SIMILARITY MEASURES OF FERMATEAN NEUTROSOPHIC SETS BASED ON THE COSINE FUNCTION AND THEIR APPLICATIONS

Abstract

In this study, we analyse the degree of hesitation, non-membership, and membership in Fermatean Neutrosophic sets (FNSs) and give similarity metrics between them based on the cosine function. Next, we utilise these similarity measures along with weighted measures amongst FNSs for diagnosing medical conditions and identifying patterns. Lastly, two instances are provided to show how effective similarity measures are at identifying patterns and making medical diagnoses.

Keywords: Measures, Fermatean Neutrosophic, Cosine Function.

Authors

Radha. R
Assistant Professor
Department of Science and Humanities
Karpagam College of Engineering
Coimbatore, Tamilnadu, India.
radharmat2020@gmail.com

Princy. R
Assistant Professor
Department of Science and Humanities
Karpagam College of Engineering
Coimbatore, Tamilnadu, India.
princy.pjs@gmail.com

Gayathri. P
Assistant Professor
Department of Science and Humanities
Karpagam College of Engineering
Coimbatore, Tamilnadu, India.
gayucbe@gmail.com

Gomathi. S
Assistant Professor
Department of Science and Humanities
Karpagam College of Engineering
Coimbatore, Tamilnadu, India.
gomathiprakash2013@gmail.com

Kavitha. A
Assistant Professor
Department of Science and Humanities
Karpagam College of Engineering
Coimbatore, Tamilnadu, India.
asrkavitha@gmail.com

Kalamani. A
Assistant Professor
Department of Science and Humanities
Karpagam College of Engineering
Coimbatore, Tamilnadu, India.
20kalamani@gmail.com

I. INTRODUCTION

The similarity measures are valuable instruments for assessing how similar two objects are to one another. Due to their widespread use in a variety of domains, including pattern recognition, machine learning, decision-making, and image processing, measures of similarity between fuzzy sets have drawn the attention of researchers. In recent years, numerous measures of similarity between fuzzy sets have been proposed and investigated [6–8, 15]. Since its introduction by Zadeh [31], fuzzy set theory has been extensively utilised to simulate uncertainty that arises in practical applications. Fuzzy sets were expanded by Atanassov [3,4] to become Atanassov's intuitionistic fuzzy sets (IFSs); numerous similarity measures between IFSs have been studied in the literature[17]. A suitable similarity measure between IFSs was proposed by Li and Cheng [16] and applied to pattern recognition challenges. Liang and Shi [16] explored the connections between several IFSs and proposed a few similarity mereasus to distinguish between them. Additionally, Mitchell [19] changed the measurements made by Li and Cheng. Szmidt and Kacprzyk [24, 25] devised a similarity measure between IFSs based on the Hamming distance, based on the extension of Hamming distance on fuzzy sets. Based on the Hausdorff distance, Hung and Yang [12] computed the distance between IFSs and produced a few similarity measures. Liu [18] created a few new similarity measures between elements and IFSs. Hung and Yang [13] proposed a similarity measures between IFSs based on the measures. The geometric distance and similarity measures of IFSs for group decision-making problems were defined by Xu and Xia [28]. The cosine similarity measure between IFSs was proposed by Ye [29]. The likelihood-based measurement of IFSs for medical diagnostic and bacteria classification problems was developed by Hung [14]. Shi and Ye [22] enhanced the IFSs' cosine similarity measures even further. The cotangent similarity measure between IFSs was suggested by Tian [27] for use in medical diagnosis. The cotangent similarity measure, which took into account the degrees of hesitation, non-membership, and membership as stated in IFSs, was further introduced by Rajarajeswari and Uma [20]. Szmidt [26] also covered the distances between IFSs and presented a family of similarity measure that took into account the degrees of hesitation, membership, and non-membership that are expressed in IFSs. Ye [30] presented two new weighted cosine similarity measures and two new cosine similarity measures based on the information conveyed by the membership, non-membership, and hesitation degrees in IFSs as well as the cosine function. The intuitionistic vector similarity measurements for medical diagnostics were provided by Son and Phong [23]. According to Antony Jansi [1], Fermatean Neutrosophic Sets were proposed.

The format of this paper is as follows: We present some fundamental ideas about IFSs and FNSs, as well as some measures of similarity between IFSs, in the following section. Based on the idea of the cosine function, we are going to suggest some similarity measures and weighted similarity measures between FNSs in Section 3. The application of FNSs' similarity measure to pattern recognition and medical diagnosis is discussed in Section 4. The benefits of the suggested similarity measures are covered in Section 5, and some closing thoughts are included in the final section to wrap up the work.

II. PRELIMINARIES

In the following, we introduce some basic concepts related to IFSs and some similarity measure between IFSs.

Definition 2.1 [3,4] :

An IFS A in X is given by,

$$A = \{< x, \mu_A(x), \vartheta_A(x) > | x \in X\} \qquad \text{------------ (1)}$$

Where $\mu_A : X \to [0,1]$ and $\vartheta_A : X \to [0,1]$, where $0 \le \mu_A(x) + \vartheta_A(x) \le 1, \forall\, x \in X$. The number $\mu_A(x)$ and $\vartheta_A(x)$ represents, respectively the membership degree and non-membership degree of the element x to the set A.

Definition 2.2 [3,4]:

For each IFS A in X, if

$$\pi_A(x) = 1 - \mu_A(x) - \vartheta_A(x), \forall x \in X. \qquad \text{---------- (2)}$$

Then $\pi_A(x)$is called the degree of indeterminacy of x to A.

Suppose that there are two IFSs $A = \{< x_j, \mu_A(x_j), \vartheta_A(x_j) > | x_j \in X\}$ and $B = \{< x_j, \mu_B(x_j), \vartheta_B(x_j) > | x_j \in X\}$ in the universe of discourse $X = \{x_1, x_2, \dots, x_n\}$.

