BS-ALGEBRAS AND ITS FUZZY IDEAL

Abstract

In this research article, a generalization of B-Algebras called BS-Algebras is introduced. We include fuzzy ideal in BS-Algebras. Some new characterizations were given.

Keywords: BS-Algebras, Fuzzy ideal, Homomorphism, Cartesian Product.

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I. INTRODUCTION

Fuzzy subsets was introduced by L.A.Zadeh[4], many research people investigated the generalization of the notion of fuzzy subset.In 1966, Imai and Iseki established two classesof abstract algebras, they are BCK-algebras and BCI-algebras[2]. J.Neggers and H.S. Kim brought the notion of B-algebras[3] which is a generalisation of BCK-algebras. As an extension, the author newly initiate the notion of BS-algebras, as a generalisation of B-algebras. In this article, the author study the concepts of BS-Algebras with its examples. The author apply the concept of fuzzy ideal in BS-Algebras and find some of their basic properties and explore some algebraic nature of fuzzy ideal in BS-algebras. The homomorphic behaviour of fuzzy ideal of BS-algebras have been investigated. Finally, Cartesian product is also applied in fuzzy ideal of BS-algebras.

II. PRELIMINARIES

Definition 2.1: *[3]* A B-algebra $\mathfrak{B} \neq$ with a constant 0 and a binary operation $*$ satisfying the following conditions

1. $\alpha * \alpha = 0$
2. $\alpha * 0 = \alpha$
3. $(\alpha * \beta) * \gamma = \alpha * (\gamma * (0 * \beta))$ for all $\alpha, \beta, \gamma \in \mathfrak{B}$

Definition 2.2: If φ_1 and φ_2 be any two fuzzy sets of $\mathfrak{B}$. Then its Intersection is defined by $\varphi_1 \cap \varphi_2 = min\{\varphi_1(\alpha), \varphi_2(\alpha)\}$ for all $\alpha \in \mathfrak{B}$

Definition 2.3: Let $A = \{\varphi_1(\alpha),\ \alpha \in \mathfrak{B}\}$ and $B = \{\varphi_2(\alpha),\ \alpha \in \mathfrak{B}\}$ be any two fuzzy sets on $\mathfrak{B}$. Then the cartesian product $A \times B = \{\varphi_1 \times \varphi_2(\alpha, \beta) : \alpha, \beta \in \mathfrak{B}\}$ which is defined by $(\varphi_1 \times \varphi_2)(\alpha, \beta) = min\{\varphi_1(\alpha), \varphi_2(\beta)\}$ where $\varphi_1 \times \varphi_2 : \mathfrak{B} \times \mathfrak{B} \rightarrow [0, 1]$ for all $\alpha, \beta \in \mathfrak{B}$.

Definition 2.4: *[1]* Let φ *be a fuzzy set* $\mathfrak{B}$ is called the doubt fuzzy bi-idealof $\mathfrak{B}$ if
1. $\varphi(1) \leq \varphi(\alpha)$
2. $\varphi(\beta * \gamma) \leq max\{\varphi(\alpha), \varphi(\alpha * (\beta * \gamma))\}$ for all $\alpha, \beta, \gamma \in \mathfrak{B}$

III.BS-ALGEBRAS ($\mathfrak{B}$)

Definition 3.1: A $\mathfrak{B} \neq \phi$ with a constant 1 and a binary operation $*$ iscalled a BS-Algebras which satisfies the conditionsof the following

1. $\alpha * \alpha = 1$
2. $\alpha * 1 = \alpha$
3. $1 * \alpha = \alpha$
4. $(\alpha * \beta) * \gamma = \alpha * (\gamma * (1 * \beta))$ for all $\alpha, \beta, \gamma \in \mathfrak{B}$

A binary relation $\leq$ on $\mathcal{B}$ can be defined by $\alpha \leq \beta$ iff $\alpha * \beta = 1$

Example 3.2

1. A set $\mathcal{B} = \{1, a, b, c\}$ which satisfies the table below

*	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

   Then $(\mathcal{B}, *, 1)$ is a BS-algebra.

2. A set $\mathcal{B} = \{1, a, b\}$ which satisfies the table below

*	1	a	b
1	1	a	b
a	a	1	a
b	b	a	1

   Then $(\mathcal{B}, *, 1)$ is a BS-algebra.

