

# GENERALIZED FIBONACCI NUMBERS AND DIFFERENCE OPERATORS

## Author

**Dr. Sanjay Harne**

Assistant Professor,  
Mata Jijabai Government Girls PG College  
Indore(India)

## I. INTRODUCTION

This chapter deals with the Fibonacci and Lucas sequences [5,6]. To consider all the sequences [7,8] of this type under one heading we are introducing generalized Fibonacci sequence  $\{G_n\}$ , which is defined by [3] —

$$\begin{aligned} G_n &= G_{n-1} + G_{n-2}, \quad n \geq 3 \\ G_1 &= a, \quad G_2 = b \end{aligned} \tag{1.1.1}$$

where,  $a$  and  $b$  are constants.

But this generalization does not include Pell sequence, associated Pell sequence, and other sequences which can be similarly defined. To discuss all such sequences under one heading we consider the sequence  $\{S_n\}$ , defined by —

$$S_{n+1} = k S_n + S_{n-1} \tag{1.1.2}$$

where,  $S_0$  and  $S_1$  are arbitrary constants and  $k$  is any number.

We shall also introduce the two companion sequences  $\{H_n\}$  and  $\{V_n\}$  defined by [4] —

$$H_{n+1} = k H_n + H_{n-1} \quad (n \geq 1), \quad H_0 = 0, \quad H_1 = 1 \tag{1.1.3}$$

$$V_{n+1} = k V_n + V_{n-1} \quad (n \geq 1), \quad V_0 = 2, \quad V_1 = k \tag{1.1.4}$$

Thus the sequence  $\{H_n\}$  and  $\{V_n\}$  are particular cases of the sequences  $\{S_n\}$ . The sequence  $\{H_n\}$  reduces to Fibonacci or Pell sequence according as  $k = 1$  or  $2$ . Similarly the sequence  $\{V_n\}$  reduce to Lucas or associated Pell sequence according as  $k = 1$  or  $2$ .

## II. DIFFERENCE OPERATORS E AND $\Delta$

In this section we shall derive some new identities for the generalized Fibonacci numbers  $G_n$ , with the help of finite difference operators  $E$  and  $\Delta$ . These operators are defined by [1] —

$$E G_n = G_{n+1} \tag{1.2.1}$$

$$\Delta G_n = G_{n+1} - G_n, \text{ and} \quad (1.2.2)$$

$$\Delta \equiv E - 1 \quad (1.2.3)$$

Also, it is known that,

$$\Delta (A_n B_m) = (E A_n) (\Delta B_m) + B_m (\Delta A_n) \quad (1.2.4)$$

Since,

$$G_{n+1} - G_n = G_{n-1}$$

Therefore, we obtain

$$\Delta G_n = G_{n-1}$$

and in general,

$$\Delta^m G_n = G_{n-m} \quad (1.2.5)$$

Using the operators we obtain the following identities :

$$G_{n-m} = \sum_{p=0}^m (-1)^p \binom{m}{p} G_{n+m-p} \quad (1.2.6)$$

$$G_{n+m} = \sum_{p=0}^m \binom{m}{p} G_{n-p} \quad (1.2.7)$$

$$\sum_{p=0}^m \binom{m}{p} G_{n+p} = \sum_{p=0}^m \binom{m}{p} 2^{m-p} G_{n-p} \quad (1.2.8)$$

$$G_{n+1}^2 - G_n^2 = G_{n+1} G_{n-1} + G_n G_{n-1} \quad (1.2.9)$$

### Proof

1. From equations (1.2.5), (1.2.3) and (1.2.1), we obtain

$$\begin{aligned} G_{n-m} &= \Delta^m G_n \\ &= (E - 1)^m G_n \\ &= \sum_{p=0}^m (-1)^p G_{n+m-p} \end{aligned}$$

which is the result of eqn. (1.2.6).

2. We have

$$\begin{aligned} G_{n+m} &= E^m G_n \\ &= (1 + \Delta)^m G_n \\ &= \sum_{p=0}^m \binom{m}{p} G_{n-p} \end{aligned}$$

which is the result of eqn. (1.2.7).