Ye [30] proposed the cosine similarity measure between IFSs A and B as following:

$$IFC^1(A,B) = \frac{1}{n}\sum_{j=1}^{n} \frac{\mu_A(x_j)\mu_B(x_j) + \vartheta_A(x_j)\vartheta_A(x_j)}{\sqrt{\mu_A{}^2(x_j) + \vartheta_A{}^2(x_j)}\sqrt{\mu_B{}^2(x_j) + \vartheta_B{}^2(x_j)}} \qquad \text{----------(3)}$$

Shi and Ye [22] further presented the cosine similarity measure by considering membership degree, non-membership degree, and hesitancy degree in IFSs as the vector space of the three terms:

$$IFC^2(A,B) = \frac{1}{n}\sum_{j=1}^{n} \frac{\mu_A(x_j)\mu_B(x_j) + \vartheta_A(x_j)\vartheta_B(x_j) + \pi_A(x_j)\pi_B(x_j)}{\sqrt{\mu_A{}^2(x_j) + \vartheta_A{}^2(x_j) + \pi_A(x_j)}\sqrt{\mu_B{}^2(x_j) + \vartheta_B{}^2(x_j) + \pi_B(x_j)}} \qquad \text{-- ------------ (4)}$$

Based on the cosine function, Ye [30] proposed two cosine similarity measures between IFSs A and B.

$$IFCS^1(A,B) = \frac{1}{n}\sum_{j=1}^{n} cos\left[\frac{\pi}{2}\begin{pmatrix} |\mu_A(x_j) - \mu_B(x_j)| \\ \vee |\vartheta_A(x_j) - \vartheta_B(x_j)| \\ \vee |\pi_A(x_j) - \pi_B(x_j)| \end{pmatrix}\right] \qquad \text{------------- (5)}$$

$$IFCS^2(A,B) = \frac{1}{n}\sum_{j=1}^{n} cos\left[\frac{\pi}{4}\begin{pmatrix} |\mu_A(x_j) - \mu_B(x_j)| \\ + |\vartheta_A(x_j) - \vartheta_B(x_j)| \\ + |\pi_A(x_j) - \pi_B(x_j)| \end{pmatrix}\right] \qquad \text{-------------- (6)}$$

On the other hand, Tian [27] proposed a cotangent similarity measure between IFSs A and B as following:

$$IFCT^1(A,B)=\frac{1}{n}\sum_{j=1}^{n} cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(\left|\mu_A(x_j)-\mu_B(x_j)\right|\vee\left|\vartheta_A(x_j)-\vartheta_B(x_j)\right|\right)\right]$$ -------------(7)

Where the symbol " ∨ " is the maximum operation.

When the three terms such as the membership degree, non-membership degree, and hesitancy degree are considered in IFSs, Rajarajeswari and Uma [20] defined the cotangent similarity measure of IFSs:

$$IFCT^2(A,B)=\frac{1}{n}\sum_{j=1}^{n} cot\left[\frac{\pi}{4}+\frac{\pi}{4}\begin{pmatrix}\left|\mu_A(x_j)-\mu_B(x_j)\right|\\ \vee\left|\vartheta_A(x_j)-\vartheta_B(x_j)\right|\\ \vee\left|\pi_A(x_j)-\pi_B(x_j)\right|\end{pmatrix}\right]$$ ------------ (8)

In the following, we introduce the weighted cosine and cotangent similarity measures between IFSs A and B, respectively [29, 22, 27, 20, 30]**.**

$$WIFC^1(A,B)=\sum_{j=1}^{n}\omega_j\frac{\mu_A(x_j)\mu_B(x_j)+\vartheta_A(x_j)\vartheta_A(x_j)}{\sqrt{{\mu_A}^2(x_j)+{\vartheta_A}^2(x_j)}\sqrt{{\mu_B}^2(x_j)+{\vartheta_B}^2(x_j)}}$$ -------------- (9)

$$WIFC^2(A,B)=\sum_{j=1}^{n}\omega_j\frac{\mu_A(x_j)\mu_B(x_j)+\vartheta_A(x_j)\vartheta_B(x_j)+\pi_A(x_j)\pi_B(x_j)}{\sqrt{{\mu_A}^2(x_j)+{\vartheta_A}^2(x_j)+\pi_A(x_j)}\sqrt{{\mu_B}^2(x_j)+{\vartheta_B}^2(x_j)+\pi_B(x_j)}}$$ ------------(10)

$$WIFCS^1(A,B)=\sum_{j=1}^{n}\omega_j cos\left[\frac{\pi}{2}\begin{pmatrix}\left|\mu_A(x_j)-\mu_B(x_j)\right|\\ \vee\left|\vartheta_A(x_j)-\vartheta_B(x_j)\right|\\ \vee\left|\pi_A(x_j)-\pi_B(x_j)\right|\end{pmatrix}\right]$$ ------------- (11)

$$WIFCS^2(A,B)=\sum_{j=1}^{n}\omega_j cos\left[\frac{\pi}{4}\begin{pmatrix}\left|\mu_A(x_j)-\mu_B(x_j)\right|\\ +\left|\vartheta_A(x_j)-\vartheta_B(x_j)\right|\\ +\left|\pi_A(x_j)-\pi_B(x_j)\right|\end{pmatrix}\right]$$ ------------- (12)

$$WIFCT^1(A,B)=\sum_{j=1}^{n}\omega_j cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(\left|\mu_A(x_j)-\mu_B(x_j)\right|\vee\left|\vartheta_A(x_j)-\vartheta_B(x_j)\right|\right)\right]$$ ------------ (13)

$$WIFCT^2(A,B)=\sum_{j=1}^{n}\omega_j cot\left[\frac{\pi}{4}+\frac{\pi}{4}\begin{pmatrix}\left|\mu_A(x_j)-\mu_B(x_j)\right|\\ \vee\left|\vartheta_A(x_j)-\vartheta_B(x_j)\right|\\ \vee\left|\pi_A(x_j)-\pi_B(x_j)\right|\end{pmatrix}\right]$$ ------------- (14)

Where $\omega_j (j = 1,2, \dots, n)$ is the weight of an element x_j, $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$ and the symbol " $\vee$ " is the maximum operation.

Definition 2.3[2]:

Let X be a universe of discourse. A Fermatean Neutrosophic set [FN Set] A on X is an object of the form: $A = \{(x, \mu_M(x), \zeta_M(x), \nu_M(x))/x \in X\}$, where $\mu_M(x), \zeta_M(x), \nu_M(x) \in [0,1]$, $0 \leq \left(\mu_M(x)\right)^3 + \left(\nu_M(x)\right)^3 \leq 1$ and $0 \leq \left(\zeta_M(x)\right)^3 \leq 1$.

Then, $0 \leq \left(\mu_M(x)\right)^3 + \left(\zeta_M(x)\right)^3 + \left(\nu_M(x)\right)^3 \leq 2$ for all $x \in X$.

$\mu_M(x)$ is the degree of membership function, $\zeta_M(x)$ is the degree of indeterminacy and $\nu_M(x)$ is the degree of non- membership function. Here $\mu_M(x)$ and $\nu_M(x)$ are dependent components and $\zeta_M(x)$ is an independent components.