3. A set $\mathcal{B} = \{1, a, b, c, d, e\}$ which satisfies the table below

*	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	1	b	d	e	c
b	b	a	1	e	c	d
c	c	d	e	1	b	a
d	d	e	c	a	1	b
e	e	c	d	b	a	1

If we put $\beta = \alpha$ in $(\alpha * \beta) * \gamma = \alpha * (\gamma * (1 * \beta))$, then we have $(\alpha * \alpha) * \gamma = \alpha * (\gamma * (1 * \alpha)) \rightarrow$ (I)

$$\Rightarrow 1 * \gamma = \alpha * (\gamma * (1 * \alpha))$$

If we put $\gamma =$ in (I) then we get $1 * \alpha = \alpha * (\alpha * (1 * \alpha)) \rightarrow (II)$ Using (i) and (I) and $\gamma = 1$ it follows that

$1 = \alpha * (1 * (1 * \alpha)), \rightarrow (III)$

We see that the four conditions (i),(ii),(iii) and (iv) are independent.

4. A set $\mathfrak{B} = \{1, a, b\}$ which satisfies the table below

*	1	a	b
1	1	a	1
a	a	1	a
b	1	a	1

Then the conditions (i) and (iv) hold, but (ii) and (iii) does not hold since $b * 1 = 1 \neq b$

5. The set $\mathfrak{B} = \{0, 1, 2\}$ which satisfies the table below

*	1	a	b
1	1	a	b
a	a	a	a
b	b	a	b

The axioms (ii), (iii) and (iv) satisfies
but it does not hold (i) since $a * a = a \neq 1$and $b * b = b \neq 1$

6. Let $\mathfrak{B} = \{1, a, b, c\}$ be a set which satisfies the table below

*	1	a	b	c
1	1	a	b	c
a	a	1	1	1
b	b	1	1	a
c	c	1	1	1

Then $(\mathfrak{B}, *, 1)$ satisfies the conditions (i), (ii) and (iii) but it does not hold (iv) since $(b * c) * 1 = a \neq b = b * (1 * (1 * c))$

Theorem 3.3: If $(\mathfrak{B}, *, 1)$ is a BS-algebras, then prove that $\beta * \gamma = \beta * (1 * (1 * \gamma))$ for all $\beta, \gamma \in \mathfrak{B}$

Proof: This comes from the condition $\alpha * 1 = \alpha$ and $(\alpha * \beta) * \gamma = \alpha * (\gamma * (1 * \beta))$
Now, $\beta * \gamma = (\beta * \gamma) * 1$ (by (ii))
$= \beta * (1 * (1 * \gamma))$ (by (iv))

Theorem 3.4: If $(\mathfrak{B}, *, 1)$ is a BS-algebra, then prove that $(\alpha * \beta) * (1 * \beta) = \alpha$ for all $\alpha, \beta \in \mathfrak{B}$

Proof: From (iv) with $\gamma = 1 * \beta$ we have $(\alpha * \beta) * (1 * \beta) = \alpha * ((1 * \beta) * (1 * \beta))$
From condition (i) $(\alpha * \beta) * (1 * \beta) = \alpha * 1$
From condition (ii),it becomes $(\alpha * \beta) * (1 * \beta) = \alpha$

Theorem 3.5: If $(\mathfrak{B}, *, 1)$ is a BS-algebra, then prove that $\alpha * \gamma = \beta * \gamma \Rightarrow \alpha = \beta$ for all $\alpha, \beta, \gamma \in \mathfrak{B}$

Proof: If $\alpha * \gamma = \beta * \gamma$, then $(\alpha * \gamma) * (1 * \gamma) = (\beta * \gamma) * (1 * \gamma)$ and by previous theorem, we get $\alpha = \beta$

Theorem 3.6: If $(\mathfrak{B}, *, 1)$ is a BS-algebras, then prove that $\alpha * (\beta * \gamma) = (\alpha * (1 * \gamma)) * \beta$ for all $\alpha, \beta, \gamma \in \mathfrak{B}$