3. We have,

$$(1 + E)^m G_n = \sum_{p=0}^m \binom{m}{p} G_{n+p}$$

Now,

$$\begin{aligned} \text{L.H.S.} &= (2 + \Delta)^m G_n \\ &= \sum_{p=0}^m \binom{m}{p} 2^{m-p} G_{n-p} \end{aligned}$$

which is the result of eqn. (1.2.8).

4. Setting  $A_n = B_m = G_n$  in eqn. (5.2.4), we get

$$\Delta (G_n G_n) = (E G_n) (\Delta G_n) + G_n (\Delta G_n)$$

which leads to the result of eqn. (1.2.9).

### III. EXPLICIT EXPRESSIONS FOR $S_n$ , $H_n$ and $V_n$

An explicit expression for the numbers  $S_n$ , may be derived by the usual method for solving difference equations. By this method we deduce that [2] –

$$S_n = X \alpha^n + Y \beta^n \tag{1.3.1}$$

where,  $\alpha$  and  $\beta$  are the roots of the equation.

$$t^2 - kt - 1 = 0 \tag{1.3.2}$$

and  $X$  and  $Y$  are suitably chosen constants depending on  $S_0$  and  $S_1$ .

From eqn. (1.3.2) we get,

$$\alpha = \frac{1}{2} [k + (k^2 + 4)^{1/2}]$$

$$\beta = \frac{1}{2} [k - (k^2 + 4)^{1/2}]$$

giving,

$$\left. \begin{aligned} \alpha + \beta &= k, \\ \alpha \beta &= -1, \\ \alpha - \beta &= (k^2 + 4)^{1/2} \end{aligned} \right\} \quad (1.3.3)$$

It follows that –

$$\left. \begin{aligned} X &= \frac{S_1 - S_0 \beta}{\alpha - \beta} \\ Y &= \frac{S_1 - S_0 \alpha}{\alpha - \beta} \end{aligned} \right\} \quad (5.3.4)$$

From eqns. (1.3.1) and (1.3.4), we obtain

$$S_n = \frac{(\alpha^n - \beta^n)S_1 + (\alpha^{n-1} - \beta^{n-1})S_0}{\alpha - \beta} \quad (1.3.5)$$

Taking  $S_0 = 0$  and  $S_1 = 1$ , the sequence  $S_n$  reduces to  $H_n$ , and from eqn. (1.3.5) we obtain,

$$H_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad (1.3.6)$$

Taking,  $S_0 = 2$  and  $S_1 = k$ , the sequence  $\{S_n\}$  is reduces to  $\{V_n\}$  and using eqn. (5.3.3), we obtain,

$$V_n = \alpha^n + \beta^n \quad (1.3.7)$$

From eqns. (1.3.5) and (1.3.6), we have

$$S_n = H_n S_1 + H_{n-1} S_0 \quad (1.3.8)$$

Now we shall derive some identities involving these numbers.

Since  $\alpha$  is a root of the equation (1.3.2), we have

$$\alpha^2 = 1 + k \alpha \tag{1.3.9}$$

therefore,

$$\alpha^{2n} = \sum_{p=0}^n \binom{n}{p} k^p \alpha^p \tag{1.3.10}$$

Similarly, we get

$$\beta^{2n} = \sum_{p=0}^n \binom{n}{p} k^p \alpha^p \tag{1.3.11}$$

Hence, from eqns. (1.3.1), (1.3.10) and (1.3.11), we obtain the identity –

$$S_{2n} = \sum_{p=0}^n \binom{n}{p} k^p S_p \tag{1.3.12}$$

Denoting  $(S_0^2 + k S_0 S_1 - S_1^2)$ , and using eqns. (1.3.1), (1.3.3) $\delta$  by and (5.3.6), the following identities have been obtained:

$$S_n^2 - S_{n-1} S_{n+1} = (-1)^n \delta \tag{1.3.13}$$

$$S_m S_{n+p} - S_{m+p} S_n = (-1)^m \delta H_p H_{n-m} \tag{1.3.14}$$

$$S_{m+n+1} = S_{m+1} H_{n+1} + S_m H_n \tag{1.3.15}$$

$$S_{m+r} H_{n+r} + (-1)^{r+1} S_m H_n = S_{m+n+r} H_r \tag{1.3.16}$$

$$H_n H_{n+m} - H_{n-s} H_{n+m+s} = (-1)^{n-s} H_s H_{s+m} \tag{1.3.17}$$

$$H_n = (-1)^{n-1} H_n \tag{1.3.18}$$

$$S_{n+1}^2 - S_{n-1}^2 = S_2 S_{2n} - S_0 S_{2n-2} \tag{1.3.19}$$

$$H_n V_n = H_{2n} \tag{1.3.20}$$

Similar results can be obtained for other sequences by suitably choosing the constants.