Definition 2.4[2]:

Let X be a non-empty set and I the unit interval [0,1]. A Fermatean Neutrosophic sets M and N of the form $M = \{(x, \mu_M(x), \zeta_M(x), \nu_M(x))/x \in X\}$ and $N = \{(x, \mu_N(x), \zeta_N(x), \nu_N(x))/x \in X\}$

- $M^C = \{(x, \nu_M(x), 1 - \zeta_M(x), \mu_M(x))/x \in X\}$.
- $M \cup = \left\{\left(x, \max(\mu_M(x), \mu_N(x)), \min\left(\zeta_M(x), \zeta_N(x)\right), \min(\nu_M(x), \nu_N(x)\right)/x \in X\right\}$.
- $M \cap = \left\{\left(x, \min(\mu_M(x), \mu_N(x)), \max\left(\zeta_M(x), \zeta_N(x)\right), \max(\nu_M(x), \nu_N(x)\right)/x \in X\right\}$

III. SOME SIMILARITY MEASURE BASED ON THE COSINE FUNCTION FOR FERMATEAN NEUTROSOPHIC SETS

1. Cosine Similarity Measure for FNSs

Let A be a FNS in an universe of discourse $X = \{x\}$, the FNS is characterized by the degree of membership $\mu_A(x)$, the degree of non-membership $\vartheta_A(x)$, and the degree of hesitation $\pi_A(x)$, $\pi_A(x) = \sqrt{1 - \left[{\mu_A}^3(x) + {\vartheta_A}^3(x)\right]}$, which can be considered as a vector representation with the three elements. Therefore, a cosine similarity measure and a weighted cosine similarity measure for FNSs are proposed in an analogous manner to the cosine similarity measure based on Bhattacharya's distance [21, 6] and cosine similarity measure for IFS [28].

Suppose that there are two FNSs

$A = \left\{< x_j, \mu_A\left(x_j\right), \vartheta_A\left(x_j\right), \pi_A\left(x_j\right) > \middle| x_j\right.$ and

$B = \{< x_j, \mu_B(x_j), \vartheta_B(x_j), \pi_B(x_j) > | x_j \in X\}$ in the universe of discourse $X = \{x_1, x_2, \dots, x_n\}$, we further propose the cosine similarity measures between FNSs as follows:

$$FNC(A,B) = \frac{1}{n}\sum_{j=1}^{n} \frac{\mu_A{}^3(x_j)\mu_B{}^3(x_j) + \vartheta_A{}^3(x_j)\vartheta_B{}^3(x_j) + \pi_A{}^3(x_j)\pi_B{}^3(x_j)}{\sqrt{\mu_A{}^6(x_j) + \vartheta_A{}^6(x_j) + \pi_A{}^6(x_j)}\sqrt{\mu_B{}^6(x_j) + \vartheta_B{}^6(x_j) + \pi_B{}^6(x_j)}} \text{ ----- (17)}$$

If we take n=1, then the cosine similarity measure between FNSs A and B becomes the correlation coefficient between FNSs A and B. Therefore, the cosine similarity measure between A and B also satisfies the following properties:

- $0 \leq FNC(A,B) \leq 1$
- $FNC(A,B) = FNC(B,A)$.
- $FNC(A,B) = 1$, if $A = B, i = 1,2, \dots, n$

Proof:

- It is obvious that the proposition is true according to the cosine value.
- It is obvious that the proposition is true.
- When $A = B$, there are $\mu_A(x_j) = \mu_B(x_j)$, $\vartheta_A(x_j) = \vartheta_B(x_j)$ and $\pi_A(x_j) = \pi_B(x_j)\ for\ j = 1,2, \dots, n$. So there is $C_{FNS}(A,B) = 1$.

Therefore, we have finished the proofs.

If we consider the weights of x_j, a weighted cosine similarity measure between FNSs A and B is proposed as follows:

$$WFNC(A,B) = \sum_{j=1}^{n} \omega_j \frac{\mu_A{}^3(x_j)\mu_B{}^3(x_j) + \vartheta_A{}^3(x_j)\vartheta_B{}^3(x_j) + \pi_A{}^3(x_j)\pi_B{}^3(x_j)}{\sqrt{\mu_A{}^6(x_j) + \vartheta_A{}^6(x_j) + \pi_A{}^6(x_j)}\sqrt{\mu_B{}^6(x_j) + \vartheta_B{}^6(x_j) + \pi_B{}^6(x_j)}}$$

-------------- (18)

Where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector of $x_j (j = 1,2, \dots, n)$, with $\omega_j \in [0,1], j = 1,2, \dots, n$, $\sum_{j=1}^{n} \omega_j = 1$. In particular, if $\omega = (1/n, 1/n, \dots, 1/n)^T$, then the weighted cosine similarity measure reduces to cosine similarity measure. That is to say, if we take $\omega_j = \frac{1}{n}, j = 1,2, \dots, n$, then there is $WFNC(A,B) = FNC(A,B)$. Obviously, the weighted cosine similarity measure of two FNSs A and B also satisfies the following properties:

- $0 \leq WFNC(A,B) \leq 1$
- $WFNC(A,B) = WFNC(B,A)$.
- $WFNC(A,B) = 1$, if $A = B, i = 1,2, \dots, n$.

Similar to the previous proof method, we can prove the above three properties.

In the following, we shall investigate the distance measure of the angle as

$d(A,B) = arccos\big(C_{FNS}(A,B)\big)$. It satisfies the following properties:

- $d(A,B) \geq 0$, if $0 \leq C_{FNS}(A,B) \leq 1$;
- $d(A,B) = \arccos(1) = 0$, if $C_{FNS}(A,A) = 1$.
- $d(A,B) = d(B,A)$, if $C_{FNS}(A,B) = C_{FNS}(B,A)$
- $d(A,C) \leq d(A,B) + d(B,C)$, if $A \subseteq B \subseteq C$ for any FNS C.

Proof:

Obviously, $d(A,B)$ satisfies the property (1) - (3). In the following, $d(A,B)$ will be proved to satisfy the property (4).

For any $C = \{< x_j, \mu_C(x_j), \vartheta_C(x_j), \pi_C(x_j) > | x_j \in X\}$, $A \subseteq B \subseteq C$, Since Equation (16) is the sum of terms, let us investigate the distance measures of the angle between the vectors:

$d_j\left(A(x_j), B(x_j)\right) = \arccos\left(FNC\left(A(x_j), B(x_j)\right)\right)$,

$d_j\left(B(x_j), C(x_j)\right) = \arccos\left(FNC\left(B(x_j), C(x_j)\right)\right)$ and