Proof: Using (iv) we obtain $(\alpha * (1 * \gamma)) * \beta = \alpha * (\beta * (1 * (1 * \gamma)))$
$= \alpha * (\beta * \gamma)$ (by thm(3.3))

Theorem 3.7: Let $(\mathfrak{B}, *, 1)$ be a BS-algebra.Then prove that for all $\alpha, \beta \in \mathfrak{B}$

(i) $\alpha * \beta = 1 \Rightarrow \alpha = \beta$
(ii)$1 * \alpha = 1 * \beta \Rightarrow \alpha = \beta$(iii)$1 * (1 * \alpha) = \alpha$

Proof: (i) Since $\alpha * \beta = 1 \Rightarrow \alpha * \beta = \beta * \beta$, by theorem (3.5), we get $\alpha = \beta$

(ii) If $1 * \alpha = 1 * \beta$, then $1 = \alpha * \alpha = (\alpha * \alpha) * 1$
$= \alpha * (1 * (1 * \alpha))$
$= \alpha * (1 * (1 * \beta))$
$= (\alpha * \beta) * 1$
$1 = \alpha * \beta$

By (i), $\alpha = \beta$

(iii) For all $\alpha \in \mathfrak{B}$, we get $1 * \alpha = (1 * \alpha) * 1$ $(b\beta(ii))$
$= 1 * (1 * (1 * \alpha))$ $(b\beta(iv))$

By (ii) part of this theorem, we have $\alpha = 1 * (1 * \alpha)$

Theorem 3.8: If $(\mathfrak{B}, *, 1)$ is a BS-algebra, then prove that $(\alpha * \beta) * \beta = \alpha * \beta^2$ for all $\alpha, \beta \in \mathfrak{B}$

Proof: From (iv), We have $(\alpha * \beta) * \beta = \alpha * (\beta * (1 * \beta))$
$= \alpha * (\beta * \beta)$
$= \alpha * \beta^2$

Theorem 3.9: If $(\mathfrak{B}, *, 1)$ is a BS-algebra, then prove that $(1 * \beta) * (\alpha * \beta) = \alpha$ for all $\alpha, \beta \in \mathfrak{B}$

Proof: From theorem (3.6), $(1 * \beta) * (\alpha * \beta) = ((1 * \beta) * (1 * \beta)) * \alpha$
$= 1 * \alpha$
$= \alpha$

Definition 3.10: A BS-algebra $(\mathfrak{B}, *, 1)$ is said to be commutative if $\alpha * (1 * \beta) = \beta * (1 * \alpha)$ for all $\alpha, \beta \in \mathfrak{B}$

Note 3.11: The BS-algebra in example:3.2 (i) is commutative but the algebra inexample:3.2 (iii)is not commutative since
$c * (1 * d) = b \neq a = d * (1 * c)$

Theorem 3.12: If $(\mathfrak{B}, *, 1)$ is commutative, then prove that $(1 * \alpha) * (1 * \beta) = \beta * \alpha$ forall $\alpha, \beta \in \mathfrak{B}$

Proof: Since $(\mathfrak{B}, *, 1)$ is commutative, then $(1 * \alpha) * (1 * \beta) = \beta * (1 * (1 * \alpha))$
$= \beta * \alpha$ (by thm(3.3))

Theorem 3.13: If $(\mathfrak{B}, *, 1)$ is commutative, then prove that $\alpha * (\alpha * \beta) = \beta$ for all $\alpha, \beta \in \mathfrak{B}$

Proof: By theorem (3.6),
Now, $\alpha * (\alpha * \beta) = (\alpha * (1 * \beta)) * \alpha$
$= (\beta * (1 * \alpha)) * \alpha$ (since $(\mathfrak{B}, *, 1)$ is commutative)
$= \beta * (\alpha * \alpha)$
$= \beta * 1$
$= \beta$

Corollary 3.14: If $(\mathfrak{B}, *, 1)$ is commutative, then the left cancellation law holds (i.e) $(\alpha * \beta) = \alpha * \beta' \Longrightarrow \beta = \beta'$

Proof: From theorem 3.13, we have $\beta = \alpha * (\alpha * \beta) = \alpha * (\alpha * \beta') = \beta'$

