EXISTENCE OF SOLUTION FOR NONLINEAR IMPLICIT FREDHOLM INTEGRODIFFERENTIAL EQUATION VIA S-ITERATION METHOD

Abstract

In this research chapter, we investigate the existence and uniqueness of the solution to a nonlinear implicit Fredholm integrodifferential equation. To analyze the problem, we utilize the $S-$ iteration method. As the study of qualitative properties typically requires differential and integral inequalities, the $S-$ iteration method proves to be equally important in the analysis of various qualitative properties, such as the continuity dependence and closeness of solutions. We provide an example that supports the established results.

Keywords: Existence, $S-$iteration, Fredholm integrodifferential equation, Continuous dependence, Parameters.

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I. INTRODUCTION

Consider the nonlinear integrodifferential equation of the type:

$$x(t) = \mathcal{G}(t) + \int_a^b \mathcal{F}\begin{pmatrix} t, s, x(s), x^{'(s)}, \cdots, x^{(n-1)}(s), x(a), x^{'(a)}, \cdots, x^{(n-1)}(a), \\ x(b), x^{'(b)}, \cdots, x^{(n-1)}(b) \end{pmatrix} ds, \qquad (1)$$

for $t \in I = [a, b]$. Let $\mathbb{R}$ stand for the set of real numbers, $E = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (n times) be the product space and $\mathbb{R}_+ = [0, \infty)$ be the given subset of $\mathbb{R}$. We assume $\mathcal{F} \in C(I^2 \times E^3, \mathbb{R})$, $\mathcal{G} \in C(I, \mathbb{R})$, and $n \geq 1$.

Many Iterative methods for certain classes of operators have been introduced by several researchers, including their convergence, equivalence of convergence, and rate of convergence, etc. (see [1, 3, 5, 8, 9, 15, 16, 17, 18, 19, 20, 21, 24, 25, 26]). Most of the iterative methods focus on both analytical and numerical approaches. Due to its simplicity and fastness, the $S-$ iteration method has attracted attention and hence, it is used in this chapter.

There is a sufficient amount of literature that deals with the special and even more general version of the equation (1) by using a variety of techniques [2, 6, 10, 11, 12, 13, 14, 22, 23, 27, 28, 29, 30, 31] and some of the references cited therein. Recently, Yunus Atalan, Faik Gürsoy and Abdul Rahim Khan [4] have studied the special version of equation (1) for different qualitative properties of solutions. Authors are inspired by the work of D. R. Sahu [24] and influenced by the work in [4].

The primary aim of this chapter is to utilize the normal $S-$iteration method to establish the existence and uniqueness of the solution for the problem (1). Additionally, we provide a result of the data dependence for the solutions of integrodifferential equation (1) through the normal $S-$iteration method.

II. EXISTENCE OF SOLUTION VIA $S-$ITERATION METHOD

In terms of continuous functions $x^{(j)}: I \to \mathbb{R}$ $(j = 0, 1, \cdots, n-1)$, we denote

$$|x(t)|_E = \sum_{j=0}^{n-1} \left|x^{(j)}(t)\right|,$$

for $\left(x(t), x^{'(t)}, \cdots, x^{(n-1)}(t)\right) \in E,\ t \in I$. We define $B = C^{n-1}(I) = C^{n-1}(I, \mathbb{R})$, is the space of all functions x which are continuously differentiable on I and endowed with the norm

$$\| x \|_B = \max_{t \in I} |x(t)|_E. \qquad (2)$$

It is easy to note that B with the norm defined by (2) forms a Banach space.

By a solution of equation (1), it mean a continuous function $x(t),\ t \in I$ which is $(n-1)$ times continuously differentiable on I and satisfies the equation (1). It is easy to observe that the solution $x(t)$ of the equation (1) and its derivatives satisfy the integral equations (see [7], p.318)

$$x^{(j)}(t) = \mathcal{G}^{(j)}(t) + \int_a^b \frac{\partial^j}{\partial t^j}\mathcal{F}\begin{pmatrix} t, s, x(s), x^{'(s)}, \cdots, x^{(n-1)}(s), x(a), x^{'(a)}, \cdots, x^{(n-1)}(a), \\ x(b), x^{'(b)}, \cdots, x^{(n-1)}(b) \end{pmatrix} ds, \qquad (3)$$

for $t \in I$ and $0 \leq j \leq n-1$

We require the following pair of known results:

Theorem 1

([24], p.194) [thm1] Let C be a nonempty closed convex subset of a Banach space X and $\mathrm{T}: \mathrm{C} \to \mathrm{C}$ a contraction operator with contract factor $\mathrm{k} \in [0,1)$ and fixed point x^*. Let α_n and β_n be two real sequences in $[0,1]$ such that $\alpha \leq \alpha_n \leq 1$ and $\beta \leq \beta_n < 1$ for all $n \in \mathbb{N}$ and for some $\alpha, \beta > 0$. For given $u_1 = v_1 = w_1 \in \mathrm{C}$, define sequences u_n, v_n and w_n in C as follows:

S-iteration process: $\begin{cases} u_{n+1} = (1-\alpha_n)Tu_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)u_n + \beta_n Tu_n, n \in \mathbb{N}. \end{cases}$

Picard iteration: $v_{n+1} = Tv_n, n \in \mathbb{N}$.

Mann iteration process: $w_{n+1} = (1-\beta_n)w_n + \beta_n Tw_n, n \in \mathbb{N}$.

Then we have the following:

- $\| u_{n+1} - x^* \| \leq k^n[1-(1-k)\alpha\beta]^n \| u_1 - x^* \|$, *for all* $n \in \mathbb{N}$.
- $\| v_{n+1} - x^* \| \leq k^n \| v_1 - x^* \|$, *for all* $n \in \mathbb{N}$.
- $\| w_{n+1} - x^* \| \leq [1-(1-k)\beta]^n \| w_1 - x^* \|$, *for all* $n \in \mathbb{N}$.

Moreover, the S-iteration process is faster than the Picard and Mann iteration processes. In particular, for $\alpha_n = 1,\ n \in \mathbb{N} \cup \{0\}$, the S-iteration process can be written as:

$$\begin{cases} u_0 \in C, \\ u_{n+1} = Ty_n, \\ y_n = (1-\beta_n)u_n + \beta_n Tu_n,\ n \in \mathbb{N} \cup \{0\}. \end{cases} \qquad (4)$$

Lemma 1

([26], p.4) [lem1] Let $\{\beta_n\}_{n=0}^{\infty}$ be a nonnegative sequence for which one assumes there exists $n_0 \in \mathbb{N} \cup \{0\}$, such that for all $n \geq n_0$ one has satisfied the inequality

$$\beta_{n+1} \leq (1-\mu_n)\beta_n + \mu_n \gamma_n, \tag{5}$$

where $\mu_n \in (0,1)$, for all $n \in \mathbb{N} \cup \{0\}$, $\sum_{n=0}^{\infty} \mu_n = \infty$ and $\gamma_n \geq 0$, $\forall\, n \in \mathbb{N} \cup \{0\}$. Then the following inequality holds

$$0 \leq \limsup_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \gamma_n. \tag{6}$$

For our convenience, we list the following hypotheses:

(H_1) The function $\mathcal{F}$ in equation (1) and its derivatives with respect t satisfy the condition

$$\begin{aligned}
&|\frac{\partial^j}{\partial t^j}\mathcal{F}(t,s,x(t),x'(t),\cdots,x^{(n-1)}(t),x(a),x'(a),\cdots,x^{(n-1)}(a),x(b),x'(b),\cdots,x^{(n-1)}(b)) \\
&-\frac{\partial^j}{\partial t^j}\mathcal{F}(t,s,y(s),y'(s),\cdots,y^{(n-1)}(s),y(a),y'(a),\cdots,y^{(n-1)}(a),y(b),y'(b),\cdots,y^{(n-1)}(b))| \\
&\leq p_j(t,s)\left[\alpha\sum_{i=0}^{n-1}\left|x^{(i)}(s)-y^{(i)}(s)\right| + \beta\sum_{i=0}^{n-1}\left|x^{(i)}(a)-y^{(i)}(a)\right| + \gamma\sum_{i=0}^{n-1}\left|x^{(i)}(b)-y^{(i)}(b)\right|\right],
\end{aligned}$$

for $j = 0,1,\cdots,n-1$, where $p_j(t,s) \in C(I^2,\mathbb{R}_+)$ and $\alpha,\beta,\gamma > 0$.

