right) > 0. \tag{5}$$

(ii). Diverges for every z, $z \neq 0$, if $r > s+1$;

The generalization of "(3)"can be given by either increasing numerator and denominator parameters, as we have shown in"(4)"or by increasing the arguments. Such series are called multiple hypergeometric series, e. g.Appell functions in two variables, Kampé de Fériet functions, Horn's functions, Srivastava's triple hypergeometric series, Srivastava–Daust series etc. Several authors worked on the reductions and transformations for these multiple hypergeometric series, see ([3],[4],[5],[6],[7], [8], [9],[10]) and references therein. The applications of multiple hypergeometric series in solving a vast number of Physical, Statistical and Mathematical problems can be found in ([11],[12],[2],[13], [14], [15]) and references therein.

The utility of the reduction formulae for certain classes of double series is discussed in several research papers, see([16], [17], [18], [19], [20]) and references therein. They have shown that certain reduction formulae for multiple hypergeometric series are applicable in solving astrophysical problems, queuing theory and related stochastic processes, physical and quantum chemical problems, boundary value problems (heat equation) and in the derivation of radial wave functions. The reduction formulae for Srivastava–Daoust hypergeometric functions have been studied and investigated in a number of papers ([21],[22],[23],[24], [25], [26], [27] and references therein).

In this chapter, we focus on investigating certain Srivastava–Daoust type reduction formulae. Certain reductions are interesting generalizations of some well–known hypergeometric functions e.g. the Euler transformation formula, the Whipple's quadratic transformation and one of the Kampé de Fériet reductions.

The Kampé de Fériet function with an arbitrary number of numerator and denominator parameters and two arguments is as follows:

$$F_{l:m;n}^{p:q;k}\left[\begin{matrix}(a_p):(b_q);(c_k); \\ (\alpha_l):(\beta_m);(\gamma_n);\end{matrix}\; x,\, y\right]$$

$$= \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r y^s}{r!s!}, \tag{6}$$

where the convergence condition is,

(i) $p+q<l+m+1,\; p+k<l+n+1,$

$|x|<\infty,\; |y|<\infty,$

or

(ii) $p+q=l+m+1,\; p+k=l+n+1,$ (7)

$$\left\{\begin{matrix} |x|^{\frac{1}{p-l}}+|y|^{\frac{1}{p-l}} < 1, & if\ p>l \\ \max\ \{|x|,|y|\} < 1, & if\ p\le l \end{matrix}\right\}$$

The Srivastava–Daoust series [2, pp. 26–28], also referred to as the generalized Lauricella function of several variables is as follows.

$$F^{A;B';\cdots;B^{(n)}}_{C:D';\cdots;D^{(n)}}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = F^{A;B';\cdots;B^{(n)}}_{C:D';\cdots;D^{(n)}}\begin{bmatrix} \left[(a_A):\theta',\ldots,\theta^{(n)}\right]: \left[(b'_{B'}):\phi'\right];\cdots;\left[(b^{(n)}_{B^{(n)}}):\phi^{(n)}\right]; & x_1,\ldots,x_n \\ \left[(c_C):\psi',\ldots,\psi^{(n)}\right]:\left[(d'_{D'}):\delta'\right];\cdots;\left[(d^{(n)}_{D^{(n)}}):\delta^{(n)}\right]; & \end{bmatrix}$$
$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \Omega(m_1,\ldots,m_n)\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}, \tag{8}$$

where

$$\Omega(m_1,\ldots,m_n) = \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta'_j+\cdots+m_n\theta^{(n)}_j}\prod_{j=1}^{B'}(b'_j)_{m_1\phi'_j}\cdots\prod_{j=1}^{B^{(n)}}(b^{(n)}_j)_{m_n\phi^{(n)}_j}}{\prod_{j=1}^{C}(c_j)_{m_1\psi'_j+\cdots+m_n\psi^{(n)}_j}\prod_{j=1}^{D'}(d'_j)_{m_1\delta'_j}\cdots\prod_{j=1}^{D^{(n)}}(d^{(n)}_j)_{m_n\delta^{(n)}_j}}. \tag{9}$$

For further details on notations and convergence conditions for “(8)”, please refer to [7] and [9, pp. 157–158]. Eq. “(8)” reduces to “(6)” when $n = 2$.

