SOLUTION OF SINGULAR PERTURBATION PROBLEMS USING FOURTH ORDER ADAPTIVE CUBIC SPLINE

Abstract

In this paper, using adaptive cubic spline, we have suggested a numerical scheme for solving a convection-diffusion problem having layer structure. The numerical scheme is derived with this spline and non-standard finite differences of the first derivative. The tridiagonal solver is used to solve the system of the numerical method. The analysis of convergence of the method is briefly discussed and the fourth order is shown. The numerical results of the examples were tabulated and compared to the existing computational results in order to support the higher accuracy of the proposed numerical scheme.

Keywords: Singular perturbation; adaptive cubic spline; convection-diffusion problems; convergence.

Author

Mamatha Kodipaka
Department of Mathematics
Vardaman College of Engineering
(Autonomous)
Kacharam, Shamshabad, Hyderabad,
Telangana, India.
k.mamatha@vardhaman.org

I. INTRODUCTION

It is well known that many physical problems with many small parameters often involve thesolution of boundary value problems. This paper deals with convection-diffusion boundary value problems involving small parameter. These problems are characterized by the inclusion of a small perturbation parametersε which multiply the second order derivative. In many fields of engineering and science, such types of problems exist such as chemical reactor theory, transport phenomena in chemistry, lubrication theory and biology.

A broad verity of books has been found in the literature for the convection – diffusion problems or singular perturbation problems (SPPs) [2,3,4,10,14]. One can refer a book on splines by Micula [9]. The survey papers [6, 8] provides a detailed research work on SPP problems. In [1], the authors suggested a difference schemes of second and fourth order based on cubic spline in compression for SPP. A variable-mesh second-order difference scheme via cubic splines is proposed to solve SPP in [5]. In the paper [7], authors usedthe artificial viscosity in B-spline collocation method to capture the layer behaviour of the problem. Phaneendra and Lalu [12] derived numerical scheme using Gaussian quadrature for the solution of SPP with one end layer, dual layer and internal layer. The authors in [13] extended the Numerov scheme to the SPP with first order derivative. Soujanya et al. [15] introduced a scheme having a fitting factor in Dahlquist scheme to get the solution of SPP having dual layers. Uniform difference schemes based on a class of splines are proposed by Stojanovic [16] for the solution of non-self-adjoint SPP.

In this paper, we present a fourth order finite difference method using adaptive cubic spline to solve singularly perturbed boundary value problems. We introduce a new parameter η in the difference scheme to achieve fourth order convergence for the proposed problem. The paper is organized as follows: In section 2, Description of the problem along with conditions for layer behavior is given. In section 3, we define the adaptive spline function. In section 4, we describe the numerical method for solving singularly perturbed boundary value problems, in Section 5, the truncation error and classification of various orders of the proposed method are given. In section 6, we discuss the convergence analysis of the method. Finally, numerical results and comparison with other methods are given in section 7.

II. DESCRIPTION OF THE METHOD

Consider the convection-diffusion boundary value problem of the form

$$\varepsilon y''(s) + p(s)y'(s) + q(s)y(s) = r(s), \tag{1}$$

$$\text{With boundary conditions } \; y(a) = \alpha, y(b) = \beta \tag{2}$$

Here $0 < \varepsilon << 1$ is a perturbation parameter. The functions $p(s), q(s), r(s)$ are assumed to be appropriately smooth in $[a, b]$ and α,β are finite constants. The layer exists in the vicinity of $s = a$ if $p(s) \geq L > 0$ over the domain $[a, b]$, where L is positive constant. The layer exists in the vicinity of $s = b$ if $P(s) \leq \overline{L} < 0$over the domain $[a, b]$,

Where $\overline{L}$ is negative constant.

III. ADAPTIVE CUBIC SPLINE

With grid points s_i in $[a,b]$, consider the mesh such that $\Omega: a = s_0 < s_1 < s_2 < \cdots <$ $s_n = b$, where $h = s_i - s_{i-1}$ for $i = 1,2\ldots N$. A function $\psi(s,\tau)$ interpolate $y(s)$ at the grid points s_i which depends on a variable τ, leads to cubic spline $\psi(s)$ in $[a,b]$ and $\psi(s)$ is named as adaptive spline function as $\tau \to 0$. Following [13], If $\psi(s,\tau)$ is an adaptive spline function then

$$\varepsilon\,\psi''(s,\tau) - p\,\psi'(s,\tau) = \frac{s-s_{i-1}}{h}(\varepsilon M_i - p m_i) + \frac{s_i - s}{h}(\varepsilon M_{i-1} - p m_{i-1}) \quad (3)$$

Where $s_{i-1} \le s \le s_i$, and $\psi'(s,\tau) = m_i, \psi''(s,\tau) = M_i$.