(H_2) $M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a) < 1$, where $M_{\mathcal{F}}$ denotes a positive constant such that for all $t,s \in I$

$$M_{\mathcal{F}} = \max \textstyle\sum_{j=0}^{n-1} p_j\ (t,s) : (t,s) \in I^2.$$

The following theorem establishes the existence and uniqueness of the solution of equation 1. (1).

Theorem 2

Assume that hypotheses $(\mathrm{H}_1)-(\mathrm{H}_2)$ hold. Let $\{\xi_k\}_{k=0}^{\infty}$ be a real sequence in $[0,1]$ satisfying $\sum_{k=0}^{\infty}\xi_k = \infty$. Then the equation (1) has a unique solution $x \in B$ and normal S −iterative method (4) (with $u_1 = x_0$) converges to $x \in B$ with the following estimate:

$$\| x_{k+1} - x \|_B \leq \frac{[M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)]^{k+1}}{e^{[1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)]\sum_{i=0}^{k}\xi_i}} \| x_0 - x \|_B. \tag{7}$$

Proof: For $x(t) \in B$, we define

$$(Tx)(t) = \mathcal{G}(t) + \int_a^b \mathcal{F}\left(t, s, x(s), x'^{(s)}, \cdots, x^{(n-1)}(s), x(a), x'^{(a)}, \cdots, x^{(n-1)}(a), x(b), x'^{(b)}, \cdots, x^{(n-1)}(b)\right) ds, \quad (8)$$

for $t \in I = [a, b]$.

Differentiating (8) on both sides with respect to t (see [7], p. 318), we have

$$(Tx)^{(j)}(t) = \mathcal{G}^{(j)}(t) + \int_a^b \frac{\partial^j}{\partial t^j} \mathcal{F}\left(t, s, x(s), x'^{(s)}, \cdots, x^{(n-1)}(s), x(a), x'^{(a)}, \cdots, x^{(n-1)}(a), x(b), x'^{(b)}, \cdots, x^{(n-1)}(b)\right) ds, \quad (9)$$

for $t \in I$ and $0 \le j \le n-1$.

Let $\{x_k\}_{k=0}^{\infty}$ and $\{x_k^{(j)}\}_{k=0}^{\infty}$, $(j = 1, \cdots, n-1)$ be iterative sequences generated by normal $S-$iteration method (4) for the operators given in (8) and (9) respectively.

We will show that $x_k \to x$ as $k \to \infty$.
From method (4), equations (3), (9) and hypotheses, we obtain

$$\begin{aligned}
&|x_{k+1}(t) - x(t)|_E \\
&= \sum_{j=0}^{n-1} \left| {x_{k+1}}^{(j)}(t) - x^{(j)}(t) \right| \\
&= \sum_{j=0}^{n-1} \left| (Ty_k)^{(j)}(t) - (Tx)^{(j)}(t) \right| \\
&= \sum_{j=0}^{n-1} | \mathcal{G}^{(j)}(t) \\
&\quad + \int_a^b \frac{\partial^j}{\partial t^j} \mathcal{F}\left(t, s, y_k(s), y'_k(s), \cdots, {y_k}^{(n-1)}(s), y_k(a), y_k^{'(a)}, \cdots, {y_k}^{(n-1)}(a), y_k(b), y_k^{'(b)}, \cdots, {y_k}^{(n-1)}(b)\right) ds \\
&\quad - \mathcal{G}^{(j)}(t) \\
&\quad - \int_a^b \frac{\partial^j}{\partial t^j} \mathcal{F}(t, s, x(s), x'(s), \cdots, x^{(n-1)}(s), x(a), x'(a), \cdots, x^{(n-1)}(a), x(b), x'(b), \cdots, x^{(n-1)}(b)) ds | \\
&= \sum_{j=0}^{n-1} \int_a^b | \frac{\partial^j}{\partial t^j} \mathcal{F}\left(t, s, y_k(s), y'_k(s), \cdots, {y_k}^{(n-1)}(s), y_k(a), y_k^{'(a)}, \cdots, {y_k}^{(n-1)}(a), y_k(b), y_k^{'(b)}, \cdots, {y_k}^{(n-1)}(b)\right) \\
&\quad - \frac{\partial^j}{\partial t^j} \mathcal{F}(t, s, x(s), x'(s), \cdots, x^{(n-1)}(s), x(a), x'(a), \cdots, x^{(n-1)}(a), x(b), x'(b), \cdots, x^{(n-1)}(b)) | ds \\
&\le \sum_{j=0}^{n-1} \int_a^b p_j\,(t, s) \left[\alpha \sum_{i=0}^{n-1} \left| (y_k)^{(i)}(s) - x^{(i)}(s) \right| + \beta \sum_{i=0}^{n-1} \left| (y_k)^{(i)}(a) - x^{(i)}(a) \right| + \gamma \sum_{i=0}^{n-1} \left| (y_k)^{(i)}(b) - x^{(i)}(b) \right| \right] ds \\
&\le M_{\mathcal{F}} \int_a^b [\, \alpha \sum_{i=0}^{n-1} \left| (y_k)^{(i)}(s) - x^{(i)}(s) \right| + \beta \sum_{i=0}^{n-1} \left| (y_k)^{(i)}(a) - x^{(i)}(a) \right| + \gamma \sum_{i=0}^{n-1} \left| (y_k)^{(i)}(b) - x^{(i)}(b) \right|] ds. \qquad (10)
\end{aligned}$$

Similarly,

$$
\begin{aligned}
&|y_k(t)-x(t)|_E \\
&=\sum_{j=0}^{n-1}\left|{y_k}^{(j)}(t)-x^{(j)}(t)\right| \\
&=\sum_{j=0}^{n-1}\left[\,(1-\xi_k)\left|{x_k}^{(j)}(t)-x^{(j)}(t)\right|+\xi_k\left|(Tx_k)^{(j)}(t)-(Tx)^{(j)}(t)\right|\right] \\
&=\left[(1-\xi_k)\sum_{j=0}^{n-1}\left|{x_k}^{(j)}(t)-x^{(j)}(t)\right|+\xi_k\sum_{j=0}^{n-1}\left|(Tx_k)^{(j)}(t)-(Tx)^{(j)}(t)\right|\right] \\
&\le \left[(1-\xi_k)\sum_{j=0}^{n-1}\left|{x_k}^{(j)}(t)-x^{(j)}(t)\right|\right. \\
&\quad +\xi_k M_{\mathcal{F}}\int_a^b\left[\,\alpha\sum_{i=0}^{n-1}\left|(x_k)^{(i)}(s)-x^{(i)}(s)\right|+\beta\sum_{i=0}^{n-1}\left|(x_k)^{(i)}(a)-x^{(i)}(a)\right|+\gamma\sum_{i=0}^{n-1}\left|(x_k)^{(i)}(b)-x^{(i)}(b)\right|\right]ds. \quad (11)
\end{aligned}
$$

Taking the supremum in the above inequalities, we get

$$
\begin{aligned}
\| x_{k+1}-x \|_B &\le M_{\mathcal{F}}\int_a^b[\,\alpha+\beta+\gamma]\,\| y_k-x \|_B\, ds \\
&= M_{\mathcal{F}}[\alpha+\beta+\gamma](b-a)\,\| y_k-x \|_B\,, \qquad (12)
\end{aligned}
$$

and

$$
\begin{aligned}
\| y_k-x \|_B &\le \left[(1-\xi_k)\,\| x_k-x \|_B+\xi_k M_{\mathcal{F}}\int_a^b(\alpha+\beta+\gamma)\,\| x_k-x \|_B\, ds\right] \\
&= \left[(1-\xi_k)\,\| x_k-x \|_B+\xi_k M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\,\| x_k-x \|_B\right] \\
&= \left[(1-\xi_k)+\xi_k M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\right]\| x_k-x \|_B \\
&= \left[1-\xi_k\big(1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\big)\right]\| x_k-x \|_B\,, \qquad (13)
\end{aligned}
$$

respectively.

