SOME NEW OSCILLATION CRITERIA FOR ALPHA-FRACTIONAL DIFFERENTIAL EQUATIONS

Abstract

In this paper, we investigate some new oscillation criteria for alpha-fractional differential equation. We obtained for the oscillation of solutions by using the uniform Lipschitz condition and convex function and also using the generalized Riccati transformation and Philo's type. Examples are provided to illustrate our theoretical results.

Keywords: Oscillation, Alpha-fractional, nonlinear

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I. INTRODUCTION

Modelling real-world phenomena is thought to benefit greatly from the application of fractional differential equation theory. In 17th century, the notion of fractional differential derivative were first appeared. It is well known that fractional differential equations are a more general form of the integer order differential equations, extending those equations to an arbitrary (non-integer) order. The definitions involve integration most frequently which is nonlocal: Riemann-Liouville derivative & Caputo derivative [1,2]. Those fractional derivatives are complicated and lack some basic properties, like the product rule and chain rule. Khalil [3], introduced a new fractional derivative called the conformable derivative or α-fractional derivative which closely resembles the classical derivative [4-6]. They are widely employed in biological, physics, electrochemistry, control theory, ecology, and viscoelasticity as well as the electromagnetic field.
[7-10].

In the last few decades, there has been a lot of interest in deriving sufficient conditions for the oscillation and non-oscillation of solutions of classes of differential equations [11-16]. Oscillatory solution plays an active role in the quantitative and qualitative theory of α-fractional differential equations. Motivated by Nehari [17], we propose the following model of the form

$$T_\alpha\big(r(\xi)T_\alpha u(\xi)\big) + p(\xi)T_\alpha u(\xi) + u(\xi)F(t,\xi) = 0,\ t \in [0,\infty),\ |xi \qquad (1)$$

here $\alpha \in (0,1]$ is a constant.

The following assumption:

(A$_1$) $r(\xi), p(\xi) \in \mathbb{C}(R_+, R_+), r(\xi), p(\xi) > 0\ and\ \int_0^\infty \frac{1}{r(s)} d_\alpha s = \infty,$

The function $F(t,\xi)$ is defined for $t \in [0,\infty), \xi \in (0,\infty)$,
(A$_2$) $F(t,\xi) \geq 0$;

(A$_3$) $F(t,\xi)$ is continuous in ξ for fixed t;

(A$_4$) In a neighborhood of every ξ in $(0,\infty), F(t,\xi)$satisfies a uniform Lipschitz condition,

(A$_5$) $F(t,\xi) \leq \delta\varphi(\xi)$, δ is constant and φ is continuous function.

A solution of (1) is oscillatory if it has arbitrarily large zeros, and otherwise it is non-oscillatory. Equation (1) said to be oscillatory if all their solutions are oscillatory.
This paper is organized as follows: We present the relevant definition and lemmas in Section II. In Section III we discuss the main results and finally we present some examples to illustrate our theoretical results.

II. PRELIMINARIES

In this section, we give some basic definitions, integrals and lemmas which are useful throughout the paper

A. Definition: 1

Given $f:[0,\infty) \to R$. Then, the conformable fractional derivative of f of order α is defined by
$T_\alpha(f)(t) = \lim_{\epsilon\to 0} \frac{f(t+\epsilon t^{1-\alpha})-f(t)}{\epsilon}$.
For every $t > 0$, $\alpha \in (0,1]$. If f is α-differentiable in some $(0,a), a > 0$ and $\lim_{t\to 0^+} f^\alpha(t)$exists, then we define
$f^\alpha(0) = \lim_{t\to 0^+} f^\alpha(t)$.

Definition: 2

$$I_a^\alpha(f)(t) = I_1^\alpha(t^{\alpha-1})(f) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx,$$

where the integral is the usual Riemann improper integral and $\alpha \in (0,1]$.