Solving Eq. (3) and using the interpolatory conditions $\psi(s_{i-1},\tau) = y_{i-1}$, $\psi(s_i,\tau) = y_i$, we have

$$\psi(s,\tau) = A_i + B_i e^{\tau z} - \frac{h^2}{\tau^3}\left[\frac{1}{2}\tau^2 z^2 + \tau z + 1\right]\left(M_i - \frac{\tau}{h}m_i\right) + \frac{h^2}{\tau^3}\left[\frac{1}{2}\tau^2(1-z)^2 - \tau\,(1\text{-}z) + 1\right] M i{-}1{-}\tau h m i{-}1 \quad (4)$$

Where $A_i(e^\tau - 1) = -x_i + x_{i-1}e^\tau - \frac{h^2}{\tau^3}\left[\left(\frac{\tau^2}{2} + \tau + 1\right) - e^\tau\right]\left(M_i - \frac{\tau}{h}m_i\right) - \frac{h^2}{\tau^3}\left[e^\tau\left(\frac{\tau^2}{2} - \tau + 1\right) - 1\right] M i{-}1{-}\tau h m i{-}1$

$$B_i(e^\tau - 1) = x_i - x_{i-1}e^\tau + \frac{2}{\tau^3}\left[\left(\frac{\tau}{2} + 1\right) - \tau\, e^\tau\right]\left(M_i - \frac{\tau}{}m_i\right) + \left[\left(\frac{\tau}{2} - 1\right) - \tau\right]\left(M_{i-1} - \frac{\tau}{}m_{i-1}\right)$$

Where $\tau = \frac{ph}{\varepsilon}$ and $z = \frac{s - s_{i-1}}{h}$

The spline function $\psi(t,\tau)$ on $[s_i, s_{i+1}]$ is acquired with replacing $i$by $(i+1)$ in Eq. (4) and utilizing the first ($\psi_i'(s_i,\tau) = \psi_{i+1}'(s_i,\tau)$) or second derivative continuity condition of $\psi(s,\tau)$ at $s = s_i$, we get the following relationship:

$$\left(M_{i+1} - \frac{\tau}{h}m_{i+1}\right)\left[e^{-\tau}\left(\frac{\tau^2}{2} + \tau + 1\right) - 1\right] + \left(M_i - \frac{\tau}{h}m_i\right)\left[e^{-\tau}\left(\frac{\tau^2}{2} - \tau - 2\right) + \left(-\frac{\tau^2}{2} - \tau + 2\right)\right] + M i{-}1{-}\tau h m i{-}1 e{-}\tau{-}1{+}\tau{-}\tau 22 = {-}\ \tau 2 h 3 e{-}\tau y i{+}1{-}e{-}\tau{+}1 y i{+}y i{-}1 \quad (5)$$

Further the relations are given below for the adaptive spline

$$(i) m_{i-1} = -h\left(\tilde{A}_1 M_{i-1} + \tilde{A}_2 M_i\right) + \frac{1}{h}(y_i - y_{i-1})$$
$$(ii) m_i = h\left(\tilde{A}_3 M_{i-1} + \tilde{A}_4 M_i\right) + \frac{1}{h}(y_i - y_{i-1})$$
$$(iii) \frac{\theta h}{2\tau} M_{i-1} = -\left(\tilde{A}_4 m_{i-1} + \tilde{A}_2 m_i\right) + \tilde{B}_1 \frac{(y_i - y_{i-1})}{h}$$

$(iv)\,\frac{\theta h}{2\tau}M_i = \left(\tilde{A}_3 m_{i-1} + \tilde{A}_1 m_i\right) + \tilde{B}_2\frac{(y_i - y_{i-1})}{h}$

Where $\tilde{A}_1 = \frac{1}{4}(1+\theta) + \frac{\theta}{2\tau}, \tilde{A}_2 = \frac{1}{4}(1-\theta) - \frac{\theta}{2\tau}, \tilde{A}_3 = \frac{1}{4}(1+\theta) - \frac{\theta}{2\tau},$

$$\tilde{A}_4 = \frac{1}{4}(1-\theta) + \frac{\theta}{2\tau}, \tilde{B}_1 = \frac{1}{2}(1-\theta), \tilde{B}_2 = -\frac{1}{2}(1+\theta), \text{ and } \theta = coth\left(\frac{\tau}{2}\right) - \frac{2}{\tau}$$

We also obtain $\tilde{A}_2 M_{i+1} + \left(\tilde{A}_1 + \tilde{A}_4\right)M_i + \tilde{A}_3 M_{i-1} = \frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}]$(6)

Remark: In the limiting case when $\tau \to 0$, we have

$$\tilde{A}_1 = \tilde{A}_4 = \frac{1}{3}, \tilde{A}_2 = \tilde{A}_3 = \frac{1}{6}, \tilde{B}_1 = \frac{1}{2}, \tilde{B}_2 = -\frac{1}{2}, \theta = 0, \frac{\theta}{\tau} = \frac{1}{6}$$

and the spline function (3) reduces to ordinary cubic spline.