Therefore, using (13) in (12), we get

$$
\| x_{k+1}-x \|_B \le \big(M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\big)\left[1-\xi_k\big(1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\big)\right]\| x_k-x \|_B. \quad (14)
$$

Thus, by applying induction on k, we get

$$
\| x_{k+1}-x \|_B \le \big(M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\big)^{k+1}\prod_{j=0}^{k}\left[\,1-\xi_j\big(1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\big)\right]\| x_0-x \|_B. \quad (15)
$$

Since $\xi_k\in[0,1]$ for all $k\in\mathbb{N}\cup 0$, the assumption (H_2) gives

$$
\begin{aligned}
&\xi_k\le 1 \quad \text{and} \quad M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)<1 \\
&\Rightarrow \xi_k M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)<\xi_k \\
&\Rightarrow \xi_k[1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)]<1,\ \forall\, k\in\mathbb{N}. \qquad (16)
\end{aligned}
$$

From the classical theory, we have

$$
1-x\le e^{-x},\ x\in[0,1].
$$

Hence, by using this fact with (16) in (15), we obtain

$$\| x_{k+1} - x \|_B \quad \le (M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a))^{k+1} e^{-\left(1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\right)\sum_{j=0}^{k}\xi_j} \| x_0 - x \|_B. \tag{17}$$

This is (7). Since $\sum_{k=0}^{\infty}\xi_k = \infty$,

$$e^{-\left(1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\right)\sum_{j=0}^{k}\xi_j} \to 0 \quad \text{as} \quad k\to\infty, \tag{18}$$

which implies $\lim_{k\to\infty} \| x_{k+1} - x \|_B = 0$. This gives $x_k \to x$ as $k \to \infty$.

Remark: It is notable that inequality (17) gives the bounds in terms of known functions, which majorizes the iterations for solution xt of equation (1) as well as its derivatives.

III. CLOSENESS OF SOLUTION VIA S −ITERATION METHOD

Now, we discuss the continuous dependency of solutions of (1) on the functions. Consider the problem (1) and the corresponding problem

$$(\overline{x}(t) = \mathcal{H}(t) + \int_a^b \mathcal{L}\left(t,s,\overline{x}(s),\overline{x}^{'(s)},\cdots,\overline{x}^{(n-1)}(s),\overline{x}(a),\overline{x}^{'(a)},\cdots,\overline{x}^{(n-1)}(a),\overline{x}(b),\overline{x}^{'(b)},\cdots,\overline{x}^{(n-1)}(b)\right)ds, \tag{19}$$

for $t \in I = [a,b]$, where $\mathcal{L} \in C(I^2 \times E^3, \mathbb{R})$, $\mathcal{H} \in C(I,\mathbb{R})$, and $n \ge 1$ is an arbitrary integer. A solution to equation (19) refers to a continuous function $\overline{x}(t)$, where t belongs to the interval I. This function must be continuously differentiable for $(n-1)$ times on I, and it must also satisfy the equation (19). It is worth noting that the solution $\overline{x}(t)$, along with its derivatives, meets the integral equations (see [7] , p.318)

$$\begin{aligned}\overline{x}^{(j)}(t) &= \mathcal{H}^{(j)}(t)\\ &+ \int_a^b \frac{\partial^j}{\partial t^j}\mathcal{L}\left(t,s,\overline{x}(s),\overline{x}^{'(s)},\cdots,\overline{x}^{(n-1)}(s),\overline{x}(a),\overline{x}^{'(a)},\cdots,\overline{x}^{(n-1)}(a),\overline{x}(b),\overline{x}^{'(b)},\cdots,\overline{x}^{(n-1)}(b)\right)ds,\end{aligned} \tag{20}$$

for $t \in I$ and $0 \le j \le n-1$

Following steps from the proof of Theorem 2, for $\overline{x}(t) \in B$ we define the operator for the equation (19)

$$\begin{aligned}(\overline{T}\overline{x})(t) &= \mathcal{H}(t)\\ &+ \int_a^b \mathcal{L}\left(t,s,\overline{x}(s),\overline{x}^{'(s)},\cdots,\overline{x}^{(n-1)}(s),\overline{x}(a),\overline{x}^{'(a)},\cdots,\overline{x}^{(n-1)}(a),\overline{x}(b),\overline{x}^{'(b)},\cdots,\overline{x}^{(n-1)}(b)\right)ds,\end{aligned} \tag{21}$$

for $t \in I = [a,b]$.

Differentiating both sides of (21) with respect to t (see [7], p. 318), we get

$$\begin{aligned}(\overline{T}\overline{x})^{(j)}(t) &= \mathcal{H}^{(j)}(t)\\ &+ \int_a^b \frac{\partial^j}{\partial t^j}\mathcal{L}\left(t,s,\overline{x}(s),\overline{x}^{'(s)},\cdots,\overline{x}^{(n-1)}(s),\overline{x}(a),\overline{x}^{'(a)},\cdots,\overline{x}^{(n-1)}(a),\overline{x}(b),\overline{x}^{'(b)},\cdots,\overline{x}^{(n-1)}(b)\right)ds,\end{aligned} \tag{22}$$

for $t \in I$ and $0 \le j \le n-1$.

The following theorem addresses the closeness of the solutions for problems (1) and (19).

Theorem 3

Consider the sequences $\{x_k\}_{k=0}^{\infty}$ and $\{\overline{x}_k\}_{k=0}^{\infty}$ generated by normal $S-$ iterative method associated with operators T in (9) and $\overline{T}$ in (22), respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup 0$. Assume that

- all the conditions of Theorem 2 hold, and $x(t)$ and $\overline{x}(t)$ are solutions of (1) and (19) respectively.
- there exist nonnegative constants ϵ_j and $\overline{\epsilon}_j$ such that

$$\left|\mathcal{G}^{(j)}(t) - \mathcal{H}^{(j)}(t)\right| \leq \epsilon_j,\ \forall\ t \in I,\ (j = 0,1,\cdots,n-1), \tag{23}$$

and

$$\begin{aligned}&|\frac{\partial^j}{\partial t^j}\mathcal{F}(t,s,x(t),x'(t),\cdots,x^{(n-1)}(t),x(a),x'(a),\cdots,x^{(n-1)}(a),x(b),x'(b),\cdots,x^{(n-1)}(b))\\ &\quad -\frac{\partial^j}{\partial t^j}\mathcal{L}(t,s,x(t),x'(t),\cdots,x^{(n-1)}(t),x(a),x'(a),\cdots,x^{(n-1)}(a),x(b),x'(b),\cdots,x^{(n-1)}(b))|\\ &\leq p_j(t,s)\overline{\epsilon}_j,\ \forall\ t \in I,\ (j = 0,1,\cdots,n-1).\end{aligned} \tag{24}$$

If the sequence $\{\overline{x}_k\}_{k=0}^{\infty}$ *converges to* $\overline{x}$*, then we have*

$$\| x - \overline{x} \|_B \leq \frac{3[M_{\mathcal{F}}(b-a)\overline{\epsilon}+\epsilon]}{1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)}, \tag{25}$$

Where

$$\epsilon = \sum_{j=0}^{n-1} \epsilon_j \ \textit{and}\ \overline{\epsilon} = \sum_{j=0}^{n-1} \overline{\epsilon}_j.$$

Proof: Suppose the sequences $\{x_k\}_{k=0}^{\infty}$ and $\{\overline{x}_k\}_{k=0}^{\infty}$ generated by normal $S-$ iterative method associated with operators T in (9)and $\overline{T}$ in (22), respectively with the real control sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$. From iterative method (4) and equations (3) with (9); (20) with (22) and hypotheses, we obtain