B. Theorem: 1

Let $\alpha \in (0,1]$ and f, g be α-differentiable at some point $t > 0$. Then,

1. $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g), \quad \forall a, b \in R,$
2. $T_\alpha(t^p)= pt^{p-\alpha}, \quad \forall p \in R,$
3. $T_\alpha(c)$ =0, c is constant,
4. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f),$
5. $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f)-fT_\alpha(g)}{g^2},$
6. If f is differentiable, then $T_\alpha(f(t)) = t^{1-\alpha}\frac{df(t)}{dt}$.

C. Lemma: 1

To prove that $T_\alpha\{\xi\{(rT_\alpha u)^2 + J(u^2,\xi)] - u(rT_\alpha u)\} \le \Phi(u^2,\xi)$.
Proof.

$$T_\alpha\{\xi\{(rT_\alpha u)^2 + J(u^2,\xi)] - u(rT_\alpha u)\} = (rT_\alpha u)^2 + J(u^2,\xi) + 2\xi(rT_\alpha u)T_\alpha(rT_\alpha u) + 2\xi uT_\alpha uT_\alpha J(u^2,\xi) + \xi\frac{\partial^\alpha}{\partial\xi^\alpha}J(u^2,\xi) - uT_\alpha(rT_\alpha u) - T_\alpha u(rT_\alpha u)$$

$$\le \Phi(u^2,\xi) - uF(u^2,\xi)\big(u - 2\xi(rT_\alpha u)\big) - T_\alpha(rT_\alpha u)\big(u - 2\xi(rT_\alpha u)\big) \le \Phi(u^2,\xi).$$

Hence the proof.

Lemma: 2

To prove that $T_\alpha\big(\xi J(\mu\xi^{1+\epsilon},\xi)\big) = \Phi(\mu\xi^{1+\epsilon},\xi) + \mu\epsilon\xi^{1+\epsilon}F(\mu\xi^{1+\epsilon},\xi)$.
Proof.

$$T_\alpha\big(\xi J(\mu\xi^{1+\epsilon},\xi)\big) = J(\mu\xi^{1+\epsilon},\xi) + \mu(1+\epsilon)\xi^{1+\epsilon}T_\alpha J(\mu\xi^{1+\epsilon},\xi) + \xi\frac{\partial^\alpha}{\partial v^\alpha}J(\mu\xi^{1+\epsilon},\xi)$$

$$= \Phi(\mu\xi^{1+\epsilon},\xi) - \mu\xi^{1+\epsilon}F(\mu\xi^{1+\epsilon},\xi) + \mu(1+\epsilon)\xi^{1+\epsilon}\, F(\mu\xi^{1+\epsilon},\xi)$$

$$= \Phi(\mu\xi^{1+\epsilon},\xi) + \mu\epsilon\xi^{1+\epsilon}F(\mu\xi^{1+\epsilon},\xi).$$

Hence the proof.

III. MAIN RESULTS

Oscillation with uniform Lipschitz condition

In the following theorem, we establish some new oscillation with uniform Lipschitz condition and convex function.

A. Theorem: 1

Let $F(t,\xi)$ be defined for $t \in [0,\infty),\ \xi \in (0,\infty)$ and satisfy the following conditions

a) $F(t,\xi) \geq 0$,

b) $F(t,\xi)$ is continuuous in $\xi$for fixed t,

c) $F(t,\xi)$ is neighborhood in every ξ in $(0,\infty), F(t,\xi)$ satify a uniform Lipschitz condition,

d) For fixed ξ, $F(t,\xi)$ is a nondecreasing function of t. If $J(t,\xi)$ is defined by

$$J(t,\xi) = I_\alpha F(s,\xi) \qquad (2)$$

And if, for some positive ϵ and all positive μ, $\xi J(\mu\xi^{1+\epsilon},\xi)$ is non-increasing for $\xi \in (a,\infty), a > 0$, then equation (2) is non-oscillatory. This condition is the best possible in the sense that the conclusion does not hold for $\epsilon = 0$.

Proof.