Note that, in many of the papers concerning reductions or transformations of Srivastava–Daoust double hypergeometric series, the parameters θ, ϕ, ψ and δ ’s appearing in“(8)” to “(9)” are given some particular constant values. For example, see [24]. With the help of one of the extension results on the Bailey transform (“(10)” and “(11)”), we can express these parameters in terms of p that can be assigned any arbitrary integer values. The results with arbitrary values of these parameters are not available in the literature till date. Moreover, it is always possible to derive general reduction formulae involving arbitrary bounded sequence $\Omega(n)$ of complex numbers in place δ_n, provided that the involved series are convergent. Further, the obvious and straightforward generalizations of the results of this paper to reductions or transformations of $(m+1)$ fold series to m–fold series can always be developed after getting the idea of applying Saalschütz summation theorem used in this chapter.

One of the two extension results on the Bailey transform [28] due to Joshi and Vyas [29]is stated as follows:

If

$$\beta_n = \sum_{r=0}^{n}\alpha_r u_{n-r}v_{n+r}t_{n+2r}w_{pn-r}z_{p'n+r} \text{ and} \tag{10}$$

$$\gamma_n = \sum_{r=n}^{\infty}\delta_r u_{r-n}v_{r+n}t_{2n+r}w_{pr-n}z_{p'r+n}$$

then, subject to convergence conditions

$$\sum_{n\geq 0}\alpha_n\gamma_n = \sum_{n\geq 0}\beta_n\delta_n \tag{11}$$

where $\alpha_r, \delta_r, u_r, v_r, w_r, t_r$ and z_r are any functions of r only and p and p' are any arbitrary integers. We use two transforms given in "(10)" to derive the reduction formulas stated in Section 2.

This chapter has three sections. The additional reduction formulae and their derivations are given in Section 2. The particular cases of certain reduction formula are discussed in Section 3.

II. REDUCTION FORMULAE FOR SRIVASTAVA–DAOUST TYPE FUNCTIONS AND THEIR DERIVATIONS

In this section, first we state the reduction formulae for Srivastava–Daoust type functions and then demonstrate the proof of reduction formulae one by one.

$$F^{D+1:1;1}_{G+1:0;0}\left[\begin{matrix}[d_D:1,1],\,[z:p+1,p]:[a:1];[k-a-z:1]; \\ [g_G:1,1],\,[k:p+1,p]:-\,;\,-\,;\end{matrix}\; x,\,x\right]$$
$$= {}_{D+2p+2}F_{G+2p+1}\left[\begin{matrix}d_D,\,k-z,\,\Delta(p;z),\Delta(p+1;k-a); \\ g_G,\,\Delta(p+1;k),\Delta(p;k-a);\end{matrix}\; x\right]. \tag{12}$$

$$F^{D+2:0;0}_{G+1:1;0}\left[\begin{matrix}[d_D:1,1],\,[z:p+1,p],[v:2,1]:\,-;-; \\ [g_G:1,1],[k:p+1,p]:\,[1+v+z-k:1];-;\end{matrix}\; -x,\,x\right]$$
$$= {}_{D+2p+2}F_{G+2p+1}\left[\begin{matrix}d_D,\,v,\,k-z,\,\Delta(p;z), \\ g_G,\,\Delta(p+1;k),\Delta(p-1;k-v),\end{matrix}\right.$$
$$\left.\begin{matrix}\Delta(p;k-v); \\ 1+v+z-k;\end{matrix}\; \frac{p^{2p}(-x)}{(p-1)^{p-1}(p+1)^{p+1}}\right]. \tag{13}$$

$$F^{D+1:1;1}_{G+1:0;0}\left[\begin{matrix}[d_D:1,1],\,[w:p-1,p]:\,[a:1]\,;[j-a-w:1]; \\ [g_G:1,1],\,[j:p-1,p]:-\,;\,-;\end{matrix}\; x,\,x\right]$$
$$= {}_{D+2p}\,F_{G+2p-1}\left[\begin{matrix}d_D,\,j-w,\,\Delta(p;w+a),\Delta(p-1;w); \\ g_G,\,\Delta(p-1;w+a),\Delta(p;j);\end{matrix}\; x\right]. \tag{14}$$