$$
\begin{aligned}
&|x_{k+1}(t) - \overline{x}_{k+1}(t)|_E \\
&= \sum_{j=0}^{n-1} \left| {x_{k+1}}^{(j)}(t) - {\overline{x}_{k+1}}^{(j)}(t) \right| \\
&= \sum_{j=0}^{n-1} \left| (Ty_k)^{(j)}(t) - \left(\overline{T}\overline{y}_k\right)^{(j)}(t) \right| \\
&= \sum_{j=0}^{n-1} | \mathcal{G}^{(j)}(t) - \mathcal{H}^{(j)}(t) \\
&\quad + \int_a^b \frac{\partial^j}{\partial t^j} \mathcal{F}\left(t, s, y_k(s), y'_k(s), \cdots, {y_k}^{(n-1)}(s), y_k(a), y_k^{'(a)}, \cdots, {y_k}^{(n-1)}(a), y_k(b), y_k^{'(b)}, \cdots, {y_k}^{(n-1)}(b)\right) ds \\
&\quad - \int_a^b \frac{\partial^j}{\partial t^j} \mathcal{L}(t, s, \overline{y}_k(s), \overline{y}_k{}'(s), \cdots, {\overline{y}_k}^{(n-1)}(s), \overline{y}_k(a), \overline{y}_k{}'(a), \cdots, {\overline{y}_k}^{(n-1)}(a), \overline{y}_k(b), \overline{y}_k{}'(b), \cdots, {\overline{y}_k}^{(n-1)}(b)) ds | \\
&\leq \sum_{j=0}^{n-1} \epsilon_j \\
&\quad + \sum_{j=0}^{n-1} \int_a^b | \frac{\partial^j}{\partial t^j} \mathcal{F}\left(t, s, y_k(s), y'_k(s), \cdots, {y_k}^{(n-1)}(s), y_k(a), y_k^{'(a)}, \cdots, {y_k}^{(n-1)}(a), y_k(b), y_k^{'(b)}, \cdots, {y_k}^{(n-1)}(b)\right) \\
&\quad - \frac{\partial^j}{\partial t^j} \mathcal{L}(t, s, \overline{y}_k(s), \overline{y}_k{}'(s), \cdots, {\overline{y}_k}^{(n-1)}(s), \overline{y}_k(a), \overline{y}_k{}'(a), \cdots, {\overline{y}_k}^{(n-1)}(a), \overline{y}_k(b), \overline{y}_k{}'(b), \cdots, {\overline{y}_k}^{(n-1)}(b)) | ds \\
&\leq \sum_{j=0}^{n-1} \epsilon_j \\
&\quad + \sum_{j=0}^{n-1} \int_a^b | \frac{\partial^j}{\partial t^j} \mathcal{F}\left(t, s, y_k(s), y'_k(s), \cdots, {y_k}^{(n-1)}(s), y_k(a), y_k^{'(a)}, \cdots, {y_k}^{(n-1)}(a), y_k(b), y_k^{'(b)}, \cdots, {y_k}^{(n-1)}(b)\right) \\
&\quad - \frac{\partial^j}{\partial t^j} \mathcal{F}(t, s, \overline{y}_k(s), \overline{y}_k{}'(s), \cdots, {\overline{y}_k}^{(n-1)}(s), \overline{y}_k(a), \overline{y}_k{}'(a), \cdots, {\overline{y}_k}^{(n-1)}(a), \overline{y}_k(b), \overline{y}_k{}'(b), \cdots, {\overline{y}_k}^{(n-1)}(b)) | ds \\
&\quad + \sum_{j=0}^{n-1} \int_a^b | \frac{\partial^j}{\partial t^j} \mathcal{F}\left(t, s, \overline{y}_k(s), \overline{y}_k^{'(s)}, \cdots, {\overline{y}_k}^{(n-1)}(s), \overline{y}_k(a), \overline{y}_k^{'(a)}, \cdots, {\overline{y}_k}^{(n-1)}(a), \overline{y}_k(b), \overline{y}_k^{'(b)}, \cdots, {\overline{y}_k}^{(n-1)}(b)\right) \\
&\quad - \frac{\partial^j}{\partial t^j} \mathcal{L}(t, s, \overline{y}_k(s), \overline{y}_k{}'(s), \cdots, {\overline{y}_k}^{(n-1)}(s), \overline{y}_k(a), \overline{y}_k{}'(a), \cdots, {\overline{y}_k}^{(n-1)}(a), \overline{y}_k(b), \overline{y}_k{}'(b), \cdots, {\overline{y}_k}^{(n-1)}(b)) | ds \\
&\leq \sum_{j=0}^{n-1} \epsilon_j + M_{\mathcal{F}} \sum_{j=0}^{n-1} \int_a^b \overline{\epsilon}_j \, ds \\
&\quad + \sum_{j=0}^{n-1} \int_a^b p_j\,(t, s) \left[\alpha \sum_{i=0}^{n-1} \left| (y_k)^{(i)}(s) - {\overline{y}_k}^{(i)}(s) \right| + \beta \sum_{i=0}^{n-1} \left| (y_k)^{(i)}(a) - {\overline{y}_k}^{(i)}(a) \right| + \gamma \sum_{i=0}^{n-1} \left| (y_k)^{(i)}(b) - {\overline{y}_k}^{(i)}(b) \right| \right] ds \\
&\leq \epsilon + M_{\mathcal{F}} \overline{\epsilon}(b - a) \\
&\quad + M_{\mathcal{F}} \int_a^b [\, \alpha \sum_{i=0}^{n-1} \left| (y_k)^{(i)}(s) - {\overline{y}_k}^{(i)}(s) \right| + \beta \sum_{i=0}^{n-1} \left| (y_k)^{(i)}(a) - {\overline{y}_k}^{(i)}(a) \right| + \gamma \sum_{i=0}^{n-1} \left| (y_k)^{(i)}(b) - {\overline{y}_k}^{(i)}(b) \right|] ds.
\end{aligned}
\tag{26}
$$

Similarly,

$$\begin{aligned}
&\left|y_k(t)-\overline{y}_k(t)\right|_E \\
&=\sum_{j=0}^{n-1}\left|{y_k}^{(j)}(t)-{\overline{y}_k}^{(j)}(t)\right| \\
&=\sum_{j=0}^{n-1}\left[(1-\xi_k)\left|{x_k}^{(j)}(t)-{\overline{x}_k}^{(j)}(t)\right|+\xi_k\left|(Tx_k)^{(j)}(t)-\left(\overline{T}\overline{x}_k\right)^{(j)}(t)\right|\right] \\
&=\left[(1-\xi_k)\sum_{j=0}^{n-1}\left|{x_k}^{(j)}(t)-{\overline{x}_k}^{(j)}(t)\right|+\xi_k\sum_{j=0}^{n-1}\left|(Tx_k)^{(j)}(t)-\left(\overline{T}\overline{x}_k\right)^{(j)}(t)\right|\right] \\
&\leq (1-\xi_k)\sum_{j=0}^{n-1}\left|{x_k}^{(j)}(t)-{\overline{x}_k}^{(j)}(t)\right|+\xi_k[\epsilon+M_{\mathcal{F}}\overline{\epsilon}(b-a)] \\
&\quad+\xi_k M_{\mathcal{F}}\int_a^b\left[\alpha\sum_{i=0}^{n-1}\left|(x_k)^{(i)}(s)-{\overline{x}_k}^{(i)}(s)\right|+\beta\sum_{i=0}^{n-1}\left|(x_k)^{(i)}(a)-{\overline{x}_k}^{(i)}(a)\right|+\gamma\sum_{i=0}^{n-1}\left|(x_k)^{(i)}(b)-{\overline{x}_k}^{(i)}(b)\right|\right]ds. \qquad (27)
\end{aligned}$$

Taking supremum in the above inequalities, we get

$$\begin{aligned}
\| x_{k+1}-\overline{x}_{k+1}\|_B &\leq \epsilon+M_{\mathcal{F}}\overline{\epsilon}(b-a)+M_{\mathcal{F}}\int_a^b[\alpha+\beta+\gamma]\| y_k-\overline{y}_k\|_B\, ds \\
&=\epsilon+M_{\mathcal{F}}\overline{\epsilon}(b-a)+M_{\mathcal{F}}[\alpha+\beta+\gamma](b-a)\| y_k-\overline{y}_k\|_B, \qquad (28)
\end{aligned}$$

and

$$\begin{aligned}
\| y_k-\overline{y}_k\|_B &\leq (1-\xi_k)\| x_k-\overline{x}_k\|_B+\xi_k\left[\epsilon+M_{\mathcal{F}}\overline{\epsilon}(b-a)+M_{\mathcal{F}}\int_a^b(\alpha+\beta+\gamma)\| x_k-\overline{x}_k\|_B\, ds\right] \\
&=(1-\xi_k)\| x_k-\overline{x}_k\|_B+\xi_k[\epsilon+M_{\mathcal{F}}\overline{\epsilon}(b-a)+M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\| x_k-\overline{x}_k\|_B] \\
&=\xi_k\left(\epsilon+M_{\mathcal{F}}\overline{\epsilon}(b-a)\right)+\left[1-\xi_k\left(1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\right)\right]\| x_k-\overline{x}_k\|_B, \qquad (29)
\end{aligned}$$

respectively.