If $a_1$and $a_2$are two consecutive zeros of a solution of (1).

$$I_\alpha^{a_1} r(s)\big(T_\alpha u(s)\big)^2 = I_\alpha^{a_1} p(s)u(s)T_\alpha u(s) + I_\alpha^{a_1} u^2(s)F(u^2(s),s). \qquad (3)$$

Since, $\xi \in (a_1,a_2)$,

$$\big(u(\xi)\big)^2 = \left(I_\alpha^{a_1}T_\alpha u(s)\right)^2 \leq \frac{(\xi-a_1)^\alpha}{\alpha r}\int_{a_1}^{a_2}\left(r\big(T_\alpha u(s)\big)\right)^2 d_\alpha s,$$

using (4), we get

$$\alpha r \leq I_{\alpha}^{a_1} \xi^{\alpha} F(u^2(s), s). \tag{4}$$

F(t, ξ) is (ξ is fixed) a increasing function of t. By (2), $J(t, \xi)$ is convex in t, and we have

$$\frac{J(u^2, \xi) - J(\mu\xi, \xi)}{u^2 - \mu\xi} \leq T_{\alpha} J(u^2, \xi),$$

$$J(\mu\xi, \xi) \geq (\mu\xi - u^2) F(u^2, \xi) + J(u^2, \xi).$$

Hence,

$$\mu\xi^{\alpha} F(u^2, \xi) \leq \xi^{\alpha-1}\big(J(\mu\xi, \xi) - u^2 F(u^2, \xi)\big),$$

where μ is arbitrary positive number and taking I_{α} from $a_1$to a_2, using (4) we get

$$\mu\alpha r \leq I_{\alpha}^{a_1} \xi^{\alpha-1}\big(J(\mu\xi, \xi) - u^2 F(u^2, \xi)\big). \tag{5}$$

Our assumption imply that $J(t, \xi)$ is decreasing in ξ for fixed t. As a result, the partial derivative $\frac{\partial^{\alpha} J(t,\xi)}{\partial \xi^{\alpha}}$ exists for almost all ξ. We introduce the function

$$\Phi(t, \xi) = J(t, \xi) + tF(t, \xi) + \xi \frac{\partial^{\alpha} J(t,\xi)}{\partial \xi^{\alpha}} \tag{6}$$

and we use the Lemma 1 and Lemma 2, we get

$$T_{\alpha}\{\xi\{(rT_{\alpha}u)^2 + J(u^2, \xi)] - u(rT_{\alpha}u)\} \leq \Phi(u^2, \xi), \tag{7}$$

$$T_{\alpha}\big(\xi J(\mu\xi^{1+\epsilon}, \xi)\big) = \Phi(\mu\xi^{1+\epsilon}, \xi) + \mu\epsilon\xi^{1+\epsilon} F(\mu\xi^{1+\epsilon}, \xi). \tag{8}$$

Since, $\xi J(\mu\xi^{1+\epsilon}, \xi)$ is decreasing for $\xi > a$, it follows from (8)

$$\Phi(\mu\xi^{1+\epsilon}, \xi) + \mu\epsilon\xi^{1+\epsilon} F(\mu\xi^{1+\epsilon}, \xi) \leq 0$$

For μ is positive and almost all $\xi > a$. So, let $\mu\xi^{1+\epsilon} = u^2$, we get

$$\Phi(u^2, \xi) + \epsilon u^2 F(u^2, \xi) \leq 0. \tag{9}$$

We using (7) in (9),

$$T_{\alpha}\{\xi\{(rT_{\alpha}u)^2 + J(u^2, \xi)] - u(rT_{\alpha}u)\} + \epsilon u^2 F(u^2, \xi) \leq 0. \tag{10}$$

Multiply by $\xi^{\alpha-1}$ and taking $I_{\alpha}^{a_1}$ between two consecutive zeros a_1, a_2 of u, and by (2), $J(0, \xi) = 0$, we have

$$a_2(r(a_2)T_{\alpha}u(a_2))^2 - a_1\big(r(a_1)T_{\alpha}u(a_1)\big)^2 + \epsilon I_{\alpha}^{a_1} \xi^{\alpha-1}\, u^2 F(u^2, \xi) \leq 0.$$