Theorem 3.15: If $(\mathfrak{B}, *, 1)$ is commutative, then prove that $(1 * \alpha) * (\alpha * \beta) = \beta * \alpha^2$ for all $\alpha, \beta \in \mathfrak{B}$

Proof: Now, $(1 * \alpha) * (\alpha * \beta) = ((1 * \alpha) * (1 * \beta)) * \alpha$ (by thm (3.6))
$= (\beta * \alpha) * \alpha$ (by thm (3.12))
$= \beta * \alpha^2$ (by thm (3.8))

IV. FUZZY IDEAL

Definition 4.1: Let A be a fuzzy set in BS-Algebra $\mathfrak{B}$ is called a fuzzy ideal of $\mathfrak{B}$ if it satisfies the conditions below

i) $\boldsymbol{\varphi}(1) \geq \boldsymbol{\varphi}(\beta)$
ii) $\boldsymbol{\varphi}(\beta) \geq \min\{\boldsymbol{\varphi}(\beta * \alpha), \boldsymbol{\varphi}(\alpha)\}$ for all $\alpha, \beta \in \mathfrak{B}$

Example 4.2: A set $\mathfrak{B} = \{1, a, b, c\}$ which satisfies the table below

*	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

Then $(\mathfrak{B}, *, 1)$ is a BS-algebra. Define a fuzzy set $\boldsymbol{\varphi}$ in $\mathfrak{B}$ by $(1) = (b) = 0.7$ and $(a) = \boldsymbol{\varphi}(c) = 0.3$ is the fuzzy ideal of $\mathfrak{B}$.

Theorem 4.3: Let $\boldsymbol{\varphi}$ be a fuzzy ideal of $\mathfrak{B}$ is a, then for all $\alpha \in \mathfrak{B}$, $\boldsymbol{\varphi}(1) \geq \boldsymbol{\varphi}(\alpha)$

Proof: Straight forward

Theorem 4.4: If $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ be two fuzzy ideals of $\mathfrak{B}$, then $\boldsymbol{\varphi}_1 \cap \boldsymbol{\varphi}_2$ is also a fuzzy ideal of $\mathfrak{B}$.

Proof:

Now, $(\boldsymbol{\varphi}_1 \cap \boldsymbol{\varphi}_2)(1) = (\boldsymbol{\varphi}_1 \cap \boldsymbol{\varphi}_2)(\alpha * \alpha)$(b$\beta$ (i))
$\geq min\{(\boldsymbol{\varphi}_1 \cap \boldsymbol{\varphi}_2)(\alpha), (\boldsymbol{\varphi}_1 \cap \boldsymbol{\varphi}_2)(\alpha)\}$
$= (\boldsymbol{\varphi}_1 \cap \boldsymbol{\varphi}_2)(\alpha)$

Therefore, $(\boldsymbol{\varphi}_1 \cap \boldsymbol{\varphi}_2)(1) \geq (\boldsymbol{\varphi}_1 \cap \boldsymbol{\varphi}_2)(\alpha)$
Also, $(\boldsymbol{\varphi}_1 \cap \boldsymbol{\varphi}_2)(\beta) = min\{\boldsymbol{\varphi}_1(\beta), \boldsymbol{\varphi}_2(\beta)\}$
$\geq min\{min\{\boldsymbol{\varphi}_1(\alpha), \boldsymbol{\varphi}_1(\beta * \alpha)\}, min\{\boldsymbol{\varphi}_2(\alpha), \boldsymbol{\varphi}_2(\beta * \alpha)\}\}$
$= min\{min\{\boldsymbol{\varphi}_1(\alpha), \boldsymbol{\varphi}_2(\alpha)\}, min\{\boldsymbol{\varphi}_1(\beta * \alpha), \boldsymbol{\varphi}_2(\beta * \alpha)\}\}$
$= min\{(\boldsymbol{\varphi}_1 \cap \boldsymbol{\varphi}_2)(\alpha), (\boldsymbol{\varphi}_1 \cap \boldsymbol{\varphi}_2)(\beta * \alpha)\}$

$(\boldsymbol{\varphi}_1 \cap \boldsymbol{\varphi}_2)(\beta) \geq min\{(\boldsymbol{\varphi}_1 \cap \boldsymbol{\varphi}_2)(\alpha), (\boldsymbol{\varphi}_1 \cap \boldsymbol{\varphi}_2)(\beta * \alpha)\}$

Hence $\boldsymbol{\varphi}_1 \cap \boldsymbol{\varphi}_2$ is a fuzzy ideal of $\mathfrak{B}$.