IV. NUMERICAL SCHEME

At the mesh point s_i, the suggested approach can be discretized by the convection-diffusion equation Eq. (1) as

$$\varepsilon\, M_i = r(s_i) - p(s_i){y_i}'(s_i) - q(s_i)y(s_i) \tag{7}$$

The above equations shall be replaced in Eq. (6) and using the following approximations for the first order derivative of s at the mesh points$s_1, s_2, \ldots, s_{N-1}$

$y'_{i-1} \approx \frac{-y_{i+1}+4y_i-3y_{i-1}}{2h}, y'_{i+1} \approx \frac{3y_{i+1}-4y_i+y_{i-1}}{2h}$

$$y'_i \approx \left(\frac{1 + 2\eta h^2 q_{i+1} + \eta h[3p_{i+1} + p_{i-1}]}{2h}\right)y_{i+1} - 2\eta\,[p_{i+1} + p_{i-1}]y_i$$
$$- \left(\frac{1 + 2\eta h^2 q_{i-1} - \eta h[p_{i+1} + 3p_{i-1}]}{2h}\right)y_{i-1} + \eta\, h[r_{i+1} - r_{i-1}]$$

We get the following system

$$L_i y_{i-1} + C_i y_i + U_i y_{i+1} = H_i \quad for\;\; i = 1,2,\ldots,N-1 \tag{8}$$

Where

$$L_i = -\,\varepsilon + \frac{3}{2}\tilde{A}_3 p_{i-1}\,h + \frac{\left(\tilde{A}_1 + \tilde{A}_4\right)p_i h}{2}[1 + 2\eta h^2 q_{i-1} - \eta h(p_{i+1} + 3p_{i-1})] - \frac{\tilde{A}_2}{2}p_{i+1}h - \tilde{A}_3 q_{i-1}h^2$$

$$C_i = 2\varepsilon - 2\tilde{A}_3 p_{i-1}\,h + 2\eta\left(\tilde{A}_1 + \tilde{A}_4\right)p_i h_i^2[p_{i+1} + p_{i-1}] + 2\tilde{A}_2 p_{i+1}h - \left(\tilde{A}_1 + \tilde{A}_4\right)q_i h^2$$

$$U_i = -\varepsilon + \frac{\tilde{A}_3}{2}p_{i-1}\,h - \frac{\left(\tilde{A}_1 + \tilde{A}_4\right)p_i h}{2}[1 + 2\eta h^2 q_{i+1} + \eta h(3p_{i+1} + p_{i-1})] - \frac{3}{2}\tilde{A}_2 p_{i+1}h - \tilde{A}_2 q_{i+1}h^2$$

$$H_i = h^2\left[\left(-\tilde{A}_2+\eta\left(\tilde{A}_1+\tilde{A}_4\right)p_ih\right)r_{i+1}-\left(\tilde{A}_1+\tilde{A}_4\right)r_i-\left(\tilde{A}_3+\eta\left(\tilde{A}_1+\tilde{A}_4\right)p_ih\right)r_{i-1}\right]$$

The tridiagonal system Eq. (2.8) is solved for i = 1, 2, …, N-1 to obtain the approximations $y_{1,}y_2,\dots,y_{N-1}$of the solution *y(s)* at $s_1, s_2,\dots\dots s_{N-1}$.

V. TRUNCATION ERROR

The scheme's local truncation error in Eq. (2.8) as follows

$T_i(h)=\varepsilon\left[(\tilde{A}_1+\tilde{A}_2+\tilde{A}_3+\tilde{A}_4)-1\right]{h_i}^2y_i^{''}+\varepsilon\left(\tilde{A}_2-\tilde{A}_3\right){h_i}^3y_i^{'''}+\left[\frac{\tilde{A}_2+\tilde{A}_3}{2}+2\eta\varepsilon\left(\tilde{A}_1+\tilde{A}_4\right)-\right.$

16(A1+A2+A3+A4)pihi4yi'''+ε12− 1+6A2+A3hi4yiiv+112A3−A2piyiiv+2pi'+qiyi'''

+6pi''+qi'yi''+2pi'''+3qi''yi'+2qi'''yi−2ri'''hi5+Ohi6 Thus for different values of $\tilde{A}_2, \tilde{A}_3, \tilde{A}_1+\tilde{A}_4$ in the scheme (8), indicates different orders:

Remarks:

- If $\tilde{A}_2=\tilde{A}_3$, for any choice of arbitrary$\tilde{A}_2, \tilde{A}_1+\tilde{A}_4$with $\left(\tilde{A}_1+\tilde{A}_4\right)+\tilde{A}_2=\frac{1}{2}$and for any value of ψ , method is obtained for second order.
- For$\tilde{A}_2=\tilde{A}_3=\frac{1}{12}$,$\left(\tilde{A}_1+\tilde{A}_4\right)=\frac{5}{6}$ and $\eta=\frac{1}{20\varepsilon}$, fourth order method is derived.