$$F^{D+2:0;0}_{G+1:1;0}\left[\begin{matrix}[d_D:1,1],\,[w:p-1,p],[v:2,1]:-;-; \\ [g_G:1,1],\,[j:p-1,p]:[1+v+w-j:1];-;\end{matrix}\; -x,\,x\right]$$
$$= {}_{D+2p+2}F_{G+2p+1}\left[\begin{matrix}d_D,\,v,\,j-w,\,\Delta(p-1;w),\,\Delta(p+1;w+v); \\ g_G,\,\Delta(p;j),\,\Delta(p;w+v),\quad 1+v+w-j;\end{matrix}\; -x\frac{(p-1)^{p-1}(p+1)^{p+1}}{p^{2p}}\right]. \tag{15}$$

$$F^{D+1:0;0}_{G+1:2;0}\left[\begin{matrix}[d_D:1,1],\,[z:p+2,\,p+1]\;\;-;-;\\ [g_G:1,1],\,[j:p-1,\,p]:[h:1],\,[2+z-j-h:1];-;\end{matrix}\;x,\,x\right]$$

$$={}_{D+3p+3}F_{G+3p+2}\left[\begin{matrix}d_D,\;\Delta(p+1;z),\Delta(p+1;h+j-1),\\ g_G,\;\Delta(p;j),\Delta(p;h+j-1),\Delta(p;1+z-h),\end{matrix}\right. \tag{16}$$

$$\left.\begin{matrix}\Delta(p+1;1+z-h);\\ 2+z-h-j,h;\end{matrix}\;x\left(\frac{(p+1)^{p+1}}{p^p}\right)^3\right].$$

$$F^{D+1:0;0}_{G+2:1;0}\left[\begin{matrix}[d_D:1,1],\,[z:p+3,\,p+2]:-;-;\\ [g_G:1,\,1],\,[j:p-1,\,p],\,[f:2,1]:[2+z-j-v:1];-;\end{matrix}\;x,\,x\right]$$

$$={}_{D+3p+5}F_{G+3p+4}\left[\begin{matrix}d_D,\;\Delta(p+2;z),\Delta(p+2;f+j-1),\\ g_G,\;\Delta(p+1;f+j-1),\Delta(p;1+z-f),\Delta(p;j),\Delta(2;f),\end{matrix}\right. \tag{17}$$

$$\left.\begin{matrix}\Delta(p+1;1+z-f);\\ 2+z-f-j;\end{matrix}\;x\left(\frac{(p+2)^{p+2}}{2p^p}\right)^2\right].$$

$$F^{D+1:0;1}_{G+2:0;0}\left[\begin{matrix}[d_D:1,1],\,[z:p+2,\,p+1]:-;\;[f+j-z-1:1];\\ [g_G:1,\,1],\,[j:p-1,\,p],\,[f:2,1]:-;-;\end{matrix}\;-x,\,x\right]$$

$$={}_{D+3p+3}F_{G+3p+2}\left[\begin{matrix}d_D,\;\Delta(p+1;z),\Delta(p+2;f+j-1),\\ g_G,\;\Delta(p;j),\Delta(p+1;f+j-1),\end{matrix}\right. \tag{18}$$

$$\left.\begin{matrix}\Delta(p;1+z-f);\\ \Delta(p-1;1+z-f),\;\Delta(2;f);\end{matrix}\;-x\frac{(p+2)^{p+2}}{2^2(p-1)^{p-1}}\right].$$

$$F^{D+1:0;0}_{G+3:0;0}\left[\begin{matrix}[d_D:1,1],\,[z:p+4,\,p+3]:-;-;\\ [g_G:1,1],\,[j:p-1,\,p],\,[f:2,1],[2+z-j-f:2,1]:-;-;\end{matrix}\;;x,\,x\right]$$

$$={}_{D+3p+7}F_{G+3p+6}\left[\begin{matrix}d_D,\;\Delta(p+3;z),\Delta(p+2;f+j-1),\\ g_G,\;\Delta(2;f),\Delta(p;j),\Delta(2;\,2+z-f-j),\Delta(p+1;f+j-1),\end{matrix}\right. \tag{19}$$