Therefore, using (29) in (28) and using hypothesis (H_2), and $\frac{1}{2}\leq\xi_k$ for all $k\in\mathbb{N}\cup 0$, the resulting inequality become

$$\begin{aligned}
\| x_{k+1}-\overline{x}_{k+1}\|_B &\leq \left[1-\xi_k\left(1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\right)\right]\| x_k-\overline{x}_k\|_B \\
&\quad+\xi_k\left(\epsilon+M_{\mathcal{F}}\overline{\epsilon}(b-a)\right)+2\xi_k\left(\epsilon+M_{\mathcal{F}}\overline{\epsilon}(b-a)\right) \\
&\leq \left[1-\xi_k\left(1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\right)\right]\| x_k-\overline{x}_k\|_B \\
&\quad+\xi_k\left(1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\right)\frac{3\left(\epsilon+M_{\mathcal{F}}\overline{\epsilon}(b-a)\right)}{\left(1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\right)}. \qquad (30)
\end{aligned}$$

We denote

$$\begin{aligned}
\beta_k &=\| x_k-\overline{x}_k\|_B\geq 0, \\
\mu_k &=\xi_k\left(1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\right)\in(0,1), \\
\gamma_k &=\frac{3\left(\epsilon+M_{\mathcal{F}}\overline{\epsilon}(b-a)\right)}{\left(1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\right)}\geq 0.
\end{aligned}$$

It is to be observed that inequality (30) satisfies all the conditions of Lemma 1, therefore, we get

$$\begin{aligned} 0 &\leq \limsup_{k\to\infty}\beta_k \leq \limsup_{k\to\infty}\gamma_k \\ \Rightarrow 0 &\leq \limsup_{k\to\infty} \| x_k - \overline{x}_k \|_B \leq \limsup_{k\to\infty} \frac{3\big(\epsilon + M_{\mathcal{F}}\overline{\epsilon}(b-a)\big)}{\big(1 - M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\big)} \\ \Rightarrow 0 &\leq \limsup_{k\to\infty} \| x_k - \overline{x}_k \|_B \leq \frac{3\big(\epsilon + M_{\mathcal{F}}\overline{\epsilon}(b-a)\big)}{\big(1 - M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\big)}. \end{aligned} \tag{31}$$

By (i), we have $\lim_{k\to\infty}x_k = x$. Using this fact and the assumption $\lim_{k\to\infty}\overline{x}_k = \overline{x}$, we get from (31) that

$$\| x - \overline{x} \|_B \leq \frac{3[M_{\mathcal{F}}(b-a)\overline{\epsilon}+\epsilon]}{1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)}. \tag{32}$$

Remark: The inequality (32) shows how the solutions of the problems (1) and (19) are related. If the functions $\mathcal{F}$ and $\mathcal{G}$ are close to $\mathcal{L}$ and $\mathcal{H}$, respectively, then not only are the solutions of the problems (1) and (19) closer to each other (i.e. $\| x - \overline{x} \|_B \to 0$), but they also depend continuously on the functions involved. Additionally, this inequality estimates the derivatives of the solutions.

Now, we focus on analyzing how solutions depend continuously on certain parameters.
Consider the problems

$$x(t) = \mathcal{G}(t) + \int_a^b \mathcal{F}\left(t,s,x(s),x^{'(s)},\cdots,x^{(n-1)}(s),x(a),x^{'(a)},\cdots,x^{(n-1)}(a),x(b),x^{'(b)},\cdots,x^{(n-1)}(b),\mu_1\right)ds, \tag{33}$$

and

$$\overline{x}(t) = \mathcal{G}(t) + \int_a^b \mathcal{F}\left(t,s,\overline{x}(s),\overline{x}^{'(s)},\cdots,\overline{x}^{(n-1)}(s),\overline{x}(a),\overline{x}^{'(a)},\cdots,\overline{x}^{(n-1)}(a),\overline{x}(b),\overline{x}^{'(b)},\cdots,\overline{x}^{(n-1)}(b),\mu_2\right)ds, \tag{34}$$

for $t \in I = [a,b]$. The functions $\mathcal{F}, \mathcal{G}$ are defined as in (1) and μ_1, μ_2 are real parameters.

A solution to equation (33) is a continuous function $x(t)$ defined on the interval I, which is differentiable $(n-1)$ times and satisfies the equation (33). We can observe that both $x(t)$ and its derivatives satisfy integral equations. (see [7], p.318)

$$\begin{aligned} x^{(j)}(t) &= \mathcal{G}^{(j)}(t) \\ &\quad + \int_a^b \frac{\partial^j}{\partial t^j}\mathcal{F}\big(t,s,x(s),x^{'(s)},\cdots,x^{(n-1)}(s),x(a),x^{'(a)},\cdots,x^{(n-1)}(a),x(b),x^{'(b)},\cdots,x^{(n-1)}(b),\mu_1\big)ds, \end{aligned} \tag{35}$$

for $t \in I$ and $0 \leq j \leq n-1$.

Now, following the steps from the proof of Theorem 2, for $x(t) \in B$, we define the operator for the equation (33)

$$\begin{aligned} (Tx)(t) &= \mathcal{G}(t) \\ &\quad + \int_a^b \mathcal{F}\left(t,s,x(s),x^{'(s)},\cdots,x^{(n-1)}(s),x(a),x^{'(a)},\cdots,x^{(n-1)}(a),x(b),x^{'(b)},\cdots,x^{(n-1)}(b),\mu_1\right)ds, \end{aligned} \tag{36}$$

for $t \in I = [a,b]$.

Taking derivatives on both sides of (36) with respect to t (see [7], p. 318), we get

$$\begin{aligned}(Tx)^{(j)}(t) &= \mathcal{G}^{(j)}(t)\\ &+\int_a^b \frac{\partial^j}{\partial t^j}\mathcal{F}\big(t,s,x(s),x^{'(s)},\cdots,x^{(n-1)}(s),x(a),x^{'(a)},\cdots,x^{(n-1)}(a),x(b),x^{'(b)},\cdots,x^{(n-1)}(b),\mu_1\big)ds,\end{aligned} \tag{37}$$

for $t\in I$ and $0\le j\le n-1$.

Similarly, for the equation (34), we define

$$\begin{aligned}\overline{x}^{(j)}(t) &= \mathcal{G}^{(j)}(t)\\ &+\int_a^b \frac{\partial^j}{\partial t^j}\mathcal{L}\left(t,s,\overline{x}(s),\overline{x}^{'(s)},\cdots,\overline{x}^{(n-1)}(s),\overline{x}(a),\overline{x}^{'(a)},\cdots,\overline{x}^{(n-1)}(a),\overline{x}(b),\overline{x}^{'(b)},\cdots,\overline{x}^{(n-1)}(b),\mu_2\right)ds,\end{aligned} \tag{38}$$

for $t\in I$ and $0\le j\le n-1$

Again, following the steps from the proof of Theorem 2, for $\overline{x}(t)\in B$, we define the operator for the equation (34)

$$\begin{aligned}(\overline{T}\overline{x})(t) &= \mathcal{G}(t)\\ &+\int_a^b \mathcal{F}\left(t,s,\overline{x}(s),\overline{x}^{'(s)},\cdots,\overline{x}^{(n-1)}(s),\overline{x}(a),\overline{x}^{'(a)},\cdots,\overline{x}^{(n-1)}(a),\overline{x}(b),\overline{x}^{'(b)},\cdots,\overline{x}^{(n-1)}(b),\mu_2\right)ds,\end{aligned} \tag{39}$$

for $t\in I=[a,b]$.

Taking derivatives on both sides of (39) with respect to t (see [7], p. 318), we get

$$\begin{aligned}\left(\overline{T}\overline{x}\right)^{(j)}(t) &= \mathcal{G}^{(j)}(t)\\ &+\int_a^b \frac{\partial^j}{\partial t^j}\mathcal{F}\left(t,s,\overline{x}(s),\overline{x}^{'(s)},\cdots,\overline{x}^{(n-1)}(s),\overline{x}(a),\overline{x}^{'(a)},\cdots,\overline{x}^{(n-1)}(a),\overline{x}(b),\overline{x}^{'(b)},\cdots,\overline{x}^{(n-1)}(b),\mu_2\right)ds,\end{aligned} \tag{40}$$

for $t\in I$ and $0\le j\le n-1$.

The next theorem asserts that the solutions depend continuously on the parameters.