Theorem 4.5. Let $\boldsymbol{\varphi}$ be a fuzzy ideals of $\mathfrak{B}$. If $\alpha * \beta \leq \gamma$, then $\boldsymbol{\varphi}(\alpha) \geq \min\{\boldsymbol{\varphi}(\beta), \boldsymbol{\varphi}(\gamma)\}$

Proof: Let $\alpha, \beta, \gamma \in \mathfrak{B}$ such that $\alpha * \beta \leq \gamma$. Then $(\alpha * \beta) * \gamma = 1$,

$(\alpha) \geq min\{\boldsymbol{\varphi}(\alpha * \beta), \boldsymbol{\varphi}(\beta)\}$
$\geq min\{min\{\boldsymbol{\varphi}((\alpha * \beta) * \gamma), \boldsymbol{\varphi}(\gamma)\}, \boldsymbol{\varphi}(\beta)\}$
$= min\{min\{\boldsymbol{\varphi}(1), \boldsymbol{\varphi}(\gamma)\}, \boldsymbol{\varphi}(\beta)\}$
$= min\{\boldsymbol{\varphi}(\gamma), \boldsymbol{\varphi}(\beta)\}$
Therefore, $(\alpha) \geq min\{\boldsymbol{\varphi}(\gamma), \boldsymbol{\varphi}(\beta)\}$

Theorem 4.6: Let $\boldsymbol{\varphi}$ be a fuzzy ideals of a BS-algebras $\mathfrak{B}$. If $\alpha \leq \beta$, then $(\alpha) \geq (\beta)$ (i.e) order reversing

Proof: Let $\alpha, \beta \in \mathfrak{B}$ such that $\alpha \leq \beta$. Then $\alpha * \beta = 1$

$(\alpha) \geq min\{\boldsymbol{\varphi}(\alpha * \beta), \boldsymbol{\varphi}(\beta)\}$(by (ii) in def 4.1)
$= min\{\boldsymbol{\varphi}(1), \boldsymbol{\varphi}(\beta)\}$
$= \boldsymbol{\varphi}(\beta) \boldsymbol{\varphi}(\alpha) \geq \boldsymbol{\varphi}(\beta)$

Hence, $\boldsymbol{\varphi}$ is order reversing.

Theorem 4.7: Let B be a crisp subset of BS-algebras $\mathfrak{B}$. Suppose a fuzzy set F = $\boldsymbol{\varphi}(\alpha)$ in $\mathfrak{B}$ defined by $\boldsymbol{\varphi}(\alpha) = \lambda$ if $\alpha \in$ Y and $\boldsymbol{\varphi}(\alpha) = \tau$ if $\alpha \notin$ Y for all $\lambda, \tau \in [0, 1]$ with $\lambda \geq \tau$. Then F is a fuzzy ideal of $\mathfrak{B}$ iff Y is a ideal of $\mathfrak{B}$

Proof:

Assume that F is a fuzzy ideal of $\mathfrak{B}$. Let $\alpha \in$ Y.
Let $\alpha, \beta \in \mathfrak{B}$ be such that $\beta * \alpha \in$ Y and $\alpha \in$ Y. Then $(\beta * \alpha) = \lambda = (\alpha)$, and hence $(\beta) \geq min\{\boldsymbol{\varphi}(\alpha), \boldsymbol{\varphi}(\beta * \alpha)\} = \lambda$.Thus $(\beta) = \lambda$ (i.e) $\beta \in$ Y.

Therefore Y is a ideal of $\mathfrak{B}$.
Conversely, assume that Y is an ideal of $\mathfrak{B}$. Let $\beta \in \mathfrak{B}$.