$$\left.\begin{matrix}\Delta(p+2;1+z-f);\\ \Delta(p+1;1+z-f);\end{matrix}\;x\frac{(p+2)^{2(p+2)}(p+3)^{(p+3)}}{2^4p^p(p+1)^{2(p+1)}}\right].$$

$$F_{G+1:1;0}^{D+1:0;1}\left[\begin{matrix}[d_D:1,1],\,[z:p+1,p]:-;[h+j-z-1:1];\\ [g_G:1,1],\,[j:p-1,p]:[h:1];\,-;\end{matrix}\; -x,\,x\right]$$
$$={}_{D+3p+1}F_{G+3p}\left[\begin{matrix}d_D,\,\Delta(p;z),\Delta(p+1;h+j-1),\Delta(p;1+z-h);\\ g_G,\,\Delta(p;j),\Delta(p;h+j-1),h,\Delta(p-1;1+z-h);\end{matrix}\; -x\frac{(p+1)^{p+1}}{(p-1)^{p-1}}\right]. \tag{20}$$

$$F_{G+1:0;0}^{D+1:0;2}\left[\begin{matrix}[d_D:1,1],\,[z:p,p-1]:-;[j-z-u:1],\,[u:1];\\ [g_G:1,1],\,[j:p-1,p]:-;-;\end{matrix}\; x,\,x\right]$$
$$={}_{D+3p-1}F_{G+3p-2}\left[\begin{matrix}d_D,\;\Delta(p-1;z),\Delta(p;j-u),\Delta(p;u+z);\\ g_G,\,\Delta(p;j),\Delta(p-1;j-u),\Delta(p-1;u+z);\end{matrix}\; x\frac{p^p}{(p-1)^{p-1}}\right]. \tag{21}$$

$$F_{G+1:0;1}^{D+2:0;0}\left[\begin{matrix}[d_D:1,1],\,[w:p-1,p],[v:2,1]:-;-;\\ [g_G:1,1],\,[k:p,p-1]:-;[1+v+w-k:1];\end{matrix}\; -x,\,x\right]$$
$$={}_{D+3p}F_{G+3p-1}\left[\begin{matrix}d_D,\,v,\,\Delta(p+1;w+v),\,\Delta(p-1;w),\\ g_G,\,\Delta(p;k),\,\Delta(p;w+v),\Delta(p-2;k-v),\end{matrix}\right. \tag{22}$$
$$\left.\begin{matrix}\Delta(p-1;k-v);\\ 1+v+w-k;\end{matrix}\; x\frac{(p-1)^{2(p-1)}(p+1)^{(p+1)}}{(p-2)^{p-2}\,p^{2p}}\right].$$

$$F_{G+1:0;0}^{D+2:1;0}\left[\begin{matrix}[d_D:1,1],\,[w:p-1,p],[v:2,1]:[k-v-w:1];-;\\ [g_G:1,1],\,[k:p+1,p]:-;-;\end{matrix}\; x,\,x\right]$$
$$={}_{D+3p+1}F_{G+3p}\left[\begin{matrix}d_D,\,v,\,\Delta(p;k-v),\,\Delta(p-1;w),\Delta(p+1;w+v);\\ g_G,\,\Delta(p+1;k),\,\Delta(p;w+v),\Delta(p-1;k-v);\end{matrix}\; x\right]. \tag{23}$$

$$F_{G+1:0;1}^{D+1:1;0}\left[\begin{matrix}[d_D:1,1],\,[w:p-1,p]:[k-e-w-1:1];-;\\ [g_G:1,1],\,[k:p-1,p-2]:-\,;\,[e:1];\end{matrix}\; -x,\,x\right]$$
$$={}_{D+3p-2}F_{G+3p-3}\left[\begin{matrix}d_D,\;\Delta(p;k+e-1),\,\Delta(p-1;w),\Delta(p-1;\,1+w-e);\\ g_G,\,\Delta(p-1;k),\,\Delta(p-2;1+w-e),\Delta(p-1;k+e-1),e;\end{matrix}\; x\frac{p^p}{(p-2)^{p-2}}\right]. \tag{24}$$

$$F_{G:2;0}^{D+1:0;0}\left[\begin{matrix}[d_D:1,\,1],\,[3a:3,\,1]:-;-;\\ [g_G:1,1]:\,[h:1],\,[3/2+3a-h:1];-;\end{matrix}\; -\frac{x}{4},\,x\right]$$
$$={}_{D+3}F_{G+2}\left[\begin{matrix}d_D,\;3a,\,2+3a-2h,\,2h-3a-1;\\ g_G,\,h,\;\;3/2+3a-h;\end{matrix}\;\frac{x}{4}\right]. \tag{25}$$