$d_j\left(A(x_j), C(x_j)\right) = \arccos\left(FNC\left(A(x_j), C(x_j)\right)\right), j = 1,2, \dots, n$ where

$$FNC\left(A(x_j), B(x_j)\right) = \sum_{j=1}^{n} \frac{{\mu_A}^3(x_j){\mu_B}^3(x_j) + {\vartheta_A}^3(x_j){\vartheta_B}^3(x_j) + {\pi_A}^3(x_j){\pi_B}^3(x_j)}{\sqrt{{\mu_A}^6(x_j) + {\vartheta_A}^6(x_j) + {\pi_A}^6(x_j)}\sqrt{{\mu_B}^6(x_j) + {\vartheta_B}^6(x_j) + {\pi_B}^6(x_j)}}$$

$$FNC\left(B(x_j), C(x_j)\right) = \sum_{j=1}^{n} \frac{{\mu_B}^3(x_j){\mu_C}^3(x_j) + {\vartheta_B}^3(x_j){\vartheta_C}^3(x_j) + {\pi_B}^3(x_j){\pi_C}^3(x_j)}{\sqrt{{\mu_B}^6(x_j) + {\vartheta_B}^6(x_j) + {\pi_B}^6(x_j)}\sqrt{{\mu_C}^6(x_j) + {\vartheta_C}^6(x_j) + {\pi_C}^6(x_j)}}$$

$$FNC\left(A(x_j), C(x_j)\right) = \sum_{j=1}^{n} \frac{{\mu_A}^3(x_j){\mu_C}^3(x_j) + {\vartheta_A}^3(x_j){\vartheta_C}^3(x_j) + {\pi_A}^3(x_j){\pi_C}^3(x_j)}{\sqrt{{\mu_A}^6(x_j) + {\vartheta_A}^6(x_j) + {\pi_A}^6(x_j)}\sqrt{{\mu_C}^6(x_j) + {\vartheta_C}^6(x_j) + {\pi_C}^6(x_j)}}$$

For three vectors $A(x_j) =< \mu_A(x_j), \vartheta_A(x_j), \pi_A(x_j) >$, $B(x_j) =< \mu_B(x_j), \vartheta_B(x_j), \pi_B(x_j) >$, $C(x_j) =< \mu_C(x_j), \vartheta_C(x_j), \pi_C(x_j) >$ in one plane, if $A(x_j) \subseteq B(x_j) \subseteq$

$C(x_j), j = 1,2, \dots, n$. Then, it is obvious that $d_j\left(A(x_j), C(x_j)\right) \le d_j\left(A(x_j), B(x_j)\right) + d_j\left(B(x_j), C(x_j)\right)$ according to the triangle inequality. Combining the inequality with Equation (16), we can obtain $d(A, C) \le d(A, B) + d(B, C)$. Thus $d(A, B)$ satisfies the property (4). So we finished the proof.

2. Similarity Measures of FNSs Based on Cosine Function

Based on the cosine function, in this section, we shall propose two cosine similarity measures between FNSs and analyse their properties.

Definition 3.2.1:

Suppose that there are two FNSs $A = \{< x_j, \mu_A(x_j), \vartheta_A(x_j), \pi_A(x_j) > | x_j \in X\}$ and $B = \{< x_j, \mu_B(x_j), \vartheta_B(x_j), \pi_B(x_j) > | x_j \in X\}$ in the universe of discourse $X = \{x_1, x_2, \dots, x_n\}$, we further propose the cosine similarity measures between FNSs as follows:

$$FNCS^1(A,B) = \frac{1}{n}\sum_{j=1}^{n} cos\left[\frac{\pi}{2}\begin{pmatrix} \left|\mu_A{}^2(x_j) - \mu_B{}^2(x_j)\right| \\ \vee \left|\vartheta_A{}^2(x_j) - \vartheta_B{}^2(x_j)\right| \\ \vee \left|\pi_A{}^2(x_j) - \pi_B{}^2(x_j)\right| \end{pmatrix}\right] \quad \text{-----------(19)}$$

$$FNCS^2(A,B) = \frac{1}{n}\sum_{j=1}^{n} cos\left[\frac{\pi}{4}\begin{pmatrix} \left|\mu_A{}^2(x_j) - \mu_B{}^2(x_j)\right| \\ +\left|\vartheta_A{}^2(x_j) - \vartheta_B{}^2(x_j)\right| \\ +\left|\pi_A{}^2(x_j) - \pi_B{}^2(x_j)\right| \end{pmatrix}\right] \quad \text{---------- (20)}$$

Where the symbol " $\vee$ " is the maximum operator.

Proposition 3.2.2:

For any two FNSs A and B in $X = \{x_1, x_2, \dots, x_n\}$, the cosine similarity measures $FNCS^k(A, B)$ $(k = 1,2)$ should satisfy the following properties (1) – (4):

- $0 \le FNCS^k(A, B) \le 1$
- $FNCS^k(A, B) = 1\ if\ and\ only\ if\ A = B$
- $FNCS^k(A, B) = FNCS^k(B, A)$
- If C is a FNS in X and $A \subseteq B \subseteq C$, then $FNCS^k(A, C) \le FNCS^k(A, B)$ and $FNCS^k(A, C) \le FNCS^k(B, C)$.

Proof:

- Since the value of the cosine function is within [0,1], the similarity measure based on the cosine function is also within [0,1]. Thus, there is $0 \le FNCS^k(A, B) \le 1$.

- For any two FNSs A and B in $X = \{x_1, x_2, \ldots, x_n\}$, if $A = B$, then ${\mu_A}^3(x_j) = {\mu_B}^3(x_j)$, ${\vartheta_A}^3(x_j) = {\vartheta_B}^3(x_j)$ and ${\pi_A}^3(x_j) = {\pi_B}^3(x_j)$ for $j = 1,2, \ldots, n$. Thus,

  $|{\mu_A}^3(x_j) - {\mu_B}^3(x_j)| = 0$, $|{\vartheta_A}^3(x_j) - {\vartheta_B}^3(x_j)| = 0$, $|{\pi_A}^3(x_j) - {\pi_B}^3(x_j)| = 0$. So, $FNCS^k(A,B) = 1, (k = 1,2)$.

  If $FNCS^k(A,B) = 1, (k = 1,2)$, this implies $|{\mu_A}^3(x_j) - {\mu_B}^3(x_j)| = 0$, $|{\vartheta_A}^3(x_j) - {\vartheta_B}^3(x_j)| = 0$, $|{\pi_A}^3(x_j) - {\pi_B}^3(x_j)| = 0$, for $j = 1,2, \ldots, n$. Since cos(0) =1. Then, there are

  ${\mu_A}^3(x_j) = {\mu_B}^3(x_j)$, ${\vartheta_A}^3(x_j) = {\vartheta_B}^3(x_j)$ and ${\pi_A}^3(x_j) = {\pi_B}^3(x_j)$ for $j = 1,2, \ldots, n$. Hence $A = B$.

- Proof is straightforward.

- If $A \subseteq B \subseteq C$, then there are $\mu_A(x_j) \le \mu_B(x_j) \le \mu_C(x_j)$, $\vartheta_A(x_j) \ge \vartheta_B(x_j) \ge \vartheta_C(x_j)$ and $\pi_A(x_j) \ge \pi_B(x_j) \ge \pi_C(x_j)$, for $j = 1,2, \ldots n$. Then, ${\mu_A}^3(x_j) \le {\mu_B}^3(x_j) \le {\mu_C}^3(x_j)$, ${\vartheta_A}^3(x_j) \ge {\vartheta_B}^3(x_j) \ge {\vartheta_C}^3(x_j)$ and ${\pi_A}^3(x_j) \ge {\pi_B}^3(x_j) \ge {\pi_C}^3(x_j)$.