VI. ANALYSIS OF CONVERGENCE

The convergence analysis of the suggested method to Eq. (1) is now being considered. The system of equations in the matrix form with the boundary conditions is

$$(D+F)W+G+T(h)=0 \tag{9}$$

Where $D=[-\varepsilon, 2\varepsilon, -\varepsilon]=\begin{bmatrix}2\varepsilon & -\varepsilon & 0 & 0 & \dots & 0\\ -\varepsilon & 2\varepsilon & -\varepsilon & 0 & \dots & 0\\ 0 & -\varepsilon & 2\varepsilon & -\varepsilon & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & \dots & \dots & 0 & -\varepsilon & 2\varepsilon\end{bmatrix}$ and

$$F=[\tilde{z}_i, \tilde{v}_i, \tilde{w}_i]=\begin{bmatrix}\tilde{v}_1 & \tilde{w}_1 & 0 & 0 & \dots & 0\\ \tilde{z}_2 & \tilde{v}_2 & \tilde{w}_2 & 0 & \dots & 0\\ 0 & \tilde{z}_3 & \tilde{v}_3 & \tilde{w}_3 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & ..\\ \dots & \dots & \dots & \dots & \dots & ..\\ 0 & \dots & \dots & 0 & \tilde{z}_{N-1} & \tilde{v}_{N-1}\end{bmatrix}$$

Where

$$\tilde{z}_i = \frac{3}{2}\tilde{A}_3 p_{i-1}h_i + \frac{(\tilde{A}_1+\tilde{A}_4)p_i h_i}{2}\left[1+2\eta {h_i}^2 q_{i-1} - \eta h_i(p_{i+1}+3p_{i-1})\right] - \frac{\tilde{A}_2}{2}p_{i+1}h_i - \tilde{A}_3 q_{i-1}{h_i}^2$$

$$\tilde{v}_i = -2\tilde{A}_3 p_{i-1}h_i + 2\eta(\tilde{A}_1+\tilde{A}_4)p_i h_i^2[p_{i+1}+p_{i-1}] + 2\tilde{A}_2 p_{i+1}h_i - (\tilde{A}_1+\tilde{A}_4)q_i {h_i}^2$$

$$\widetilde{w}_i = \frac{\tilde{A}_3}{2}p_{i-1}h_i - \frac{(\tilde{A}_1+\tilde{A}_4)p_i h_i}{2}\left[1+2\eta {h_i}^2 q_{i+1} + \eta h_i(3p_{i+1}+p_{i-1})\right] - \frac{3}{2}\tilde{A}_2 p_{i+1}h_i - \tilde{A}_2 q_{i+1}{h_i}^2 \text{for } i=1,2,3,\dots,N-1$$

and $G = [q_1 - \tilde{z}_1\alpha, q_2, q_3, \dots, q_{N-1} - \widetilde{w}_{N-1}\beta]$

Where

$$q_i = {h_i}^2\left[(-\tilde{A}_2 + \eta(\tilde{A}_1+\tilde{A}_4)p_i h_i)r_{i+1} - (\tilde{A}_1+\tilde{A}_4)r_i - (\tilde{A}_3 + \eta(\tilde{A}_1+\tilde{A}_4)p_i h_i)r_{i-1}\right]$$

for $i = 2,3,\dots,N-1$.

$T(h) = O(h^6)$ for $\tilde{A}_2 = \tilde{A}_3 = \frac{1}{12}, (\tilde{A}_1+\tilde{A}_4) = \frac{5}{6}$ and $\eta = \frac{1}{20\varepsilon}$

and $W = [W_1, W_2, W_3, \dots, W_{N-1}]^T, T(h) = [T_1, T_2, \dots, T_{N-1}]^T, O = [0,0,\dots,0]^T$ are associated vectors of Eq. (9)

Let $w = [w_1, w_2, \dots, w_{N-1}]^T \cong W$ which satisfies the equation

$$(D+F)w + G = 0 \quad (10)$$

Let $e_i = w_i - W_i, i = 1,2,3,4,\dots,N-1$ be the discretization error so that $E = [e_1, e_2, \dots, e_{N-1}]^T = w - W$.