#### IV. GEOMETRICAL PROPERTIES OF $\{S_n\}$ AND $\{H_n\}$

##### Theorem 1

Area of the triangle having vertices at the points with rectangular Cartesian coordinates  $(S_n, S_{n+r})$ ,  $(S_{n+a}, S_{n+a+r})$ ,  $(S_{n+b}, S_{n+b+r})$  is independent of  $n$ .

**Proof**

Twice the area of the specified triangle is equal to the absolute value of the determinant.

$$\Delta = \begin{vmatrix} S_n & S_{n+r} & 1 \\ S_{n+a} & S_{n+a+r} & 1 \\ S_{n+b} & S_{n+b+r} & 1 \end{vmatrix}$$

Using (1.3.15) for the second column the determinant can be written as –

$$H_{r+1} \begin{vmatrix} S_n & S_{n+r} & 1 \\ S_{n+a} & S_{n+a} & 1 \\ S_{n+b} & S_{n+b} & 1 \end{vmatrix} + H_r \begin{vmatrix} S_n & S_{n+r} & 1 \\ S_{n+a} & S_{n+a-1} & 1 \\ S_{n+b} & S_{n+b-1} & 1 \end{vmatrix}$$

The first determinant is obviously zero; in the second on alternately subtracting the second and first columns from each other, the suffixes can be reduced and finally we get –

$$\Delta = \pm H_r \begin{vmatrix} S_1 & S_0 & 1 \\ S_{a+1} & S_a & 1 \\ S_{b+1} & S_b & 1 \end{vmatrix},$$

according as n is odd or even.

On expanding the determinant along the third column we obtain,

$$\Delta = \pm H_r [ (S_{a+1} S_b - S_a S_{b+1}) - (S_1 S_b - S_0 S_{b+1}) + (S_1 S_a - S_0 S_{a+1}),$$

which on using (1.3.14), reduces to –

$$\Delta = \pm H_r [H_b - H_a - (-1)^a H_{b-a}] \delta \tag{1.4.1}$$

Thus the area of the specified triangle is independent of n.

**Particular Case**

Taking  $T_0 = 0, T_1 = 1, k = 1$  and  $e = h, a = 2h, b = 4h$ , we find that the area of the triangle whose vertices are  $(F_n, F_{n+h}), (F_{n+2h}, F_{n+3h}), (F_{n+4h}, F_{n+5h})$  is equal to –

$$\frac{1}{2} F_h (F_{rh} - 2F_{2h}) \tag{1.4.2}$$

Duncan has proved that the area of this triangle is,

$$\frac{1}{2} [F_h(F_{4h} - F_{2h}) - (F_{3h} F_{4h} - F_{2h} F_{5h})] ,$$

which on using (5.3.17) simplifies to the expression given in (1.4.2).

**Theorem 2**

Lines drawn through the origin with direction ratios  $T_n, T_{n+a}, T_{n+b}$ , where,  $a$  and  $b$  are any integers are always co-planar for every value of  $n$ .

**Proof**

Direction ratios of any three such lines are  $S_e, S_{e+a}, S_{e+b}, S_f, S_{f+a}, S_{f+b}, S_g, S_{g+a}, S_{g+b}$ . these will be coplanar if

$$\begin{vmatrix} S_e & S_{e+a} & S_{e+b} \\ S_f & S_{f+a} & S_{f+b} \\ S_g & S_{g+a} & S_{g+b} \end{vmatrix} = 0 \tag{1.4.3}$$

On using the relation (1.3.15) the left-hand side of (1.4.3) can be written as the sum of four determinants each of which is zero. Hence, the theorem is proved.