  Thus, we have

  $|{\mu_A}^3(x_j) - {\mu_B}^3(x_j)| \le |{\mu_A}^3(x_j) - {\mu_C}^3(x_j)|$,

  $|{\mu_B}^3(x_j) - {\mu_C}^3(x_j)| \le |{\mu_A}^3(x_j) - {\mu_C}^3(x_j)|$,

  $|{\vartheta_A}^3(x_j) - {\vartheta_B}^3(x_j)| \le |{\vartheta_A}^3(x_j) - {\vartheta_C}^3(x_j)|$,

  $|{\vartheta_B}^3(x_j) - {\vartheta_C}^3(x_j)| \le |{\vartheta_A}^3(x_j) - {\vartheta_C}^3(x_j)|$,

  $|{\pi_A}^3(x_j) - {\pi_B}^3(x_j)| \le |{\pi_A}^3(x_j) - {\pi_C}^3(x_j)|$ and

  $|{\pi_B}^3(x_j) - {\pi_C}^3(x_j)| \le |{\pi_A}^3(x_j) - {\pi_C}^3(x_j)|$.

Hence, $FNCS^k(A,C) \le FNCS^k(A,B)$ and $FNCS^k(A,C) \le FNCS^k(B,C)$ for $k = 1,2$, as the cosine function is a decreasing function with the interval $[0, \pi/2]$.

Thus, the proofs of these properties are completed.

In many situations, the weight of the elements $x_j \in X$ should be taken into account. For example, in Multiple Attribute Decision Making (MADM), the considered attributes usually have different importance, and thus need to be assigned different weights. As a result, two weighted cosine similarity measure between FNSs A and B is proposed as follows:

$$WFNCS^1(A,B) = \sum_{j=1}^{n} \omega_j \cos\left[\frac{\pi}{2}\begin{pmatrix} \left|\mu_A{}^3(x_j) - \mu_B{}^3(x_j)\right| \\ \vee \left|\vartheta_A{}^3(x_j) - \vartheta_B{}^3(x_j)\right| \\ \vee \left|\pi_A{}^3(x_j) - \pi_B{}^3(x_j)\right| \end{pmatrix}\right] \quad \text{------------- (21)}$$

$$WFNCS^2(A,B) = \sum_{j=1}^{n} \omega_j \cos\left[\frac{\pi}{4}\begin{pmatrix} \left|\mu_A{}^3(x_j) - \mu_B{}^3(x_j)\right| \\ +\left|\vartheta_A{}^3(x_j) - \vartheta_B{}^3(x_j)\right| \\ +\left|\pi_A{}^3(x_j) - \pi_B{}^3(x_j)\right| \end{pmatrix}\right] \quad \text{------------- (22)}$$

Where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector of $x_j (j = 1,2, \dots, n)$, with $\omega_j \in [0,1], j = 1,2, \dots, n$, $\sum_{j=1}^{n} \omega_j = 1$ and the symbol " $\vee$ " is the maximum operator. In particular, if $\omega = (1/n, 1/n, \dots, 1/n)^T$, then the weighted cosine similarity measure reduces to cosine similarity measure. That is to say, if we take $\omega_j = \frac{1}{n}, j = 1,2, \dots, n$, then there is

$WFNCS^k(A,B) = WFNCS^1(A,B)$ $(k = 1,2)$.

Obviously, the weighted cosine similarity measures also satisfy the axiomatic requirements of similarity measures in Proposition 2.

Proposition 3.2.3:

For any two FNSs A and B in $X = \{x_1, x_2, \dots, x_n\}$, the cosine similarity measures $WFNCS^k(A,B)$ $(k = 1,2,3,4)$ should satisfy the following properties (1) – (4):

- $0 \leq WFNCS^k(A,B) \leq 1$
- $WFNCS^k(A,B) = 1 \; if \; and \; only \; if \; A = B$
- $WFNCS^k(A,B) = WFNCS^k(B,A)$
- If C is a FNS in X and $A \subseteq B \subseteq C$, then W $FNCS^k(A,C) \leq WFNCS^k(A,B)$ and $WFNCS^k(A,C) \leq WFNCS^k(B,C)$.

By using similar proof in Proposition 1, we can give the proofs of these properties (1) – (4).

3. Similarity Measures of FNSs based on the Cotangent Function

In this section, we shall propose two cotangent similarity measures between FNSs.

Definition 3.3.1:

Suppose that there are two FNSs $A = \{< x_j, \mu_A(x_j), \vartheta_A(x_j), \pi_A(x_j) > | x_j \in X\}$ and $B = \{< x_j, \mu_B(x_j), \vartheta_B(x_j), \pi_B(x_j) > | x_j \in X\}$ in the universe of discourse $X = \{x_1, x_2, \dots, x_n\}$, we further propose the cotangent similarity measures between FNSs as follows:

$$FNCT^1(A,B) = \frac{1}{n}\sum_{j=1}^{n} cot\left[\frac{\pi}{4} + \frac{\pi}{4}\begin{pmatrix} \left|\mu_A{}^3(x_j) - \mu_B{}^3(x_j)\right| \\ \vee \left|\vartheta_A{}^3(x_j) - \vartheta_B{}^3(x_j)\right| \\ \vee \left|\pi_A{}^3(x_j) - \pi_B{}^3(x_j)\right| \end{pmatrix}\right] \quad \text{--------- (23)}$$

$$FNCT^2(A,B) = \frac{1}{n}\sum_{j=1}^{n} cot\left[\frac{\pi}{4} + \frac{\pi}{12}\begin{pmatrix} \left|\mu_A{}^3(x_j) - \mu_B{}^3(x_j)\right| \\ +\left|\vartheta_A{}^3(x_j) - \vartheta_B{}^3(x_j)\right| \\ +\left|\pi_A{}^3(x_j) - \pi_B{}^3(x_j)\right| \end{pmatrix}\right] \quad \text{------ (24)}$$

Where the symbol " ∨ " is the maximum operator.

Proposition 3.3.2:

For any two FNSs A and B in $X = \{x_1, x_2, \dots, x_n\}$, the cotangent similarity measures $FNCT^k(A,B)$ $(k = 1{,}2)$ should satisfy the following properties (1) – (4):

- $0 \le FNCT^k(A,B) \le 1$
- $FNCT^k(A,B) = 1\ if\ and\ only\ if\ A = B$
- $FNCT^k(A,B) = FNCT^k(B,A)$
- If C is a FNS in X and $A \subseteq B \subseteq C$, then $FNCT^k(A,C) \le FNCT^k(A,B)$ and $FNCT^k(A,C) \le FNCT^k(B,C)$.