Theorem 4

Consider the sequences $\{x_k\}_{k=0}^{\infty}$ and $\{\overline{x}_k\}_{k=0}^{\infty}$ generated by normal $S-$ iterative method associated with operators T in (37)and $\overline{T}$ in (40), respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2}\le\xi_k$ for all $k\in\mathbb{N}\cup\{0\}$. Assume that

- the hypothesis (H_2) holds.
- the function $\mathcal{F}$ satisfy the conditions:

$$
\begin{aligned}
&|\frac{\partial^j}{\partial t^j}\mathcal{F}(t,s,x(t),x'(t),\cdots,x^{(n-1)}(t),x(a),x'(a),\cdots,x^{(n-1)}(a),x(b),x'(b),\cdots,x^{(n-1)}(b),\mu_1)\\
&\qquad -\frac{\partial^j}{\partial t^j}\mathcal{F}(t,s,y(s),y'(s),\cdots,y^{(n-1)}(s),y(a),y'(a),\cdots,y^{(n-1)}(a),y(b),y'(b),\cdots,y^{(n-1)}(b),\mu_1)|\\
&\leq p_j(t,s)\left[\alpha\sum_{i=0}^{n-1}\left|x^{(i)}(s)-y^{(i)}(s)\right|+\beta\sum_{i=0}^{n-1}\left|x^{(i)}(a)-y^{(i)}(a)\right|+\gamma\sum_{i=0}^{n-1}\left|x^{(i)}(b)-y^{(i)}(b)\right|\right], \qquad (41)\\
&|\frac{\partial^j}{\partial t^j}\mathcal{F}(t,s,x(t),x'(t),\cdots,x^{(n-1)}(t),x(a),x'(a),\cdots,x^{(n-1)}(a),x(b),x'(b),\cdots,x^{(n-1)}(b),\mu_1)\\
&\qquad -\frac{\partial^j}{\partial t^j}\mathcal{F}(t,s,x(t),x'(t),\cdots,x^{(n-1)}(t),x(a),x'(a),\cdots,x^{(n-1)}(a),x(b),x'(b),\cdots,x^{(n-1)}(b),\mu_2)|\\
&\leq p_j(t,s)|\mu_1-\mu_2|, \qquad (42)
\end{aligned}
$$

for $j=0,1,\cdots,n-1$, where $p_j(t,s)\in C(I^2,\mathbb{R}_+)$ and $\alpha,\beta,\gamma>0$.

Suppose $x(t)$ and $\overline{x}(t)$ are solutions of (33) and (34) respectively and if the sequence $\{\overline{x}_k\}_{k=0}^{\infty}$ converges to $\overline{x}$, then we have

$$\| x-\overline{x} \|_B \leq \frac{3[M_{\mathcal{F}}|\mu_1-\mu_2|(b-a)]}{1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)}. \qquad (43)$$

Proof: Suppose the sequences $\{x_k\}_{k=0}^{\infty}$ and $\{\overline{x}_k\}_{k=0}^{\infty}$ generated by normal $S-$ iterative method associated with operators T in (37)and $\overline{T}$ in (40), respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2}\leq\xi_k$ for all $k\in\mathbb{N}\cup\{0\}$. From iterative method (4) and equations (35) with (37); (38) with (40) and hypotheses, we obtain

$$
\begin{aligned}
&|x_{k+1}(t)-\overline{x}_{k+1}(t)|_E \\
&=\sum_{j=0}^{n-1}\left|x_{k+1}{}^{(j)}(t)-\overline{x}_{k+1}{}^{(j)}(t)\right| \\
&=\sum_{j=0}^{n-1}\left|(Ty_k)^{(j)}(t)-\left(\overline{T}\overline{y}_k\right)^{(j)}(t)\right| \\
&=\sum_{j=0}^{n-1}|\mathcal{G}^{(j)}(t)-\mathcal{G}^{(j)}(t) \\
&\quad+\int_a^b\frac{\partial^j}{\partial t^j}\mathcal{F}\left(t,s,y_k(s),y'_k(s),\cdots,y_k{}^{(n-1)}(s),y_k(a),y_k^{'(a)},\cdots,y_k{}^{(n-1)}(a),y_k(b),y_k^{'(b)},\cdots,y_k{}^{(n-1)}(b),\mu_1\right)ds \\
&\quad-\int_a^b\frac{\partial^j}{\partial t^j}\mathcal{F}(t,s,\overline{y}_k(s),\overline{y}_k{}'(s),\cdots,\overline{y}_k{}^{(n-1)}(s),\overline{y}_k(a),\overline{y}_k{}'(a),\cdots,\overline{y}_k{}^{(n-1)}(a),\overline{y}_k(b),\overline{y}_k{}'(b),\cdots,\overline{y}_k{}^{(n-1)}(b),\mu_2)ds| \\
&\leq\sum_{j=0}^{n-1}\int_a^b|\frac{\partial^j}{\partial t^j}\mathcal{F}\left(t,s,y_k(s),y'_k(s),\cdots,y_k{}^{(n-1)}(s),y_k(a),y_k^{'(a)},\cdots,y_k{}^{(n-1)}(a),y_k(b),y_k^{'(b)},\cdots,y_k{}^{(n-1)}(b),\mu_1\right) \\
&\quad-\frac{\partial^j}{\partial t^j}\mathcal{F}(t,s,\overline{y}_k(s),\overline{y}_k{}'(s),\cdots,\overline{y}_k{}^{(n-1)}(s),\overline{y}_k(a),\overline{y}_k{}'(a),\cdots,\overline{y}_k{}^{(n-1)}(a),\overline{y}_k(b),\overline{y}_k{}'(b),\cdots,\overline{y}_k{}^{(n-1)}(b),\mu_2)|ds \\
&\leq\sum_{j=0}^{n-1}\int_a^b|\frac{\partial^j}{\partial t^j}\mathcal{F}\left(t,s,y_k(s),y'_k(s),\cdots,y_k{}^{(n-1)}(s),y_k(a),y_k^{'(a)},\cdots,y_k{}^{(n-1)}(a),y_k(b),y_k^{'(b)},\cdots,y_k{}^{(n-1)}(b),\mu_1\right) \\
&\quad-\frac{\partial^j}{\partial t^j}\mathcal{F}(t,s,\overline{y}_k(s),\overline{y}_k{}'(s),\cdots,\overline{y}_k{}^{(n-1)}(s),\overline{y}_k(a),\overline{y}_k{}'(a),\cdots,\overline{y}_k{}^{(n-1)}(a),\overline{y}_k(b),\overline{y}_k{}'(b),\cdots,\overline{y}_k{}^{(n-1)}(b),\mu_1)|ds \\
&\quad+\sum_{j=0}^{n-1}\int_a^b|\frac{\partial^j}{\partial t^j}\mathcal{F}\left(t,s,\overline{y}_k(s),\overline{y}_k^{'(s)},\cdots,\overline{y}_k{}^{(n-1)}(s),\overline{y}_k(a),\overline{y}_k^{'(a)},\cdots,\overline{y}_k{}^{(n-1)}(a),\overline{y}_k(b),\overline{y}_k^{'(b)},\cdots,\overline{y}_k{}^{(n-1)}(b),\mu_1\right) \\
&\quad-\frac{\partial^j}{\partial t^j}\mathcal{F}(t,s,\overline{y}_k(s),\overline{y}_k{}'(s),\cdots,\overline{y}_k{}^{(n-1)}(s),\overline{y}_k(a),\overline{y}_k{}'(a),\cdots,\overline{y}_k{}^{(n-1)}(a),\overline{y}_k(b),\overline{y}_k{}'(b),\cdots,\overline{y}_k{}^{(n-1)}(b),\mu_2)|ds \\
&\leq M_{\mathcal{F}}\int_a^b|\mu_1-\mu_2|\,ds \\
&\quad+\sum_{j=0}^{n-1}\int_a^b p_j\,(t,s)\left[\alpha\sum_{i=0}^{n-1}\left|(y_k)^{(i)}(s)-\overline{y}_k{}^{(i)}(s)\right|+\beta\sum_{i=0}^{n-1}\left|(y_k)^{(i)}(a)-\overline{y}_k{}^{(i)}(a)\right|+\gamma\sum_{i=0}^{n-1}\left|(y_k)^{(i)}(b)-\overline{y}_k{}^{(i)}(b)\right|\right]ds \\
&\leq M_{\mathcal{F}}|\mu_1-\mu_2|(b-a) \\
&\quad+M_{\mathcal{F}}\int_a^b[\,\alpha\sum_{i=0}^{n-1}\left|(y_k)^{(i)}(s)-\overline{y}_k{}^{(i)}(s)\right|+\beta\sum_{i=0}^{n-1}\left|(y_k)^{(i)}(a)-\overline{y}_k{}^{(i)}(a)\right|+\gamma\sum_{i=0}^{n-1}\left|(y_k)^{(i)}(b)-\overline{y}_k{}^{(i)}(b)\right|]ds. \qquad (44)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\left|y_k(t)-\overline{y}_k(t)\right|_E \\
&=\sum_{j=0}^{n-1}\left|y_k{}^{(j)}(t)-\overline{y}_k{}^{(j)}(t)\right| \\
&=\sum_{j=0}^{n-1}[\,(1-\xi_k)\left|x_k{}^{(j)}(t)-\overline{x}_k{}^{(j)}(t)\right|+\xi_k\left|(Tx_k)^{(j)}(t)-\left(\overline{T}\overline{x}_k\right)^{(j)}(t)\right|] \\
&=\left[(1-\xi_k)\sum_{j=0}^{n-1}\left|x_k{}^{(j)}(t)-\overline{x}_k{}^{(j)}(t)\right|+\xi_k\sum_{j=0}^{n-1}\left|(Tx_k)^{(j)}(t)-\left(\overline{T}\overline{x}_k\right)^{(j)}(t)\right|\right] \\
&\leq(1-\xi_k)\sum_{j=0}^{n-1}\left|x_k{}^{(j)}(t)-\overline{x}_k{}^{(j)}(t)\right|+\xi_k[M_{\mathcal{F}}|\mu_1-\mu_2|(b-a)] \\
&\quad+\xi_k M_{\mathcal{F}}\int_a^b[\,\alpha\sum_{i=0}^{n-1}\left|(x_k)^{(i)}(s)-\overline{x}_k{}^{(i)}(s)\right|+\beta\sum_{i=0}^{n-1}\left|(x_k)^{(i)}(a)-\overline{x}_k{}^{(i)}(a)\right|+\gamma\sum_{i=0}^{n-1}\left|(x_k)^{(i)}(b)-\overline{x}_k{}^{(i)}(b)\right|]ds. \qquad (45)
\end{aligned}
$$