Let $\alpha, \beta \in \mathfrak{B}$. If $\beta * \alpha \in$ Y and $\alpha \in$ Y, then $\beta \in$ Y.
Hence, $(\beta) = \lambda = min\{\boldsymbol{\varphi}(\beta * \alpha), \boldsymbol{\varphi}(\alpha)\}$

If $\beta * \alpha \notin Y$ and $\alpha \notin Y$, then clearly $\boldsymbol{\varphi}(\beta) \geq min\{\boldsymbol{\varphi}(\alpha), \boldsymbol{\varphi}(\beta * \alpha)\}$
If exactly one of $\beta * \alpha$ and α belong to Y, then exactly one of $(\beta * \alpha), (\alpha)$is equal to τ. Therefore, $(\beta) \geq \tau = min\{\boldsymbol{\varphi}(\alpha), \boldsymbol{\varphi}(\beta * \alpha)\}$
Consequently, A is a fuzzy ideal of.

Theorem 4.8: Let $\boldsymbol{\varphi}$ be an ideal of $\mathfrak{B}$ then the set $\boldsymbol{\upsilon}$ (: x) is an ideal of $\mathfrak{B}$ for every x $\in$ [0, 1]

Proof:

Suppose that $\boldsymbol{\varphi}$ is an fuzzy ideal of $\mathfrak{B}$. For x $\in$ [0, 1]. Let $\alpha, \beta \in \mathfrak{B}$ be such that $\beta * \alpha \in \boldsymbol{\upsilon}$ (: x) and $\alpha \in \boldsymbol{\upsilon}$ ($\boldsymbol{\varphi}$: x).Then $\boldsymbol{\varphi}(\beta) \geq min\{\boldsymbol{\varphi}(\alpha), \boldsymbol{\varphi}(\beta * \alpha)\}$.
Then $\beta \in \boldsymbol{\upsilon}$ (: x). Hence $\boldsymbol{\upsilon}$ ($\boldsymbol{\varphi}$: x) is an ideal of $\mathfrak{B}$.

Definition 4.9: Let f : $\mathfrak{B} \to \mathfrak{B}'$ be the two BS-algebras. Let Y be a fuzzy set in $\mathfrak{B}'$. Then the inverse image of Y is defined as
$f^{-1}(\boldsymbol{\varphi})(\alpha) = \boldsymbol{\varphi}(f(\alpha))$. The set $f^{-1}(B) = \{f^{-1}(\boldsymbol{\varphi})(\alpha) : \alpha \in \mathfrak{B}'\}$ *is a fuzzy set.*

Theorem 4.10: Let f: $\mathfrak{B} \to \mathfrak{B}'$ be a homomorphism of BS-algebras. If Y is a fuzzy ideal of $\mathfrak{B}'$, then the pre-image $f^{-1}(Y)$ in $\mathfrak{B}'$ is a fuzzy ideal of $\mathfrak{B}$

Proof:

For all $\alpha \in \mathfrak{B}'$, $f^{-1}(\boldsymbol{\varphi})(\alpha) = \boldsymbol{\varphi}(f(\alpha)) \leq \boldsymbol{\varphi}(1) = \boldsymbol{\varphi}(f(1)) = f^{-1}(\square)(1)$
Therefore, $f^{-1}(\boldsymbol{\varphi})(\alpha) \leq f^{-1}(\boldsymbol{\varphi})(1)$
Let $\alpha, \beta \in \mathfrak{B}'$. Then $f^{-1}(\boldsymbol{\varphi})(\alpha) = \boldsymbol{\varphi}(f(\alpha))$
$\geq min\{\boldsymbol{\varphi}(f(\alpha) * f(\beta)), \boldsymbol{\varphi}(f(\beta))\}$
$\geq min\{\boldsymbol{\varphi}(f(\alpha * \beta)), \boldsymbol{\varphi}(f(\beta))\}$
$= min\{f^{-1}(\boldsymbol{\varphi})(\alpha * \beta), f^{-1}(\boldsymbol{\varphi})(\beta)\}$

Therefore, $f^{-1}(\boldsymbol{\varphi})(\alpha) \geq min\{f^{-1}(\boldsymbol{\varphi})(\alpha * \beta), f^{-1}(\boldsymbol{\varphi})(\beta)\}$
Hence $f^{-1}(Y) = \{f^{-1}(\boldsymbol{\varphi})(\alpha) : \alpha \in \mathfrak{B}'\}$ is a fuzzy ideal of $\mathfrak{B}$