**Theorem 3**

The points having Cartesian coordinates  $(S_n, S_{n+a}, S_{n+b})$ , where,  $a$  and  $b$  are any integers and  $n = 1,2,3,\dots$ , are always co-planar and the plane through these points passes through the origin, and its equation is independent of  $n$ .

**Proof**

Equation to the plane passing through any three such points is –

$$\begin{vmatrix} x & y & z & 1 \\ S_e & S_{e+a} & S_{e+b} & 1 \\ S_f & S_{f+a} & S_{f+b} & 1 \\ S_g & S_{g+a} & S_{g+b} & 1 \end{vmatrix} = 0 \tag{1.4.4}$$

where,  $e, f$  and  $g$  are particular values of  $n$ .

Here the coefficient of  $x$  is

$$\begin{aligned} &= [(S_{f+a} S_{g+b} - S_{f+b} S_{g+a}) - (S_{e+a} S_{g+b} - S_{e+b} S_{g+a}) \\ &\quad + (S_{e+a} S_{f+b} - S_{e+b} S_{f+a})] \\ &= (-1)^a [H_{b-a}(-1)^f H_{g-f} - (-1)^e H_{g-e} + (-1)^e H_{f-e}] \delta \end{aligned}$$

The coefficient of  $y$  is obtained on putting  $a = 0$  in the coefficient of  $x$ ; the coefficient of  $z$  is obtained from the coefficient of  $y$  on replacing  $b$  by  $a$ ; the constant term is zero as is already proved in (1.4.3).

Thus the equation to the plane simplifies to -

$$(-1)^a H_{b-a} x - H_b y + H_a z = 0 \quad (1.4.5)$$

This equation is independent of n. Also it does not depend on the initial values  $T_0$  and  $T_1$ .

### Particular Case

From (1.4.5) we obtain that the points  $(F_e, F_{e+2}, F_{e+5})$ ,  $e = 1,2,3,\dots$ ;  $(L_f, L_{f+2}, L_{f+5})$ ,  $f = 1,2,3,\dots$ ;  $(G_g, G_{g+2}, G_{g+5})$ ,  $g = 1,2,3,\dots$ ; all lie on the plane  $2x - 5y + z = 0$ .

### Theorem 4

The planes –

$$S_n x + S_{n+a} y + S_{n+b} z + S_{n+r} = 0,$$

where, a, b, r are any integers and  $n = 1,2,3,\dots$ ; all intersect in a given line whose equation is independent of n.

### Proof

Let two such planes be –

$$\begin{aligned} S_e x + S_{e+a} y + S_{e+b} z + S_{e+r} &= 0, \\ S_f x + S_{f+a} y + S_{f+b} z + S_{f+r} &= 0. \end{aligned} \quad (1.4.6)$$

The equations to the line of intersection of the parallel planes through the origin are

$$\frac{x}{S_{e+a} S_{f+b} - S_{e+b} S_{f+a}} = \frac{y}{S_e S_{f+b} - S_{e+b} S_f} = \frac{z}{S_e S_{f+b} - S_{e+b} S_f}$$

On using (5.3.14) and preceding as in (5.4.5) we obtain the equation of the line of intersection of the parallel planes through the origin as –

$$\frac{x}{(-1)^a H_{b-a}} = \frac{-y}{H_b} = \frac{z}{H_a}$$

The line of intersection of the planes given by (5.4.6) meets the plane  $z = 0$ , at the point given by –

$$\frac{x}{(-1)^a H_{r-a}} = \frac{-y}{H_r} = \frac{1}{H_a}$$

Thus the equation to the line of intersection of the planes given by (1.4.6) becomes

$$\frac{(-1)^a H_a x - H_{r-a}}{H_{b-a}} = \frac{H_a y + H_r}{-H_b} = \frac{z}{H_a}, \quad (1.4.7)$$

which is independent of n.