From (6), we get

$$\epsilon\mu\alpha r + a_2(r(a_2)T_\alpha u(a_2))^2 - a_1\big(r(a_1)T_\alpha u(a_1)\big)^2 \le \epsilon I_\alpha^{a_1}\xi^{\alpha-1}J(\mu\xi,\xi). \quad (11)$$

Here $A = \epsilon\mu\alpha r$, if $a, a_1, \dots, a_m$ are consecutive zeros of u and using (11),

$$mA \le a\big(r(a)T_\alpha u(a)\big)^2 - a_m\big(r(m)T_\alpha u(a_m)\big)^2 + \epsilon I_\alpha^a \xi^{\alpha-1}J(\mu\xi,\xi)$$

$$m \le \frac{a\big(r(a)T_\alpha u(a)\big)^2 + \epsilon I_\alpha^a\xi^{\alpha-1}J(\mu\xi,\xi)}{A} \quad (12)$$

The convexity of $J(t,\xi)$, we obtain that

$$J(\lambda\xi^{1+\epsilon},\xi) \ge J(\mu\xi,\xi) + (\lambda\xi^\epsilon - \mu)\xi F(\mu\xi,\xi) \quad (13)$$

Here λ is any positive number, If we assume $\lambda = \mu a^{-\epsilon}$, then $\lambda a^\epsilon - \mu \ge 0$. Since $F(t,\xi)$ is increasing function of t, (2) become that $J(t,\xi) \le \frac{t^\alpha}{\alpha}F(t,\xi)$. So,

$(\lambda\xi^\epsilon - \mu)\xi F(\mu\xi,\xi) \ge (\lambda\xi^\epsilon - \mu)\mu^{-1}J(\mu\xi,\xi)$,

(13) become,

$$a^\epsilon J(\mu a^{-\epsilon}\xi^{1+\epsilon},\xi) \ge \xi^\epsilon J(\mu\xi,\xi). \quad (14)$$

By assumption, $\xi J(\mu a^{-\epsilon}\xi^{1+\epsilon},\xi)$ is decreasing for $\xi > a$.Hence,
$\xi J(\mu a^{-\epsilon}\xi^{1+\epsilon},\xi) \le a\, J(\mu a, a)$, and (14) leads to
$J(\mu\xi,\xi) \le \left(\frac{a}{\xi}\right)^{1+\epsilon} J(\mu a, a)$.

Thus,

$I_\alpha J(\mu\xi,\xi) \le \int_a^\infty J(\mu\xi,\xi)d_\alpha\xi \le \frac{a^{2\alpha-1}}{\epsilon+2(1-\alpha)} J(\mu a, a)$,

and (12) yields,

$$m \le \frac{a}{A}\left(\big(r(a)T_\alpha u(a)\big)^2 + J(\mu a, a)\right)$$

for the number of zeros which a solution of (1), which vanishes at $\xi = a$, can have in any interval (a,b). Hence, all solutions are non-oscillatory, the theorem is proved.

Oscillation with Riccati transformation

In the following theorem, we are using the Riccati techniques and Philo’s type to demonstrate the new oscillation.

Theorem: 2

If there exists a function $g \in \mathbb{C}^{\alpha}([0,\infty),\mathbb{R})$ such that

$$\limsup_{\xi\to\infty} I_{\alpha}^{\xi_1}\left(\psi(s)-\frac{p^2(s)\phi(s)}{4r(s)}\right)=\infty, \tag{15}$$

here $\phi(s)=\exp(-2I_{\alpha}^{\xi}g(s), \psi(s)= \phi(s)(r(s)g^2(s)-p(s)g(s)+\delta\varphi(s)-T_{\alpha}(r(s)g(s))$.
Then every solution u of (1) is oscillatory.