Proof:

- Since,

  $$\frac{\pi}{4} \le \left(\frac{\pi}{12}\left(3 + \left|\mu_A{}^3(x_j) - \mu_B{}^3(x_j)\right| + \left|\vartheta_A{}^3(x_j) - \vartheta_B{}^3(x_j)\right| + \left|\pi_A{}^3(x_j) - \pi_B{}^3(x_j)\right|\right)\right) \le \frac{\pi}{2},$$

  It is obvious that the cotangent function $FNCT^k(A,B)$ are within 0 and 1.

- It is obvious that the proposition is true.

- When $A = B$, then obviously $FNCT^k(A,B) = 1$. On the other hand if $FNCT^k(A,B) = 1$ then,

  $\mu_A{}^3(x_j) = \mu_B{}^3(x_j), \vartheta_A{}^3(x_j) = \vartheta_B{}^3(x_j)\ and \pi_A{}^3(x_j) = \pi_B{}^3(x_j)\ for j = 1{,}2, \dots, n.$

  This implies $A = B$.

- If $A \subseteq B \subseteq C$ then we can write $\mu_A(x_j) \le \mu_B(x_j) \le \mu_C(x_j)$, $\vartheta_A(x_j) \ge \vartheta_B(x_j) \ge \vartheta_C(x_j)$ and $\pi_A(x_j) \ge \pi_B(x_j) \ge \pi_C(x_j)$, for $j = 1{,}2, \dots n$. Then, $\mu_A{}^3(x_j) \le \mu_B{}^3(x_j) \le \mu_C{}^3(x_j)$, $\vartheta_A{}^3(x_j) \ge \vartheta_B{}^3(x_j) \ge \vartheta_C{}^3(x_j)$ and $\pi_A{}^3(x_j) \ge \pi_B{}^3(x_j) \ge \pi_C{}^3(x_j)$.

The cotangent function is decreasing function within the interval $\left[\frac{\pi}{4},\frac{\pi}{2}\right]$.

Hence we can write $FNCT^k(A,C) \leq FNCT^k(A,B)$ and $FNCT^k(A,C) \leq FNCT^k(B,C)$.

Thus, the proofs of these properties are completed.

In many situations, the weight of the elements $x_j \in X$ should be taken into account. For example, in Multiple Attribute Decision Making (MADM), the considered attributes usually have different importance, and thus need to be assigned different weights. As a result, two weighted cotangent similarity measure between FNSs A and B is proposed as follows:

$$WFNCT^1(A,B) = \sum_{j=1}^{n} \omega_j cot\left[\frac{\pi}{4} + \frac{\pi}{4}\begin{pmatrix} \left|\mu_A{}^3(x_j) - \mu_B{}^3(x_j)\right| \\ \vee \left|\vartheta_A{}^3(x_j) - \vartheta_B{}^3(x_j)\right| \\ \vee \left|\pi_A{}^3(x_j) - \pi_B{}^3(x_j)\right| \end{pmatrix}\right] \quad \text{---------- (25)}$$

$$WFNCT^2(A,B) = \sum_{j=1}^{n} \omega_j cot\left[\frac{\pi}{4} + \frac{\pi}{12}\begin{pmatrix} \left|\mu_A{}^3(x_j) - \mu_B{}^3(x_j)\right| \\ +\left|\vartheta_A{}^3(x_j) - \vartheta_B{}^3(x_j)\right| \\ +\left|\pi_A{}^3(x_j) - \pi_B{}^3(x_j)\right| \end{pmatrix}\right] \quad \text{-------- (26)}$$

Where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $x_j (j = 1,2,\ldots,n)$, with $\omega_j \in [0,1], j = 1,2,\ldots,n$, $\sum_{j=1}^{n} \omega_j = 1$ and the symbol " $\vee$ " is the maximum operator. In particular, if

$\omega = (1/n, 1/n, \ldots, 1/n)^T$, then the weighted cotangent similarity measure reduces to cotangent similarity measure. That is to say, if we take $\omega_j = \frac{1}{n}, j = 1,2,\ldots,n,$ then there is $WFNCT^k(A,B) = WFNCT^1(A,B)$ $(k = 1,2)$.

Proposition 3.3.3:

For any two FNSs A and B in $X = \{x_1, x_2, \ldots, x_n\}$, the cosine similarity measures $WSFCS^k(A,B)$ $(k = 1,2,3,4)$ should satisfy the following properties (1) – (4):

- $0 \leq WFNCT^k(A,B) \leq 1$
- $WFNCT^k(A,B) = 1 \; if \; and \; only \; if \; A = B$
- $WFNCT^k(A,B) = WFNCT^k(B,A)$
- If C is a FNS in X and $A \subseteq B \subseteq C$, then W $FNCT^k(A,C) \leq WFNCT^k(A,B)$ and $WFNCT^k(A,C) \leq WFNCT^k(B,C)$.

By using similar proof in Proposition 3, we can give the proofs of these properties (1) – (4)

IV. APPLICATIONS

In this section, the cosine and cotangent similarity measures for FNSs are applied to pattern recognition and medical diagnosis to illustrate the feasibility of the proposed methods and deliver a comparative analysis

1. Example 1: Pattern Recognition

Let us consider, a three known patterns $A_i(i = 1,2,3)$, which are represented by the FNSs: $A_i(i = 1,2,3)$ in the feature space $X = \{x_1, x_2, x_3\}$ as

$$A_1 = \{(0.8,0.1,0.2), (0.7,0.2,0.3), (0.7,0.4,0.3)\}$$

$$A_2 = \{(0.7,0.2,0.2), (0.8,0.2,0.3), (0.6,0.5,0.4)\}$$

$$A_3 = \{(0.8,0.3,0.3), (0.7,0.2,0.2), (0.8,0.4,0.4)\}$$

Consider an unknown pattern $A \in FNSs(X)$ that will be recognized, where

$$A = \{(x_1, 0.5,0.5,0.4), (x_2, 0.5,0.2,0.3), (x_3, 0.9,0.1,0.2)\}$$

The purpose of this problem is classify the pattern A in one classes A_1, A_2 and A_3. For it, the proposed similarities degrees have been computed from A to $A_i(i = 1,2,3)$ and are given in Table 1.