**Particular case**

The planes whose equations are –

$$F_e x + F_{e+1} y + F_{e+3} z + F_{e+4} = 0, e = 1,2,3,\dots,$$

$$L_f x + L_{f+1} y + L_{f+3} z + L_{f+4} = 0, f = 1,2,3,\dots,$$

$$B_g x + B_{g+1} y + B_{g+3} z + B_{g+4} = 0, g = 1,2,3,\dots,$$

all intersect in the line  $\frac{x+2}{1} = \frac{y+3}{2} = \frac{z}{-1}$

**V. A THIRD-ORDER DETERMINANT INVOLVING THE NUMBERS  $S_n$**

From (1.4.3) it follows that

$$\begin{vmatrix} S_a & S_{a+m} & S_{a+m+n} \\ S_b & S_{b+m} & S_{b+m+n} \\ S_r & S_{r+m} & S_{r+m+n} \end{vmatrix} = 0 \tag{1.5.1}$$

for all integers a, b, r, m and n.

We shall now evaluate the determinant,

$$\Delta_1 = \begin{vmatrix} S_a + e & S_{a+m} + e & S_{a+m+n} + e \\ S_b + e & S_{b+m} + e & S_{b+m+n} + e \\ S_r + e & S_{r+m} + e & S_{r+m+n} + e \end{vmatrix}$$

where, e is an arbitrary constant and a, b, r, m and n are any integers.

On writing the determinant as the sum of eight determinants; using the equation (1.5.1) and the property that a determinant vanishes if two columns are identical, we obtain,

$$\begin{aligned} \Delta_1 &= \begin{vmatrix} S_a & S_{a+m} & e \\ S_b & S_{b+m} & e \\ S_r & S_{r+m} & e \end{vmatrix} + \dots + \dots \\ &= e H_m \begin{vmatrix} S_a & S_{a-1} & 1 \\ S_b & S_{b-1} & 1 \\ S_r & S_{r-1} & 1 \end{vmatrix} + \dots + \dots \end{aligned}$$

The first determinant by using (1.3.14) can be written as –

$$= \delta e H_m \left[ (-1)^{r-1} H_{b-r} + (-1)^{a-1} H_{r-a} + (-1)^{b-1} H_{-b+a} \right]$$

Hence,

$$\Delta_1 = \delta g [ (-1)^b H_{r-b} - (-1)^a H_{r-p} + (-1)^a H_{b-a} ] [H_m - H_{m+n} + (-1)^m H_n] \quad (5.5.2)$$

## VI. FOURTH-ORDER DETERMINANTS

We shall now evaluate the determinant,

$$\Delta_2 = \begin{vmatrix} S_{n+3} & S_{n+2} & S_{n+1} & S_n \\ S_{n+2} & S_{n+3} & S_n & S_{n+1} \\ S_{n+1} & S_n & S_{n+3} & S_{n+2} \\ S_n & S_{n+1} & S_{n+2} & S_{n+3} \end{vmatrix},$$

Hence, we obtain,

$$\begin{aligned} \Delta_2 &= [(S_{n+3} + S_{n+2})^2 - (S_{n+1} + S_n)^2] [(S_{n+3} - S_{n+2})^2 - (S_{n+1} - S_n)^2] \\ &= (S_{n+4}^2 - S_{n+2}^2)(S_{n+1}^2 - S_{n-1}^2) \\ &= (S_2 S_{2n+6} - S_0 S_{2n+4}) (S_2 S_{2n} - S_0 S_{2n-2}) \end{aligned} \quad (1.6.1)$$

on using (1.3.19).

## VII. EVALUATION OF A CIRCULANT

We now evaluate the circulant

$$\Delta_3 = \begin{vmatrix} S_n & S_{n+g} & \dots & S_{n+(m-1)g} \\ S_{n+(m-1)g} & S_n & \dots & S_{n+(m-2)g} \\ \dots & \dots & \dots & \dots \\ S_{n+g} & S_{n+2g} & \dots & S_n \end{vmatrix}$$