By deducting Eq. (9) from Eq. (10), the error equation is developed as

$$(D+F)E = T(h) \quad (11)$$

Let $|p(s)| \le \xi_1$ and $|q(s)| \le \xi_2$ where ξ_1, ξ_2 are positive constants. If $F_{i,j}$ be the $(i,j)^{th}$ element of F, then

$$|p_{i,i+1}| = |\widetilde{w}_i| \le \varepsilon + \left[h\left[(\tilde{A}_3 - \widetilde{3A}_2) - (\tilde{A}_1+\tilde{A}_4)\right]\xi_1 - h^2\left[\tilde{A}_2\xi_2 + 4(\tilde{A}_1+\tilde{A}_4)\eta\xi_1^2\right] - 2h^3(\tilde{A}_1+\tilde{A}_4)\eta\xi_1\xi_2\right] \text{for } i = 1,2,\dots,N-2$$

$$|p_{i,i-1}| = |\tilde{z}_i| \le \varepsilon + \left[h\left[(3\tilde{A}_3 - \tilde{A}_2) + (\tilde{A}_1+\tilde{A}_4)\right]\xi_1 - h^2\left[\tilde{A}_2\xi_2 + 4(\tilde{A}_1+\tilde{A}_4)\eta\xi_1^2\right] + 2h^3(\tilde{A}_1+\tilde{A}_4)\eta\xi_1\xi_2\right] \text{for } i = 2,3,\dots,N-1.$$

Thus for sufficiently small h_i, we have

$$|p_{i,i+1}| < \varepsilon, i = 1,2,\ldots,N-2 \tag{12}$$

$$|p_{i,i-1}| < \varepsilon, i = 2,3,\ldots,N-1 \tag{13}$$

Hence $(D+F)$is irreducible [15].
Let $\bar{S}_i$ be the sum of the entries of the i^{th} row of $(D+F)$ matrix, then we obtain

$$\bar{S}_i = \varepsilon + \frac{h}{2}\left[\tilde{A}_2 p_{i+1} - (\tilde{A}_1+\tilde{A}_4)p_i - 3\tilde{A}_3 p_{i-1}\right] - h^2\left[\tilde{A}_2 q_{i+1} - \frac{(\tilde{A}_1+\tilde{A}_4)}{2}\eta p_i(p_{i+1}+3p_{i-1}) + (\tilde{A}_1+\tilde{A}_4)q_i\right] - h^3(\tilde{A}_1+\tilde{A}_4)\eta p_i q_{i+1} \quad \text{for } i = 1$$

$$\bar{S}_i = -h^2\left(\tilde{A}_3 q_{i-1} + (\tilde{A}_1+\tilde{A}_4)q_i + \tilde{A}_2 q_{i+1}\right) + h^3(\tilde{A}_1+\tilde{A}_4)\eta p_i(q_{i-1}-q_{i+1}) \quad \text{for } i = 2,3,\ldots,N-2$$

$$\bar{S}_i = \varepsilon + \frac{h}{2}\left[3\tilde{A}_2 p_{i+1} + (\tilde{A}_1+\tilde{A}_4)p_i - \tilde{A}_3 p_{i-1}\right] - h^2\left[\tilde{A}_3 q_{i-1} - \frac{(\tilde{A}_1+\tilde{A}_4)}{2}\eta p_i(3p_{i+1}+p_{i-1}) + (\tilde{A}_1+\tilde{A}_4)q_i\right] + h^3(\tilde{A}_1+\tilde{A}_4)\eta p_i q_{i-1} \quad \text{for } i = N-1$$

Let $\xi_{1^*} = \min_{1\le i\le N}|p(s_i)|$ and $\xi_1^* = \max_{1\le i\le N}|p(s_i)|$,$\xi_{2^*} = \min_{1\le i\le N}|q(s_i)|$ and $\xi_2^* = \max_{1\le i\le N}|q(s_i)|$.
We have $0 < \varepsilon << 1 \& \varepsilon \propto O(h)$, it has been confimed that for sufficiently small $h, (D + F$is monotone [14,15].

Hence $(D+F)^{-1}$ exists and $(D+F)^{-1} \ge 0$.
Thus from Eq. (11) we get

$$\|E\| \le \|(D+F)^{-1}\|\|T\| \tag{14}$$

Let $(D+F)^{-1}_{i,k}$ be the $(i,k)^{th}$ element of $(D+F)^{-1}$ and we define

$$\|(D+F)^{-1}\| = \max_{1\le i\le N-1}\sum_{k=1}^{N-1}(D+F)^{-1}_{i,k} \quad \text{and} \quad \|T(h)\| = \max_{1\le i\le N-1}|T(h)| \tag{15}$$

Since $(D+F)^{-1}_{i,k} \ge 0$ and $\sum_{k=1}^{N-1}(D+F)^{-1}_{i,k}\bar{S}_k = 1fori = 1,2,\ldots,N-1$.