1. **Derivations of the results (12) to (25):** To obtain the reduction formulae "(12)" to "(25)" listed in Section II, we set different expressions for $\alpha_r, \delta_{r,} u_{r,}, v_r, w_r, t_r$ and z_r in "(10)", which yields γ_n and, closed form for β_n when the Saalschütz summation theorem [28, p. 243, "(III.2)"] is applied. The final results are obtained with the help of "(11)". Note that, D and G are positive integers, while p and p' are arbitrary integers.

   - Choosing $\alpha_n = \frac{(a)_n x^n}{n!}$, $\delta_n = \frac{(k-a-z)_n x^n}{n!}$, $z_n = \frac{(z)_n}{(k)_n}$ and $p = p'$ in "(10)" and using "(11)", we get the reduction "(12)".
   - Selecting $\alpha_n = \frac{(-x)^n}{(1+v+z-k)_n n!}$, $u_n = \frac{x^n}{n!}$, $z_n = \frac{(z)_n}{(k)_n}$ and $p = p'$ in "(10)" and using "(11)", we obtain the reduction "(13)".
   - Letting $\alpha_n = \frac{(a)_n x^n}{n!}$, $u_n = \frac{(j-a-w)_n x^n}{n!}$, $w_n = \frac{(w)_n}{(j)_n}$ and $p = p'$ in "(10)" and using "(11)", we get the result "(14)".
   - Selecting $\alpha_n = \frac{(-x)^n}{(1+v+w-j)_n n!}$, $u_n = \frac{x^n}{n!}$, $w_n = \frac{(w)_n}{(j)_n}$, $v_n = (v)_n$ and $p = p'$ in "(10)" and using "(11)", we obtain the reduction "(15)".
   - Selecting $\alpha_n = \frac{x^n}{(2+z-h-j)_n (h)_n n!}$, $u_n = \frac{x^n}{n!}$, $w_n = \frac{1}{(j)_n}$, $z_n = (z)_n$ and $p = p'$ in "(10)" and using "(11)", we obtain the result "(16)".
   - Choosing $\alpha_n = \frac{x^n}{(2+z-v-j)_n n!}$, $u_n = \frac{x^n}{n!}$, $w_n = \frac{1}{(j)_n}$, $v_n = \frac{1}{(f)_n}$, $z_n = (z)_n$ and $p' = p+2$ in "(10)" and using "(11)", we obtain the reduction "(17)".
   - Letting $\alpha_n = \frac{(-x)^n}{n!}$, $u_n = \frac{(f+j-z-1)_n x^n}{n!}$, $w_n = \frac{1}{(j)_n}$, $v_n = \frac{1}{(f)_n}$, $z_n = (z)_n$ and $p' = p+1$ in "(10)" and using "(11)", we obtain the result "(19)".
   - Selecting $\alpha_n = \frac{x^n}{n!}$, $u_n = \frac{x^n}{n!}$, $w_n = \frac{1}{(j)_n}$, $v_n = \frac{1}{(2+z-f-j)_n (f)_n}$, $z_n = (z)_n$ and $p' = p+3$ in "(10)" and using "(11)", we obtain the reduction "(20)".
   - Letting $\alpha_n = \frac{(-x)^n}{(h)_n n!}$, $u_n = \frac{(j+h-z-1)_n x^n}{n!}$, $w_n = \frac{1}{(j)_n}$, $z_n = (z)_n$ and $p' = p$ in "(10)" and using "(11)", we obtain the result "(21)".
   - Taking $\alpha_n = \frac{x^n}{n!}$, $u_n = \frac{(u)_n (j-u-z)_n x^n}{n!}$, $w_n = \frac{1}{(j)_n}$, $z_n = (z)_n$ and $p' = p-1$ in "(10)" and using "(11)", we obtain the result "(22)".