Let  $w$  be any one of the  $m$  numbers,

$$w_r = \cos \frac{2r\pi}{m} + i \sin \frac{2r\pi}{m}, \quad (r = 1, 2, 3, \dots, m)$$

so that,

$$w^m - 1 = 0$$

$$\text{Therefore, } T_1 = w_1 + w_2 + w_2 + \dots + w_m = 0$$

$$T_2 = w_1 \cdot w_2 + \dots = 0$$

$$\dots \dots \dots = 0$$

$$T_m = w_1 w_2 w_3 w_4 \dots w_m = (-1)^{m+1}$$

Hence,

$$\prod_{r=1}^m (y - w_r z) = y^m - z^m \quad (1.7.1)$$

Therefore, as discussed in -

$$\begin{aligned} \Delta_3 &= \prod_{r=1}^m (S_n + w_r S_{n+k} + \dots + w_r^{m-1} S_{n+(m-1)g}) \\ &= \prod_{r=1}^m \left[ \frac{C\alpha^n (1 - w_r^m \alpha^{mg})}{1 - w_r \alpha^g} + \frac{D\beta^n (1 - w_r^m \beta^{mg})}{1 - w_r \beta^g} \right] \\ &= \prod_{r=1}^m \left[ \frac{(T_n - T_{n+mg}) - (-1)^g w_r (S_{n-k} - S_{n+(m-1)g})}{(1 - w_r \alpha^g)(1 - w_r \beta^g)} \right] \\ &= \frac{(S_n - S_{n+mk})^m - (-1)^{mk} (S_{n-g} - S_{n+(m-1)g})^m}{(1 - \alpha^{mg})(1 - \beta^{mg})} \\ &= \frac{(S_n - S_{n+mk})^m - (-1)^{mk} (S_{n-g} - S_{n+(m-1)g})^m}{1 + (-1)^{mg} - w_{mg}} \quad (1.7.2) \end{aligned}$$

### VIII.A THIRD-ORDER DETERMINANT WITH EACH ELEMENT AS THE PRODUCTS OF TWO NUMBERS

We shall evaluate,

$$\Delta_4 = \begin{vmatrix} H_n \cdot S_{m+n} & H_{n+a} \cdot S_{m+n+a} & H_{n+a+b} \cdot S_{m+n+a+b} \\ H_{n+r} \cdot S_{m+n+r} & H_{n+r+a} \cdot S_{m+n+r+a} & H_{n+r+a+b} \cdot S_{m+n+r+a+b} \\ H_{n+s} \cdot S_{m+n+s} & H_{n+s+a} \cdot S_{m+n+s+a} & H_{n+s+a+b} \cdot S_{m+n+s+a+b} \end{vmatrix}$$

and shall show that  $\Delta_4$ , is independent of n

On using (1.3.16), we can write,

$$H_{n+a} S_{m+n+a} + (-1)^{a+1} H_n S_{m+n} = H_a S_{m+2n+a}$$

Hence, multiplying the first column by  $(-1)^{a+1}$ ,  $(-1)^{a+b+1}$  and adding to the second and third columns respectively, we obtain,

$$\Delta_4 = H_a H_{a+b} \begin{vmatrix} H_n \cdot S_{m+n} & S_{m+2n+a} & S_{m+2n+a+b} \\ H_{n+r} \cdot S_{m+n+r} & S_{m+2n+2r+a} & S_{m+2n+2r+a+b} \\ H_{n+s} \cdot S_{m+n+s} & S_{m+2n+2s+b} & S_{m+2n+2s+a+b} \end{vmatrix}$$

$$\Delta_4 = H_a H_b H_{a+b} \begin{vmatrix} H_n \cdot S_{m+n} & S_{m+2n+a} & S_{m+2n+a-1} \\ H_{n+r} \cdot S_{m+n+r} & S_{m+2n+2r+a} & S_{m+2n+2r+a-1} \\ H_{n+s} \cdot S_{m+n+s} & S_{m+2n+2s+a} & S_{m+2n+2s+a-1} \end{vmatrix}$$

on using (1.3.15),

Now alternately subtracting the third and second columns from one another, we obtain,

$$\begin{aligned} \Delta_4 &= H_a H_b H_{a+b} (-1)^{m+a} \begin{vmatrix} H_n \cdot S_{m+n} & S_0 & S_1 \\ H_{n+r} \cdot S_{m+n+r} & S_{2r} & S_{2r+1} \\ H_{n+s} \cdot S_{m+n+s} & S_{2s} & S_{2s+1} \end{vmatrix} \\ &= H_a H_b H_{a+b} (-1)^{m+a} [H_n S_{m+n} H_{2s-2r} - H_{n+r} S_{m+n+r} H_{2s} + \\ &\quad + H_{n+s} S_{m+n+s} H_{2r}] \delta \end{aligned}$$

on using (1.3.14)