Taking supremum in the above inequalities, we get

$$\begin{aligned}\| x_{k+1}-\overline{x}_{k+1} \|_B &\leq M_{\mathcal{F}}|\mu_1-\mu_2|(b-a)+M_{\mathcal{F}}\int_a^b [\alpha+\beta+\gamma] \| y_k-\overline{y}_k \|_B \, ds \\ &= M_{\mathcal{F}}|\mu_1-\mu_2|(b-a)+M_{\mathcal{F}}[\alpha+\beta+\gamma](b-a) \| y_k-\overline{y}_k \|_B , \end{aligned} \tag{46}$$

and

$$\begin{aligned}\| y_k-\overline{y}_k \|_B &\leq (1-\xi_k) \| x_k-\overline{x}_k \|_B+\xi_k[M_{\mathcal{F}}|\mu_1-\mu_2|(b-a)+M_{\mathcal{F}}\int_a^b(\alpha+\beta+\gamma) \| x_k-\overline{x}_k \|_B \, ds] \\ &= (1-\xi_k) \| x_k-\overline{x}_k \|_B+\xi_k[M_{\mathcal{F}}|\mu_1-\mu_2|(b-a)+M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a) \| x_k-\overline{x}_k \|_B] \\ &= \xi_k(M_{\mathcal{F}}|\mu_1-\mu_2|(b-a))+[(1-\xi_k)+\xi_k M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a))] \| x_k-\overline{x}_k \|_B \\ &= \xi_k\big(M_{\mathcal{F}}|\mu_1-\mu_2|(b-a)\big)+\big[1-\xi_k\big(1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\big)\big] \| x_k-\overline{x}_k \|_B , \end{aligned} \tag{47}$$

respectively.

Therefore, using (47) in (46) and using hypothesis (H_2), and $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup 0$, the resulting inequality become

$$\begin{aligned}\| x_{k+1}-\overline{x}_{k+1} \|_B &\leq \big[1-\xi_k\big(1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\big)\big] \| x_k-\overline{x}_k \|_B \\ &\quad +\xi_k\big(M_{\mathcal{F}}|\mu_1-\mu_2|(b-a)\big)+2\xi_k\big(M_{\mathcal{F}}|\mu_1-\mu_2|(b-a)\big) \\ &\leq \big[1-\xi_k\big(1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\big)\big] \| x_k-\overline{x}_k \|_B \\ &\quad +\xi_k\big(1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\big)\frac{3\big(M_{\mathcal{F}}|\mu_1-\mu_2|(b-a)\big)}{\big(1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\big)}. \end{aligned} \tag{48}$$

We denote

$$\begin{aligned}\beta_k &= \| x_k-\overline{x}_k \|_B \geq 0, \\ \mu_k &= \xi_k\big(1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\big) \in (0,1), \\ \gamma_k &= \frac{3\big(M_{\mathcal{F}}|\mu_1-\mu_2|(b-a)\big)}{\big(1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\big)} \geq 0. \end{aligned}$$

Now, it is to be observed that, the inequality (48) satisfies all the conditions of Lemma 1, therefore, we get

$$\begin{aligned} 0 &\leq \limsup_{k\to\infty}\beta_k \leq \limsup_{k\to\infty}\gamma_k \\ \Rightarrow 0 &\leq \limsup_{k\to\infty} \| x_k-\overline{x}_k \|_B \leq \limsup_{k\to\infty}\frac{3\big(M_{\mathcal{F}}|\mu_1-\mu_2|(b-a)\big)}{\big(1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\big)} \\ \Rightarrow 0 &\leq \limsup_{k\to\infty} \| x_k-\overline{x}_k \|_B \leq \frac{3\big(M_{\mathcal{F}}|\mu_1-\mu_2|(b-a)\big)}{\big(1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\big)}. \end{aligned} \tag{49}$$

By (i), we have $\lim_{k\to\infty}x_k = x$. Using this fact and the assumption $\lim_{k\to\infty}\overline{x}_k = \overline{x}$, we get from (49) that

$$\| x-\overline{x} \|_B \leq \frac{3[M_{\mathcal{F}}|\mu_1-\mu_2|(b-a)]}{1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)}. \tag{50}$$

Remark: The concept of "dependence of solutions on parameters" refers to how the properties of a solution change when certain scalar parameters are varied. It is important to note that the initial conditions do not involve any parameters. However, the dependence on parameters plays a crucial role in many physical problems.

IV. EXAMPLE

We consider the following integral equation:

$$x(t) = \frac{t+e^{-t}}{3} + \int_0^1 \frac{3t-2s}{5}\left[\frac{s-\sin\left(x(s)\right)}{2} + \frac{x(0)+x(1)}{3}\right] ds,\ t \in [0,1]. \tag{51}$$

Comparing this equation with proposed equation (1) for $n = 1$, we get

$$\mathcal{G}(t) = \frac{t+e^{-t}}{3} \in C(I = [0,1], \mathbb{R});$$

$$\mathcal{F}\big(t, s, x(s), x(0), x(1)\big) = \frac{3t-2s}{5}\left[\frac{s-\sin\left(x(s)\right))}{2} + \frac{x(0)+x(1)}{3}\right] \in C(I^2 \times \mathbb{R}^3, \mathbb{R}).$$

Now, we have

$$\begin{aligned} |\mathcal{F}\big(t,s,x,x(0),x(1)\big) \quad & -\mathcal{F}\big(t,s,y(s),y(0),y(1)\big)| \\ &= \left|\frac{3t-2s}{5}\left[\frac{s-\sin\big(x(s)\big))}{2} + \frac{x(0)+x(1)}{3}\right] - \frac{3t-2s}{5}\left[\frac{s-\sin\big(y(s)\big))}{2} + \frac{y(0)+y(1)}{3}\right]\right| \\ &\le \left|\frac{3t-2s}{5}\right|\left[\left|\frac{s-\sin\big(x(s)\big))}{2} - \frac{s-\sin\big(y(s)\big)}{2}\right| + \left|\frac{x(0)+x(1)}{3} - \frac{y(0)+y(1)}{3}\right|\right] \\ &\le \left|\frac{3t-2s}{5}\right|\left[\frac{1}{2}\left|\sin\big(x(s)\big) - \sin\big(y(s)\big)\right| + \frac{1}{3}|x(0)-y(0)| + \frac{1}{3}|x(1)-y(1)|\right]. \end{aligned} \tag{52}$$