- Selecting $\alpha_n = \frac{(-x)^n}{n!}$, $u_n = \frac{x^n}{(1+v+w-k)_n\, n!}$, $w_n = (w)_n$, $z_n = \frac{1}{(k)_n}$, $v_n = (v)_n$ and $p' = p-1$ in "(10)" and using "(11)", we obtain the reduction "(23)".
- Taking $\alpha_n = \frac{(k-v-w)_n\, x^n}{n!}$, $u_n = \frac{x^n}{n!}$, $w_n = (w)_n$, $z_n = \frac{1}{(k)_n}$, $v_n = (v)_n$ and $p' = p$ in "(10)" and using "(11)", we obtain the reduction "(24)".
- Letting $\alpha_n = \frac{(k+e-w-1)_n (-x)^n}{n!}$, $u_n = \frac{x^n}{(e)_n\, n!}$, $w_n = (w)_n$, $z_n = \frac{1}{(k)_n}$ and $p' = p-2$ in "(10)" and using "(11)", we obtain the result "(25)".
- Choosing $\alpha_n = \frac{\left(-\frac{x}{4}\right)^n}{(h)_n \left(\frac{3}{2}+3a-h\right)_n n!}$, $u_n = \frac{x^n}{n!}$, $t_n = (3a)_n$ in "(10)" and using "(11)", we obtain the reduction "(26)".

III. PARTICULAR CASES OF INVESTIGATED RESULTS

The reductions stated in previous section i.e. "(12)" to "(25)" have arbitrary variables p, D and G. By assigning different integer values to these variables p, D and G, we obtain the well–known results, e.g., the Kampé de Fériet reduction formula, the Euler transformation formula and the Whipple's quadratic transformation formula as recorded in [2, p. 28, "(34)"] and [30, p. 60] and[31, p. 633, "(E.4.3)"], respectively, are obtained.

1. Choosing $p = 0$ in "(12)", a Kampé de Fériet reduction formula [2, p. 28, "(34)"] follows.
2. Selecting $p = D = G = 0$ in "(13)", the well–known Whipple's quadratic transformation [31, p. 633, "(E.4.3)"] follows.
3. The "(12)" is a generalization of both the Euler transformation[30, p. 60] and the Kampé de Fériet reduction given by [2, p. 28, "(34)"]which follows respectively, when $p = D = G = 0$ and $p = 1$.
4. Eq."(15)" is a generalization of both Whipple's quadratic transformation[31, p. 633, "(E.4.3)"]and Horn's H_4 reduction:

$$H_4\left[v,\, w; 1+w+v-j, j\, ; -x,\, x\right]$$

$$= {}_4F_3\left[\begin{matrix} v,\ j-w,\ \Delta(2; w+v); \\ 1+w+v-j,\ w+v,\ j; \end{matrix}\ -4x\right], \tag{26}$$

   which follows respectively, when $p = D = G = 0$ and $p = 1,\ D = G = 0$.
5. When $p = D = G = 0$ in "(16)",the Whipple's quadratic transformation given by [31, p. 633, "(E.4.3)"] follows.
6. The "(20)" generalizes and unifies the Horn's H_4 reduction as mentioned in part "(iv)" of this section and the Kampé de Fériet reduction mentioned in[2, p. 28, "(34)"], which follows when $p = 1,\ D = G = 0$ and $p = 0$, respectively.

7. The "(21)" generalizes the Kampé de Fériet reduction mentioned in [2, p. 28, "(34)"], which follows when $p=1$ or $p=0$.
8. Choosing $p=D=G=0$ in "(23)", the Whipple's quadratic transformation given [31, p. 633,"(E.4.3)"] follows.
9. When $h=a+\frac{2}{3}$ in "(25)", we obtain a Srivastava–Daoust transformation due to [21, p. 21].Note that, the Gauss theorem cannot be applied in the right side of "(3.1)", see [21, p. 24] since convergence conditions of Gauss theorem are violated.
   Further, for $D=G=0$, "(25)" also generalizes the Bailey's cubic transformation given by [32, p.190] for ${}_3F_2(x)$.

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