Proof.
Suppose that u is a non-oscillatory solution of (1). We define the Riccati transformation,

$$W(\xi)=\phi(\xi)\left(\frac{r(\xi)T_{\alpha}u(\xi)}{u(\xi)}+r(\xi)g(\xi)\right),$$

$$T_{\alpha}W(\xi)\leq -\psi(\xi)-\frac{W^2(\xi)}{r(\xi)\phi(\xi)}-\frac{p(\xi)W(\xi)}{r(\xi)}, \tag{16}$$

Taking $I_{\alpha}^{\xi_1}$ on both side

$$I_{\alpha}^{\xi_1}\, T_{\alpha}W(\xi)\leq -I_{\alpha}^{\xi_1}\,\psi(\xi)-I_{\alpha}^{\xi_1}\left(\frac{W^2(\xi)}{r(\xi)\phi(\xi)}+\frac{p(\xi)W(\xi)}{r(\xi)}\right)$$

$$W(\xi_1)\geq W(\xi)+I_{\alpha}^{\xi_1}\left(\psi(\xi)-\frac{p^2(s)\phi(s)}{4r(s)}\right).$$

Taking $\limsup\limits_{\xi\to\infty}$, we get

$$\limsup_{\xi\to\infty} I_{\alpha}^{\xi_1}\left(\psi(s)-\frac{p^2(s)\phi(s)}{4r(s)}\right)\leq W(\xi_1)<\infty,$$

which leads to contradictions (15).

In the sequel, we say that a function $H(\xi,s)\in\mathbb{C}(\mathbb{D},\mathbb{R}_+)$ satisfying $H(\xi,\xi)=0, H(\xi,s)>0$ for $\xi>s\geq 0$, where $\mathbb{D}=\{(\xi,s):\xi>s\geq 0\}$, $h_1,h_2\in\mathbb{C}(\mathbb{D},\mathbb{R}_+)$, $k(s)\in\mathbb{C}^{\alpha}(\mathbb{R}_+,\mathbb{R}_+)$ Furthermore, H has continuous derivatives

$$\frac{\partial^{\alpha}\big(H(\xi,s)k(s)\big)}{\partial\xi^{\alpha}}=h_1(\xi,s)\sqrt{H(\xi,s)k(s)}\ ,\quad \frac{\partial^{\alpha}\big(H(\xi,s)k(s)\big)}{\partial s^{\alpha}}=-h_2(\xi,s)\sqrt{H(\xi,s)k(s)}\ ,\ (\xi,s)\in\mathbb{D},$$

Theorem: 3

Suppose that there exists a function $g\in\mathbb{C}^{\alpha}([0,\infty),\mathbb{R})$ such that

$$\limsup_{\xi\to\infty}\frac{1}{H(\xi,\xi_0)k(\xi_0)}\,I_{\alpha}^{\xi_1}\,H(\xi,s)k(s)\left(\psi(s)-\frac{1}{4}\left(\frac{p(s)}{r(s)}+\frac{h_2(\xi,s)s^{1-\alpha}}{\sqrt{H(\xi,s)k(s)}}\right)^2 r(s)\phi(s)\right)=\infty. \tag{17}$$

Then every solution u of (1) is oscillatory.

Proof.

Suppose that u is a non-oscillatory solution of (1). Multiply by $H(\xi,s)k(s)$and taking $I_{\alpha}^{\xi_1}$ in (16), we get

$$I_{\alpha}^{\xi_1}H(\xi,s)k(s)T_{\alpha}W(\xi)\leq -I_{\alpha}^{\xi_1}H(\xi,s)k(s)\psi(\xi)-I_{\alpha}^{\xi_1}H(\xi,s)k(s)\left(\frac{W^2(\xi)}{r(\xi)\phi(\xi)}+\frac{p(\xi)W(\xi)}{r(\xi)}\right)$$

$$-H(\xi,\xi_0)k(\xi_0)W(\xi_0)\leq -I_{\alpha}^{\xi_1}H(\xi,s)k(s)\psi(\xi)-I_{\alpha}^{\xi_1}H(\xi,s)k(s)\left(\frac{W^2(\xi)}{r(\xi)\phi(\xi)}+\left(\frac{p(\xi)}{r(\xi)}+\frac{h_2(\xi,s)s^{1-\alpha}}{\sqrt{H(\xi,s)k(s)}}\right)W(\xi)\right)$$