Table 1: The Similarity Measures between $A_i(i = 1, 2, 3)$ And A

Similarity Measures	(A_1, A)	(A_2, A)	(A_3, A)
$FNS^1(A_i, A)$	0.8704	0.8320	**0.8992**
$FNCS^1(A_i, A)$	0.8747	0.8360	**0.9041**
$FNCS^2(A_i, A)$	0.9104	0.8863	**0.9328**
$FNCT^1(A_i, A)$	0.5967	0.5533	**0.654**
$FNCT^2(A_i, A)$	0.7695	0.7327	**0.7898**

From the numerical results presented in Table 1, we know that the degree of similarity between A_3 and A is the largest one as derived by five similarity measures. That is, all the 5 similarity measures assign the unknown class A to the known class A_3 according to the principle of maximum degree of similarity between FNSs. Compared with Garg's correlation coefficient method [10], we can get the same result that all the 5 similarity measures assign the unknown class A to the known class A_3 according to the principle of maximum degree of similarity between FNSs.

If we consider the weight of $x_i(i = 1,2,3)$ are 0.5, 0.3 and 0.2 respectively. Then we use the proposed weighted similarities measures have been computed from A to $A_i(i = 1,2,3)$ and are given in Table 2.

Table 2: The Weighted Similarity Measures between $A_i(i = 1,2,3)$ And A

Similarity Measures	(A_1, A)	(A_2, A)	(A_3, A)
$WFNS^1(A_i, A)$	0.8244	0.8145	**0.8692**
$WFNCS^1(A_i, A)$	0.8631	0.8622	**0.8808**
$WFNCS^2(A_i, A)$	0.8938	0.8975	**0.9221**
$WFNCT^1(A_i, A)$	0.5818	0.5877	**0.6161**
$WFNCT^2(A_i, A)$	0.7503	0.7426	**0.7753**

From the numerical results presented in Table 2, we know that the weighted similarity measures between A_3 and A is the largest one as derived by five similarity measures. That is, all the 5 similarity measures assign the unknown class A to the known class A_3 according to the principle of maximum degree of similarity between FNSs. Compared with Garg's correlation coefficient method [10], we can get the same result that all the 5 similarity measures assign the unknown class A to the known class A_3 according to the principle of maximum degree of similarity between FNSs.

2. Example 2: Medical Diagnosis

Let us consider a set of diagnosis $D = \{D_1(Viral\ fever), D_2(typhoid),$

$D_3(Stomach\ Problem), D_4(Malaria), D_5(Chest\ Problem)\}$ and a set of symptoms

$S = \{s_1(Temperature), s_2(Head\ Ache), s_3(Stomach\ Pain), s_4(Cough), s_5(Chest\ Pain)\}$. Suppose that a patient, with respect to all symptoms, can be depicted by the following FNS:

$$P(Patient) = \{(s_1, 0.8, 0.5, 0.2), (s_2, 0.6, 0.4, 0.4), (s_3, 0.7, 0.5, 0.1)$$
$$(s_4, 0.9,0.3,0.2), (s_5, 0.7, 0.6, 0.3)\}$$

And then each diagnoses $D_i(i = 1,2,3,4,5)$ can be viewed as FNSs with respect to all the symptoms as follows:

$$D_1(Viral\ Fever) = \{(s_1, 0.4,0.1,0.9), (s_2, 0.7, 0.3, 0.2), (s_3, 0.5, 0.6, 0.1)$$
$$(s_4, 0.2, 0.5, 0.4), (s_5, 0.1, 0.7, 0.5)\}$$

$D_2(Typhoid) = \{(s_1, (0.6, 0.2, 0.1), (s_2, 0.8, 0.3, 0.2), (s_3, 0.5, 0.5, 0.3),$
$(s_4, 0.7, 0.2, 0.4), (s_5, 0.2, 0.7, 0.4)\}$

$D_3(Stomach\ Problem) = \{(s_1, 0.5, 0.3, 0.2), (s_2, 0.7, 0.5, 0.3), (s_3, 0.8, 0.2, 0.5),$
$(s_4, 0.7, 0.3, 0.1), (s_5, 0.9, 0.1, 0.3)\}$

$D_4(Malaria) = \{(s_1, 0.7, 0.4, 0.3), (s_2, 0.6, 0.5, 0.4), (s_3, 0.4, 0.6, 0.2),$
$(s_4, 0.8, 0.2, 0.4), (s_5, 0.3, 0.2, 0.1)\}$

$D_5(Chest\ Problem) = \{(s_1, 0.9, 0.2, 0.1), (s_2, 0.7, 0.6, 0.4), (s_3, 0.3, 0.2, 0.1),$
$(s_4, 0.6, 0.3, 0.1), (s_5, 0.8,0.2, 0.5)\}$

The purpose of this problem is classify the pattern P in one classes $D_i(i = 1,2,3,4,5)$. For this, the proposed similarities measures have been computed from P to $D_i(i = 1,2,3,4,5)$ and are given in Table 3.

Table 3: The Similarity Measures between $D_i(i = 1, 2, 3, 4, 5)$ And P

Similarity Measures	(D_1, P)	(D_2, P)	(D_3, P)	(D_4, P)	(D_5, P)
$FNS^1(D_i, P)$	0.5811	0.8863	0.9288	0.9420	**0.9469**
$FNCS^1(D_i, P)$	0.7440	0.8752	0.8911	**0.9208**	0.8685
$FNCS^2(D_i, P)$	0.7644	0.9236	0.9200	**0.9355**	0.9224
$FNCT^1(D_i, P)$	0.5083	0.6002	0.6328	**0.7018**	0.6005
$FNCT^2(D_i, P)$	0.6628	0.7741	0.7700	**0.8155**	0.7717

From the numerical results presented in Table 3, expect for the $FNS^1(D_i, P)(i = 1,2,3,4,5)$, we know that the similarity measures between D_4 and P is the largest one as derived by five similarity measures. That is, the four similarity measures assign the unknown class P to the known class D_4 according to the principle of the maximum degree of similarity between FNSs. Compared with Garg's correlation coefficients method [10] we can get same result that the four similarity measures assign the unknown class P to the known class D_4 according to the principle of the maximum degree of similarity between FNSs expect for the $FNS^1(D_i, P)(i = 1,2,3,4,5)$.

If we consider the weight of $s_i(i = 1,2,3,4,5)$ is $0.15, 0.20, 0.25, 0.16, 0.13$ respectively. Then we apply the proposed weighted similarities measures, which have been computed from P to $D_i(i = 1,2,3,4,5)$ and are given in Table 4.

Table 4: The Weighted Similarity Measures between $D_i(i = 1, 2, 3, 4, 5)$ and P

Similarity Measures	(D_1, P)	(D_2, P)	(D_3, P)	(D_4, P)	(D_5, P)
$WFNS^1(D_i, P)$	0.5608	0.7994	0.8251	0.8235	**0.8517**
$WFNCS^1(D_i, P)$	0.6889	0.7881	0.8020	**0.8204**	0.7699
$WFNCS^2(D_i, P)$	0.7083	0.8280	0.8201	**0.8382**	0.8206
$WFNCT^1(D_i, P)$	0.4830	0.5465	0.5780	**0.6245**	0.5304
$WFNCT^2(D_i, P)$	0.6149	0.6986	0.6892	**0.7327**	0.6865

From the numerical results presented in table 4, we get the following results:

- For similarity measures$WFNS^1(D_i, P)(i = 1,2,3,4,5)$, the degree of similarity between D_5 and P is the largest one, so the pattern P should belong to the class of known diagnoses D_5 according to the principle of the maximum degree of similarity between FNSs.