Now with the help of equations (1.3.5), (1.3.6) and (1.3.7) we can write,

$$H_{n+s} S_{m+n+s} H_{2r} = [S_{m+2n+2s+2r} - S_{m+2n+2s-2r} + (-1)^{n+s} (S_{m-2r} - S_{m+2r})] / (\alpha-\beta)^2$$

Hence, we get,

$$\begin{aligned} \Delta_4 &= \frac{(-1)^{m+n+a} H_a H_b H_{a+b}}{w_2 + 2} [(S_{m+2r-2s} - S_{m+2s-2r}) + \\ &\quad + (-1)^s (S_{m-2r} - S_{m+2r}) - (-1)^r (S_{m-2s} - S_{m+2s})] \delta \quad (5.8.1) \end{aligned}$$

which is independent of n.

## IX. A FOURTH-ORDER DETERMINANT WITH ELEMENTS AS PRODUCTS OF TWO NUMBERS

We shall now show that,

$$\Delta_5 = \begin{vmatrix} H_a S_{a+m} & H_{a+o} S_{a+m+o} & H_{a+p} S_{a+m+p} & H_{a+q} S_{a+m+q} \\ H_b S_{b+m} & H_{b+o} S_{b+m+o} & H_{b+p} S_{b+m+p} & H_{b+q} S_{b+m+q} \\ H_r S_{r+m} & H_{r+o} S_{r+m+o} & H_{r+p} S_{r+m+p} & H_{r+q} S_{r+m+q} \\ H_s S_{s+m} & H_{s+o} S_{s+m+o} & H_{s+p} S_{s+m+p} & H_{s+q} S_{s+m+q} \end{vmatrix} = 0 \quad (1.9.1)$$

For all integers p, q, r, s, m, o, p and q.

Multiplying the first column by  $(-1)^{a+1}$ ,  $(-1)^{b+1}$ ,  $(-1)^{c+1}$  and adding to the second, third and fourth columns respectively; using equation (1.3.16) the determinant reduces to

$$H_o H_p H_q = \begin{vmatrix} H_a S_{a+m} & S_{2a+m+o} & S_{2a+m+p} & S_{2a+m+q} \\ H_b S_{b+m} & S_{2b+m+o} & S_{2b+m+p} & S_{2b+m+q} \\ H_r S_{r+m} & S_{2r+m+o} & S_{2r+m+p} & S_{2r+m+q} \\ H_s S_{s+m} & S_{2s+m+o} & S_{2s+m+p} & S_{2s+m+q} \end{vmatrix}$$

Expanding along the first column and using the equation (1.5.1), we find that the determinant vanishes. The result (1.9.1) can be extended for the nth order determinants

## REFERENCES

- [1] Dewey, D.C. "Chains of Equivalent Fibonacciwise Triangles", *The Fibonacci Quarterly*, 5, (1967), 87-88.
- [2] Ferrar, W.L. : *Algebra*, Oxford University, (1963), 23.
- [3] Horadam, A.F. : "A Generalized Fibonacci Sequence", *Amer. Math. Monthly*, 68, (1961), 456-459.
- [4] Thomson Milne, L.M. : "The Calculus of Finite Difference", London, MacMillan (1960).
- [5] Vorobyov, N.N. : "The Fibonacci Numbers", D.C. Health Company, Boston (1963).
- [6] Waddill, M.E. and Sacks, L. : "Another Generalized Fibonacci Sequence", *The Fibonacci Quarterly*, 5, (1967), 209-222.
- [7] Zeitlin, D. : "On Identities for Fibonacci Numbers", *American Mathematical Monthly*, 70, (1963), 987-991.
- [8] Zeitlin, D. : "Power Identities for Sequences Defined by  $w_{n+2} = dw_{n+1} - cw_n$ ", *The Fibonacci Quarterly*, 2, (1965), 241-256.