$$-H(\xi,\xi_0)k(\xi_0)W(\xi_0)\leq -I_{\alpha}^{\xi_1}H(\xi,s)k(s)\left(\psi(\xi)+\frac{1}{4}\left(\frac{p(s)}{r(s)}+\frac{h_2(\xi,s)s^{1-\alpha}}{\sqrt{H(\xi,s)k(s)}}\right)^2 r(s)\phi(s)\right)$$

$$\limsup_{\xi\to\infty}\frac{1}{H(\xi,\xi_0)k(\xi_0)}I_{\alpha}^{\xi_1}H(\xi,s)k(s)\left(\psi(s)-\frac{1}{4}\left(\frac{p(s)}{r(s)}+\frac{h_2(\xi,s)s^{1-\alpha}}{\sqrt{H(\xi,s)k(s)}}\right)^2 r(s)\phi(s)\right)\leq W(\xi_0)<\infty,$$

which leads to contradictions. Hence the proof.

B. Corollary: 1

Assume that the conditions of Theorem 3 hold with (17) replaced by

$$\limsup_{\xi\to\infty}\frac{1}{H(\xi,\xi_0)k(\xi_0)}I_{\alpha}^{\xi_1}H(\text{\xi},s)k(s)\psi(s)=\infty,\text{ and}$$

$$\limsup_{\xi\to\infty}\frac{1}{H(\xi,\xi_0)k(\xi_0)}I_{\alpha}^{\xi_1}H(\xi,s)k(s)\left(\frac{p(s)}{r(s)}+\frac{h_2(\xi,s)s^{1-\alpha}}{\sqrt{H(\xi,s)k(s)}}\right)^2 r(s)\phi(s)<\infty.$$

Then every solution u of (1) is oscillatory.

IV. EXAMPLES

A. Example: 1

Consider the conformable differential equation,

$$T_{1/2}\left(T_{1/2}u(\xi)\right)+\frac{1}{2\xi^{1/2}}T_{1/2}u(\xi)+u^2(\xi)\frac{(\xi sin\xi-cos\xi)}{sin^2\xi}=0, \tag{18}$$

here $\alpha=1/2$, $r(s)=1, p(s)=\frac{1}{2s^{1/2}}$, $\delta=1, \varphi(s)=s\sin s-\cos s$, $g(s)=-1/2s$, $\phi(s)=s$ and

$\psi(s)=s(-1/2s+s\ \ sins-\cos s)$.

$$\limsup_{\xi\to\infty}I_{\alpha}^{\xi_1}\left(s(-1/2s+s\ \ sins-\cos s)-\frac{s^{7/4}}{16}\right)=\infty.$$

Hence, all the conditions of Theorem 2 are satisfied. Therefore, every solution of (18) is oscillatory. In fact, $u(\xi) = \sin\xi$ is one such solution of (18).

Example:2
Consider the conformable differential equation,

$$T_{1/2}\left(\xi^{1/2} T_{1/2} u(\xi)\right) - T_{1/2} u(\xi) - u^2(\xi)\xi^{3/2} e^{-\xi} = 0, \quad (19)$$

here $\alpha = 1/2, r(s) = \xi^{1/2}, p(s) = -1, \delta = 1, \varphi(s) = s^{3/2}, g(s) = 1/2s, \phi(s) = 1/s^2$ and $\psi(s) = 1/s^2 \left(-1/4s^{3/2} + 3/4s + s^{3/2}\right)$.

$$\limsup_{\xi\to\infty} I_\alpha^{\xi_1}\left(\psi(s) - \frac{p^2(s)\phi(s)}{4r(s)}\right) < \limsup_{\xi\to\infty} \int_{\xi_1}^{\xi} 1/s^2 \, d_\alpha s < \infty,$$

Hence, all the conditions of Theorem 2 are not satisfied. In fact, $u(\xi) = e^{\xi}$ is a non-oscillatory solution of (19).

V. CONCLUSION

In this article, we have identified some new oscillation or non-oscillation criteria for alpha-fractional differential equation. Since the obtained results are general forms of earlier works they would help for the investigate in future studies. Examples are provided that to illustrate our theoretical results.

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