Theorem 4.11: Let f : $\mathfrak{B} \to \mathfrak{B}'$ be an onto homomorphism of BS-algebra. Then Y isa fuzzy ideal of $\mathfrak{B}'$, if $f^{-1}(Y)$ in $\mathfrak{B}'$ is a fuzzy ideal of $\mathfrak{B}'$

Proof:

For any $u \in \mathfrak{B}'$, there exists $\alpha \in \mathfrak{B}'$ such that $f(\alpha) = u$
Then $(u) = (f(\alpha)) = f^{-1}(\boldsymbol{\varphi})(\alpha) \leq f^{-1}(\boldsymbol{\varphi})(1) = \boldsymbol{\varphi}(f(1)) = \boldsymbol{\varphi}(1)$
Therefore, $(u) \leq (1)$
Let $u, v \in \mathfrak{B}'$. Then $f(\alpha) = u$ and $f(\beta) = v$ for some $\alpha, \beta \in \mathfrak{B}'$.

Thus, $(u) = (f(\alpha)) = f^{-1}(\boldsymbol{\varphi})(\alpha)$
$\geq min\{f^{-1}(\boldsymbol{\varphi})(\alpha * \beta), f^{-1}(\boldsymbol{\varphi})(\beta)\}$
$= min\{\boldsymbol{\varphi}(f(\alpha * \beta)), \boldsymbol{\varphi}(f(\beta))\}$
$= min\{\boldsymbol{\varphi}(f(\alpha) * f(\beta)), \boldsymbol{\varphi}(f(\beta))\}$
$= min\{\boldsymbol{\varphi}(u * v), \boldsymbol{\varphi}(v)\}$

Therefore, $(u) \geq min\{\boldsymbol{\varphi}(u * v), \boldsymbol{\varphi}(v)\}$
Then Y is a fuzzy ideal of $\mathfrak{B}'$

Theorem 4.12: Let X and Y be fuzzy ideals of , then $X \times Y$ is a fuzzy ideal of $\mathfrak{B} \times \mathfrak{B}$

Proof: For any $(\alpha, \beta) \in \mathfrak{B} \times \mathfrak{B}$, we have

$(\boldsymbol{\varphi}_1 \times \boldsymbol{\varphi}_2)(1, 1) = min\{\boldsymbol{\varphi}_1(1), \boldsymbol{\varphi}_2(1)\}$
$\geq min\{\boldsymbol{\varphi}_1(\alpha), \boldsymbol{\varphi}_2(\beta)\}$ for all $\alpha, \beta \in \mathfrak{B}'$
$= (\boldsymbol{\varphi}_1 \times \boldsymbol{\varphi}_2)(\alpha, \beta)$

Therefore, $(\boldsymbol{\varphi}_1 \times \boldsymbol{\varphi}_2)(1, 1) \geq (\boldsymbol{\varphi}_1 \times \boldsymbol{\varphi}_2)(\alpha, \beta)$
Let $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathfrak{B}' \times \mathfrak{B}'$. Then

$$(\boldsymbol{\varphi}_1 \times \boldsymbol{\varphi}_2)(\alpha_1, \beta_1) = min\{\boldsymbol{\varphi}_1(\alpha_1), \boldsymbol{\varphi}_2(\beta_1)\}$$
$$\geq min\{min\{\boldsymbol{\varphi}_1(\alpha_1 * \alpha_2), \boldsymbol{\varphi}_1(\alpha_2)\}, min\{\boldsymbol{\varphi}_2(\beta_1 * \beta_2), \boldsymbol{\varphi}_2(\beta_2)\}\}$$
$$= min\{min\{\boldsymbol{\varphi}_1(\alpha_1 * \alpha_2), \boldsymbol{\varphi}_2(\beta_1 * \beta_2)\}, min\{\boldsymbol{\varphi}_1(\alpha_2), \boldsymbol{\varphi}_2(\beta_2)\}\}$$
$$= min\{(\boldsymbol{\varphi}_1 \times \boldsymbol{\varphi}_2)((\alpha_1 * \alpha_2), (\beta_1 * \beta_2)), (\boldsymbol{\varphi}_1 \times \boldsymbol{\varphi}_2)(\alpha_2, \beta_2)\}$$