Hence

$$(D+F)^{-1}_{i,1} \le \frac{1}{\bar{S}_1} < \frac{1}{h^2\left[[\tilde{A}_2+(\tilde{A}_1+\tilde{A}_4)]\xi_{2^*} - 4(\tilde{A}_1+\tilde{A}_4)\eta\xi_1^2\right]} \tag{16}$$

$$(D+F)^{-1}_{i,N-1} \le \frac{1}{\bar{S}_{N-1}} < \frac{1}{h^2\left[[\tilde{A}_3+(\tilde{A}_1+\tilde{A}_4)]\xi_{2^*} - 4(\tilde{A}_1+\tilde{A}_4)\eta\xi_1^2\right]} \tag{17}$$

Furthermore

$$\sum_{k=2}^{N-2}(D+F)^{-1}_{i,k} \le \frac{1}{\min_{2\le k\le N-2}\bar{S}_k} < \frac{1}{h^2(\tilde{A}_1+\tilde{A}_2+\tilde{A}_3+\tilde{A}_4)\xi_{2^*}}, \quad i = 2,3,\ldots,N-2 \tag{18}$$

By the help of Eqs. (15) - (18) and using Eq. (14) we get

$$\|E\| \leq O(h^4). \quad (19)$$

Hence the method given in Eq. (8) is fourth order convergent for
$\tilde{A}_2 = \tilde{A}_3 = \frac{1}{12}, (\tilde{A}_1 + \tilde{A}_4) = \frac{5}{12}$ and $\eta = -\frac{1}{20\varepsilon}$

VII. NUMERICAL EXPERIMENTS

Two point singular perturbation problems are investigated based on adaptive splineto establish the vitality of our proposed method computationally. These illustrations were considered because they were extensively explored in the research and had accurate solutions that could be compared. Maximum absolute errors in the solution are tabulated and compared with the existing method results which have demonstrated improvement.

Example 1: $-\varepsilon y''(s) + \frac{1}{s+1} y'(s) + \frac{1}{s+2} y(s) = f(s), y(1) = e + 2, y(0) = 1 + 2^{\frac{-1}{\varepsilon}}$
where $f(s) = (-\varepsilon + \frac{1}{s+1} + \frac{1}{s+2}) e^s + \frac{1}{s+2} 2^{\frac{-1}{\varepsilon}} (s+1)^{1+\frac{1}{\varepsilon}}$

Example 2: $-\varepsilon y''(s) - y'(s) = 0, y(0) = 1, y(1) = exp\left(\frac{-1}{\varepsilon}\right)$
$y(s) = exp\left(\frac{-s}{\varepsilon}\right)$ is the exact solution

Example 3: $\varepsilon y''(s) + 2(2s-1) y'(s) - 4y(s) = 0$ with $y(0) = 1, y(1) = 1$
The exact solution is

$$y(s) = \frac{-e^{\frac{1}{2\varepsilon} - \frac{(1-2s)^2}{2\varepsilon}} \left(2e^{\frac{(1-2s)^2}{2\varepsilon}} \sqrt{2\pi} werf\left(\frac{1-2s}{\sqrt{2\varepsilon}}\right) - e^{\frac{(1-2s)^2}{2\varepsilon}} \sqrt{2\pi} erf\left(\frac{1-2s}{\sqrt{2\varepsilon}}\right) - 2\varepsilon\right)}{e^{\frac{1}{2\varepsilon}} \sqrt{2\pi} erf\left(\frac{1}{\sqrt{2\varepsilon}}\right) + 2\sqrt{\varepsilon}}.$$

Example 4: $\varepsilon y''(s) - 2(2s-1) y(s) - 4y(s) = 0$ with $y(0) = 1, y(1) = 1$
This problem exhibits dual layers at $s = 0$ and $s = 1$

VIII. DISCUSSIONS AND CONCLUSION

In this chapter, we suggested a numerical scheme to solve a convection-diffusion problems using adaptive cubic spline. We introduce a new parameter η in the difference scheme to achieve fourth order convergence for the suggested problem. We have obtained a three-term relation with the help of difference scheme which involves a parameter ω. Using this, we have solved the tridiagonal scheme obtained by the method using discrete invariant imbedding. The convergence of the method has been discussed for the standard test examples chosen from the literature. MAEs are provided to illustrate the effectiveness of the method and also presented point wise errors in the solution of the examples for the comparison shown in tables 1-.4 to support the method. The proposed fourth order algorithm has been found to produce better results.

Table 1: The MAEs in Example 1 for $A_1 = A_4 = \frac{1}{3}$ and $A_2 = A_3 = \frac{1}{6}$