- For similarity measures $WFNCS^1(D_i, P), WFNCS^2(D_i, P), WFNCT^1(D_i, P),\ WFNCT^2(D_i, P), i = 1,2,3,4,5,$ the degree of similarity between D_4 and P is the largest one, so the pattern P should belong to the class of known diagnoses D_4 according to the principle of the maximum degree of similarity between FNSs. At the same time, for this case compared with Garg's correlation coeeficients method [10], we can get the same result that the pattern P should belong to the class of the known diagnoses D_4 according to the principle of the maximum degree of similarity between FNSs.

V. CONCLUSION

In this paper, we presented another form of five similarity measures between FNSs based on the cosine function between FNSs by considering the degree of membership, degree of non-membership and the degree of hesitation in FNSs. Then, we applied these similarity measures and weighted similarity measures between FNSs to pattern recognition and medical diagnosis. Finally, two illustrative examples are given to demonstrate the efficiency of the similarity measures for pattern recognition and medical diagnosis.

REFERENCES

[1] I.Antonov,On a new geometrical interpretation of the intuitionistic fuzzy sets, Notes on Intuitionistic Fuzzy Sets 1(1) 564 (1995), 29–31.
[2] C. Antony Crispin Sweety, R. Jansi, Fermatean Neutrosophic Sets, International Journal of Advanced Research in Computer and Communication Engineering Vol.10, Issue 6, June 2021.
[3] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy sets and Systems 20(1) (1986), 87-96.
[4] K. T. Atanassov, More on Intuitionistic fuzzy sets, Fuzzy sets and Systems. 1989; 33:37- 46.
[5] Bhattacharya A. On a measure of divergence of two multinomial populations. Sankhya. 1946; 7:401-406.
[6] Bustince H, Barrenechea E, Pagola M. Restricted equilance functions, Fuzzy Sets Syst. 2006; 157: 2333-2346.

[7] Bustince H, Barrenechea E, Pagola M. Image thresholding using restricted equilance functions and maximizing the measures of similarity. Fuzzy Sets Syst. 2007; 158:496-516.
[8] Bustince H, Barrenechea E, Pagola M. Relationship between restricted dissimilarity functions, restricted equivalence functions and normal E-N functions: image thresholding invariant. Pattern Recognit Lett. 2008; 29: 525-536.
[9] Fatma Kutlu Gundogu and Cengiz Kahraman, Spherical Fuzzy Sets and Spherical Fuzzy Topsis method, Journal of Intelligent & Fuzzy Systems.
[10] Garg H. A novel correlation coefficients between Pythagorean fuzzy sets and its applications to decision-making processed. Int J Intell Syst. 2016; 31(12): 1234-1252.
[11] Guiwu Wei and Yu Wei, Similarity measures of Pythagorean fuzzy sets based on the cosine function and their applications, Int J Intell Syst. 2018; 33: 634-652.
[12] Hung WL, Yang MS. Similarity measures of intuitionistic fuzzy sets based on Hausdorff distance. Pattern Recognit Lett. 2004; 25: 1603-1611.
[13] Hung WL, Yang MS. Similarity measures of intuitionistic fuzzy sets based on Lp metric. Int J Approxiamate Reasoning. 2007; 46:120-136.
[14] Hung KC. Applications of medical information: using an enhanced likelihood measured approach based on intuitionistic fuzzy sets. IIE Trans Healthcare Syst Eng. 2012; 2(3):224-231.
[15] Lee SH, Pedrycz W, Sohn G. Design of similarity and dissimilarity measures for fuzzy sets on the basis of diatance measure. INt J Fuzzy Syst. 2009; 11(2): 67-72.
[16] Li D, Cheng C. New similarity measures of intuitionistic fuzzy sets and application to pattern recognition. Pattern Recognit Lett. 2002; 23: 221-225.
[17] Li YH, Olson DL, Zheng Q. Similarity measures between intuitionistic fuzzy (vague) sets:a comparative analysis. Pattern Recognit Lett. 2007; 28: 278-285.
[18] Liu HW. New similarity measures between intuitionistic fuzzy sets and between elements. Math Comput Modell. 2005;42: 61-70.
[19] Mitchell HB. On the Dengfeng- Chuntian similarity measures and its application to pattern recognition. Pattern Recognit Lett. 2003; 24: 3103-3104.
[20] Rajarajeswari P, Uma N. Intuitionistic fuzzy multi similarity measure based on cotangent function . Int J Eng Res Technol. 2013; 2(11):1323-1329.
[21] Salton G, McGill MJ. Introduction to Modern Information Retrieval. New York, NY: McGraw-Hill; 1983.
[22] Shi LL, Ye J. Study on fault diagnosis on turbine using an improved cosine similarity measure for vague sets. J Appl Sci. 2013; 13(10): 1781-1786.
[23] Son LH, Phong PH. On the performance evaluation of intuitionistic vector similarity measures for medical diagnosis. J Intell Fuzzy Syst. 2016; 31(3): 15 97-1608.
[24] Szmidt E, Kacprzyk J. Distances between intuitionistic fuzzy sets. Fuzzy Sets Syst. 2000; 114:505-518.
[25] Szmidt E, Kacprzyk J. A concept of similarity measure for intuitionistic fuzzy sets and its use in group decision-making. Lect Notes Comp Sci(Subseries LNAI) 2005; 3558:272-282.
[26] Szmidt E. Distances and similarities in intuitionistic fuzzy sets. Studies in Fuzziness and Soft Computing. Berlin, Germany: Springer; 2014.
[27] Tian MY. A new fuzzy similarity based on cotangent function for medical diagnosis. Adv Model Optim. 2013;15(2):151-156.
[28] Xu ZS, Xia MM. Some new similarity measures for intuitionistic fuzzy values and their application in group decision making. J Syst Sci Eng. 2010; 19: 430-452.
[29] J. Ye, "Cosine similarity measures for intuitionistic fuzzy sets and their applications", Mathematical and Computer Modelling, Vol. 53, No. 1, pp. 91-97, 2011.
[30] J. Ye. Similarity measures for intuitionistic fuzzy sets based on cosine function for the decision making of mechanical design schemes. J Intell Fuzzy Syst. 2016; 30(1): 1597-
1608.
[31] Zadeh LA. Fuzzy sets. Inform Control. 1965; 8:338-356.