Taking sup norm, we obtain

$$\begin{aligned} \parallel \mathcal{F}\big(t,s,x,x(0),x(1)\big) - \mathcal{F}\big(t,s,y(s),y(0),y(1)\big) \parallel \quad & \le \sup_{t,s\in I}\left|\frac{3t-2s}{5}\right|\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{3}\right) \parallel x-y \parallel \\ &\le \frac{3}{5}\left(\frac{7}{6}\right) \parallel x-y \parallel \\ &= \frac{7}{10} \parallel x-y \parallel, \end{aligned} \tag{53}$$

Where $M_{\mathcal{F}} = \frac{3}{5}, \quad \alpha = \frac{1}{2}, \quad \beta = \frac{1}{3}, \quad \gamma = \frac{1}{3}.$

Therefore, we the estimate

$$M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a) = \frac{3}{5}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{3}\right)(1-0) = \frac{7}{10} \times 1 = \frac{7}{10} < 1. \tag{54}$$

We define the operator $T: B = C(I, \mathbb{R}) \to B = C(I, \mathbb{R})$ by

$$(Tx)(t) = \frac{t+e^{-t}}{3} + \int_0^1 \frac{3t-2s}{5}\left[\frac{s-\sin\left(x(s)\right)}{2} + \frac{x(0)+x(1)}{3}\right] ds,\ t \in [0,1]. \tag{55}$$

Since, all the conditions of Theorem 2 are satisfied, we can conclude that the sequence x_k associated with the normal $S-$iterative method (4) for the operator T in (55) converges to a unique solution $x \in B$.

Further, we also have for any $x_0 \in B$

$$\begin{aligned} \| x_{k+1} - x \|_B &\leq \frac{[M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)]^{k+1}}{e^{[1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)]\sum_{i=0}^{k}\xi_i}} \| x_0 - x \| \\ &\leq \frac{[\frac{7}{10}]^{k+1}}{e^{\left[1-\frac{7}{10}\right]\sum_{i=0}^{k}\xi_i}} \| x_0 - x \| \\ &\leq \frac{(\frac{7}{10})^{k+1}}{e^{\left(\frac{3}{10}\right)\sum_{i=0}^{k}\xi_i}} \| x_0 - x \| \\ &\leq \frac{(\frac{7}{10})^{k+1}}{e^{\left(\frac{3}{10}\right)\sum_{i=0}^{k}\xi_i}} \| x_0 - x \| \\ &\leq \frac{(\frac{7}{10})^{k+1}}{e^{\left(\frac{3}{10}\right)\sum_{i=0}^{k}\frac{1}{1+i}}} \| x_0 - x \|, \end{aligned} \tag{56}$$

for $\xi_i = \frac{1}{1+i} \in [0,1]$. The estimate obtained from (56) is a bound for the truncation error at the k-th iteration of computation.

Next, we consider the perturbed integral equation:

$$\overline{x}(t) = \frac{t+2e^{-t}}{3} + \int_0^1 \frac{3t-2s}{5}\left[\frac{s-\sin\left(\overline{x}(t)\right)}{2} + \frac{\overline{x}(0)+\overline{x}(1)}{3} - s + \frac{1}{7}\right] ds,\ t \in [0,1]. \tag{57}$$

Similarly, comparing it with the equation (19) for $n = 1$, we have

$$\mathcal{H}(t) = \frac{t+2e^{-t}}{3}, \quad \mathcal{L}\big(t, s, \overline{x}(s), \overline{x}(0), \overline{x}(1)\big) = \frac{3t-2s}{5}\left[\frac{s-\sin\left(\overline{x}(t)\right)}{2} + \frac{\overline{x}(0)+\overline{x}(1)}{3} - s + \frac{1}{7}\right].$$

Now, we define the mapping $\overline{T}: B = C(I, \mathbb{R}) \to B = C(I, \mathbb{R})$ by

$$\left(\overline{T}x\right)(t) = \frac{t+2e^{-t}}{3} + \int_0^1 \frac{3t-2s}{5}\left[\frac{s-\sin\left(\overline{x}(t)\right)}{2} + \frac{\overline{x}(0)+\overline{x}(1)}{3} - s + \frac{1}{7}\right] ds,\ t \in [0,1]. \tag{58}$$

According to Theorem 2, all conditions of the perturbed integral equation are satisfied. Consequently, the sequence $\overline{x}_k$ related to the normal $S-$iterative method (4) for the operator $\overline{T}$ in (58) converges to a unique solution $\overline{x} \in B$.

The estimates below are what we have now:

$$|\mathcal{G}(t) \quad -\mathcal{H}(t)| = \left|\frac{t+e^{-t}}{3} - \frac{t+2e^{-t}}{3}\right| = \left|\frac{t+2e^{-t}-t-2e^{-t}}{3}\right| = \frac{e^{-t}}{3} \leq \frac{1}{3} = \epsilon_1,\ t \in I = [0,1], \tag{59}$$

$$\begin{aligned} |\mathcal{F}\big(t, s, x, x(0), x(1)\big) \quad &-\mathcal{L}\big(t, s, x(s), x(0), x(1)\big)| \\ &= \left|\frac{3t-2s}{5}\left[\frac{s-\sin\left(x(s)\right))}{2} + \frac{x(0)+x(1)}{3}\right] - \frac{3t-2s}{5}\left[\frac{s-\sin\left(x(t)\right)}{2} + \frac{x(0)+x(1)}{3} - s + \frac{1}{7}\right]\right| \\ &= \left|\frac{3t-2s}{5}\right| \left|s - \frac{1}{7}\right| \\ &\leq \frac{18}{35} = \overline{\epsilon}_1. \end{aligned} \tag{60}$$

Let us consider two sequences $\{x_k\}_{k=0}^{\infty}$ and $\{\overline{x}_k\}_{k=0}^{\infty}$ generated by the normal $S-$ iterative method associated with operators T in (55) and $\overline{T}$ in (58), respectively. Here,

x_k approaches a limit x as $k \to \infty$ and $\overline{x}_k$ approaches a limit $\overline{x}$ as $k \to \infty$. Let $\{\xi_k\}_{k=0}^{\infty}$ be a real sequence in the interval $[0,1]$ such that $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$. From Theorem 3, we can conclude that:

$$\begin{aligned} \| x - \overline{x} \|_B &\leq \frac{3[M_{\mathcal{F}}(b-a)\overline{\epsilon}_1 + \epsilon_1]}{1 - M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)} \\ \Rightarrow \| x - \overline{x} \|_B &\leq \frac{3\left[\frac{3}{5}(1-0)\frac{18}{35} + \frac{1}{3}\right]}{1 - \frac{3}{5}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{3}\right)(1-0)} \\ \Rightarrow \| x - \overline{x} \|_B &\leq \frac{\frac{229}{175}}{\frac{3}{10}} = \frac{458}{105}. \end{aligned} \tag{61}$$

The statement highlights the extent to which solutions are influenced by the functions involved and how closely related these solutions are.

Now, we demonstrate how the solutions are dependent on real parameters. Let's examine the integral equations that involve real parameters:

$$x(t) = \frac{t + e^{-t}}{3} + \int_0^1 \frac{3t - 2s}{5}\left[\frac{s - \sin(x(s))}{2} + \frac{x(0) + x(1)}{3}\right] ds + \mu_1,\ t \in [0,1] \tag{62}$$

and

$$x(t) = \frac{t + e^{-t}}{3} + \int_0^1 \frac{3t - 2s}{5}\left[\frac{s - \sin(x(s))}{2} + \frac{x(0) + x(1)}{3}\right] ds + \mu_2,\ t \in [0,1]. \tag{63}$$

Therefore, by using similar arguments and referencing Theorem 4, one can have

$$\begin{aligned} \| x - \overline{x} \|_B &\leq \frac{3[M_{\mathcal{F}}|\mu_1 - \mu_2|(b-a)]}{1 - M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)} \\ \Rightarrow \| x - \overline{x} \|_B &\leq \frac{3\left[\frac{3}{5}|\mu_1 - \mu_2|(1-0)\right]}{\frac{3}{10}} \\ \Rightarrow \| x - \overline{x} \|_B &\leq \frac{3\frac{3}{5}|\mu_1 - \mu_2|}{\frac{3}{10}} \\ \Rightarrow \| x - \overline{x} \|_B &\leq \frac{3\frac{3}{5}|\mu_1 - \mu_2|}{\frac{3}{10}} = \frac{9}{5} \times \frac{10}{3}|\mu_1 - \mu_2| \\ \Rightarrow \| x - \overline{x} \|_B &\leq 6|\mu_1 - \mu_2|. \end{aligned} \tag{64}$$

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