Therefore, $(\boldsymbol{\varphi}_1 \times \boldsymbol{\varphi}_2)(\alpha_1, \beta_1) \geq min\{(\boldsymbol{\varphi}_1 \times \boldsymbol{\varphi}_2)((\alpha_1 * \alpha_2), (\beta_1 * \beta_2)), (\boldsymbol{\varphi}_1 \times \boldsymbol{\varphi}_2)(\alpha_2, \beta_2)\}$

Hence, X × Y is a fuzzy ideal of $\mathfrak{B} \times \mathfrak{B}$

Theorem 4.13: Let X and Y be the two fuzzy sets in $\mathfrak{B}$ such that X × Y is a fuzzy ideal of $\mathfrak{B} \times \mathfrak{B}$, then

i) Either $\boldsymbol{\varphi}_1(1) \geq \boldsymbol{\varphi}_1(\alpha)$ or $\boldsymbol{\varphi}_2(1) \geq \boldsymbol{\varphi}_2(\alpha)$ $\forall\ \alpha \in \mathfrak{B}'$
ii) If $\boldsymbol{\varphi}_1(1) \geq \boldsymbol{\varphi}_1(\alpha)$ $\forall\ \alpha \in \mathfrak{B}'$, then either $\boldsymbol{\varphi}_2(1) \geq \boldsymbol{\varphi}_1(\alpha)$ or $\boldsymbol{\varphi}_2(1) \geq \boldsymbol{\varphi}_2(\alpha)$
iii) If $\boldsymbol{\varphi}_2(1) \geq \boldsymbol{\varphi}_2(\alpha)$ $\forall\ \alpha \in \mathfrak{B}'$, then either $\boldsymbol{\varphi}_1(1) \geq \boldsymbol{\varphi}_1(\alpha)$ or $\boldsymbol{\varphi}_1(1) \geq \boldsymbol{\varphi}_2(\alpha)$

Proof. (i) Assume that $\boldsymbol{\varphi}_1(\alpha) > \boldsymbol{\varphi}_1(1)$ and $\boldsymbol{\varphi}_2(\beta) > \boldsymbol{\varphi}_2(1)$ for some $\alpha, \beta \in \mathfrak{B}'$.

Then $(\boldsymbol{\varphi}_1 \times \boldsymbol{\varphi}_2)(\alpha, \beta) = min\{\boldsymbol{\varphi}_1(\alpha), \boldsymbol{\varphi}_2(\beta)\}$
$> min\{\boldsymbol{\varphi}_1(1), \boldsymbol{\varphi}_2(1)\}$
$= (\boldsymbol{\varphi}_1 \times \boldsymbol{\varphi}_2)(1, 1)$
$\Longrightarrow (\boldsymbol{\varphi}_1 \times \boldsymbol{\varphi}_2)(\alpha, \beta) > (\boldsymbol{\varphi}_1 \times \boldsymbol{\varphi}_2)(1, 1)\ \forall\ \alpha, \beta \in \mathfrak{B}'$,

Which is a contradiction.

Hence (i) is proved.

(ii) Again assume that $\boldsymbol{\varphi}_2(1) < \boldsymbol{\varphi}_1(\alpha)$ and $\boldsymbol{\varphi}_2(1) < \boldsymbol{\varphi}_2(\beta)$ $\forall\ \alpha, \beta \in \mathfrak{B}'$

Then$(\boldsymbol{\varphi}_1 \times \boldsymbol{\varphi}_2)(1, 1) = min\{\boldsymbol{\varphi}_1(1), \boldsymbol{\varphi}_2(1)\}$
$= \boldsymbol{\varphi}_2(1)$
Now,$(\boldsymbol{\varphi}_1 \times \boldsymbol{\varphi}_2)(\alpha, \beta) = min\{\boldsymbol{\varphi}_1(\alpha), \boldsymbol{\varphi}_2(\beta)\}$
$> \boldsymbol{\varphi}_2(1)$
$= (\boldsymbol{\varphi}_1 \times \boldsymbol{\varphi}_2)(1, 1)$, which is a contradiction.

Hence (ii) is proved

(iii) The proof is similar to (ii)

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