$\varepsilon \backslash h$	1/64	1/128	1/256	1/512	1/1024
Proposed Method					
2^{-4}	3.35(-4)	8.36(-5)	2.09(-5)	5.22(-6)	1.31(-6)
2^{-5}	7.82(-4)	1.94(-4)	4.84(-5)	1.21(-5)	3.03(-6)
2^{-6}	1.76(-3)	4.24(-4)	1.05(-4)	2.63(-5)	6.56(-6)
2^{-7}	4.53(-3)	9.43(-4)	2.23(-4)	5.51(-5)	1.37(-5)
2^{-8}	2.39(-2)	2.83(-3)	5.09(-4)	1.16(-4)	2.82(-5)
2^{-9}	1.36(-1)	1.95(-2)	1.96(-3)	2.86(-4)	6.01(-5)
Results in [11]					
2^{-4}	8.12(-4)	2.03(-4)	5.07(-5)	1.26(-5)	3.17(-6)
2^{-5}	3.53(-3)	8.79(-4)	2.19(-4)	5.48(-5)	1.37(-5)
2^{-6}	1.50(-2)	3.68(-3)	9.17(-4)	2.29(-4)	5.72(-5)
2^{-7}	6.75(-2)	1.54(-2)	3.77(-3)	9.37(-4)	2.34(-4)
2^{-8}	2.66(-1)	6.83(-2)	1.55(-2)	3.81(-3)	9.48(-4)
2^{-9}	6.92(-1)	2.68(-1)	6.87(-2)	1.56(-2)	3.83(-3)

Table 2: The MAEs in Example 2 for $A_1 = A_4 = \frac{1}{3}$ and $A_2 = A_3 = \frac{1}{6}$

$\varepsilon \backslash h$	1/64	1/128	1/256	1/512	1/1024
Proposed Method					
2^{-3}	1.24(-07)	7.76(-09)	4.85(-10)	3.05(-11)	2.60(-12)
2^{-4}	2.00(-06)	1.25(-07)	7.80(-09)	4.87(-10)	3.06(-11)
2^{-5}	3.24(-05)	2.00(-06)	1.25(-07)	7.80(-09)	4.87(-10)
2^{-6}	5.42(-04)	3.24(-05)	2.00(-06)	1.25(-07)	7.80(-09)
2^{-7}	0.0075	5.42(-04)	3.24(-05)	2.00(-06)	1.25(-07)
2^{-8}	0.0586	0.0075	5.42(-04)	3.24(-05)	2.00(-06)
2^{-9}	0.2255	0.0586	0.0075	5.42 (-04)	3.24(-05)
Results in [11]					
2^{-3}	4.77(-4)	1.19(-4)	2.98(-5)	7.45(-6)	1.86(-6)
2^{-4}	1.92(-3)	4.79(-4)	1.19(-4)	2.99(-5)	7.48(-6)
2^{-5}	7.87(-3)	1.92(-3)	4.79(-4)	1.19(-4)	2.99(-5)
2^{-6}	3.45(-2)	7.87(-3)	1.92(-3)	4.79(-4)	1.19(-4)
2^{-7}	6.75(-2)	1.54(-2)	3.77(-3)	9.37(-4)	2.34(-4)
2^{-8}	3.51(-1)	1.35(-1)	3.45(-2)	7.87(-3)	1.92(-3)
2^{-9}	6.00(-1)	3.51(-1)	1.35(-1)	3.45(-2)	7.87(-3)

Table 3: The MAEs in Example 3

□\ h	1/32	1/64	1/128	1/ 256	1/ 512	1/1024
Results in [12]						
2^{-5}	5.9701(-3)	3.3654(-3)	1.7391(-3)	8.7449(-4)	4.3697(-4)	2.1822(-4)
2^{-6}	5.3525(-3)	3.2322(-3)	1.7219(-3)	8.7336(-4)	4.3719(-4)	2.1834(-4)
2^{-7}	1.1177(-2)	2.9851(-3)	1.6827(-3)	8.6953(-4)	4.3725(-4)	2.1848(-4)
2^{-8}	2.5867(-2)	2.6763(-3)	1.6161(-3)	8.6093(-4)	4.3668(-4)	2.1860(-4)
2^{-9}	4.7842(-2)	5.5886(-3)	1.4925(-3)	8.4134(-4)	4.3477(-4)	2.1862(-4)
2^{-10}	7.5829(-2)	1.2934(-2)	1.3381(-3)	8.0805(-4)	4.3046(-4)	2.1834(-4)
Proposed Method						
2^{-5}	8.6799(-6)	5.4123(-7)	3.3837(-8)	2.1151(-9)	1.3218(-10)	8.1866(-12)
2^{-6}	2.4578(-5)	1.5331(-6)	9.5748(-8)	5.9831(-9)	3.7392(-10)	2.3330(-11)
2^{-7}	6.7327(-5)	4.3399(-6)	2.7061(-7)	1.6919(-8)	1.0575(-09)	6.6083(-11)
2^{-8}	1.9559(-4)	1.2289(-5)	7.6654(-7)	4.7874(-8)	2.9915(-09)	1.8695(-10)
2^{-9}	5.6901(-4)	3.3663(-5)	2.1700(-6)	1.3531(-7)	8.4593(-09)	5.2877(-10)
2^{-10}	1.5105(-3)	9.7794(-5)	6.1444(-6)	3.8327(-7)	2.3937(-08)	1.4958(-09)

Table 4: MAEs in Example 4

$\varepsilon\backslash h$	1/32	1/64	1/128	1/ 256	1/ 512	1/1024
Proposed Method						
2^{-5}	6.6213(-03)	4.9949(-04)	2.9956(-05)	1.8527(-06)	1.1549(-07)	7.2202(-09)
2^{-6}	5.3873(-02)	7.0956(-03)	5.2237(-04)	3.1287(-05)	1.9343(-06)	1.2056(-07)
2^{-7}	2.1476(-01)	5.6298(-02)	7.3146(-03)	5.3245(-04)	3.1875(-05)	1.9704(-06)
2^{-8}	4.6087(-01)	2.2024(-01)	5.7467(-02)	7.4197(-03)	5.3714(-04)	3.2149(-05)
2^{-9}	6.7876(-01)	4.6685(-01)	2.2288(-01)	5.8040(-02)	7.4712(-03)	5.3941(-04)
2^{-10}	8.2386(-01)	6.8315(-01)	4.6973(-01)	2.2419(-01)	5.8325(-02)	7.4966(-03)

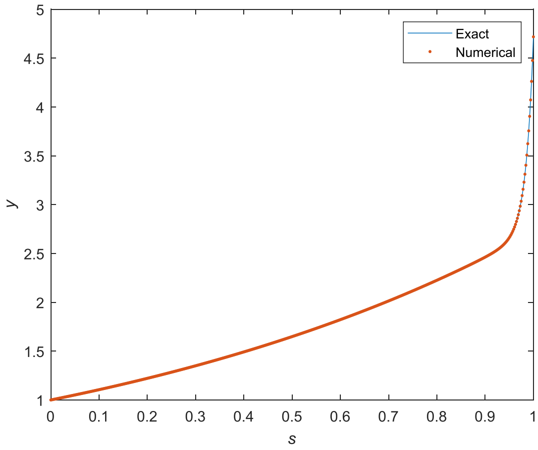

Figure 1: Solution Profile for $\varepsilon = 2^{-7}$with $h = 2^{-9}$

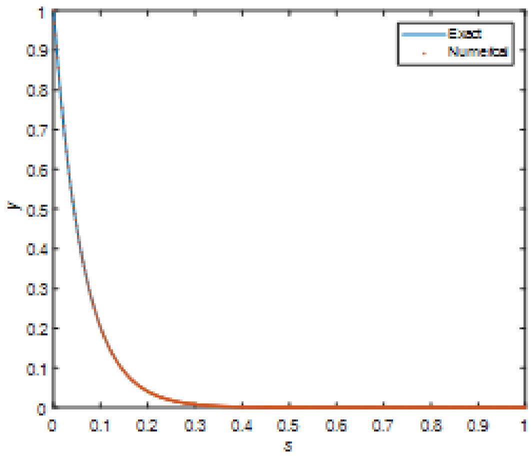

Figure 2: Solution Profile for $\varepsilon = 2^{-4}$with $h = 2^{-9}$

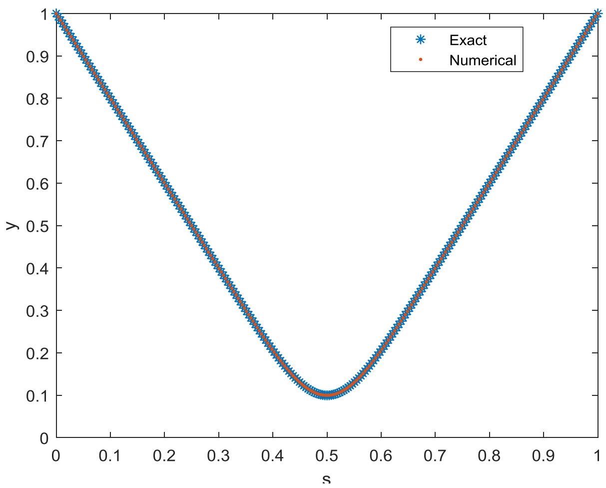

Figure 3: Solution Profile for$\varepsilon = 2^{-6}$with $h = 2^{-8}$

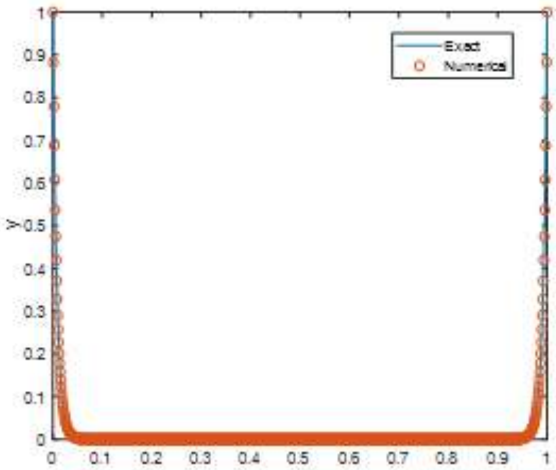

Figure 4: Solution Profile for $\varepsilon = 2^{-6}$ with $h = 2^{-10}$

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