LINEAR DIFFERENTIAL EQUATION WITH FIXED COEFFICIENTS AND ITS APPLICATION

Abstract

In this chapter, we shall discuss a linear differential equation and its forms applied in many fields of engineering. Including the following sections

- Definition of the Linear Differential Equation and Its solution.
- The way to find a Complementary Solution
- The way to find a Particular Solution
- Simple Problems
- Application
- Data

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I. INTRODUCTION

The foremost vital problem in classical as well as quantum physics or mechanics, the solution of linear differential equation is given by classical manner, and in several works of literature, different formal methods are given. In this chapter, we discuss some special methods to find solutions of linear differential equation with constant coefficients, and later we discuss their application, which is very useful in various fields of science and engineering.

Definition and Solution:

1. The linear differential equation with constant coefficients is given by, [1], [3]

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0 y = Q(x), Q(x) \neq 0$$

Or d/dx= D, then $a_n D^n y + a_{n-1} D^{n-1} y + \cdots + a_0 y = Q(x), Q(x) \neq 0 \ldots\ldots$ (1)
Where $a_0, a_{1,} a_2, \ldots\ldots a_n$ are the constants.

This equation is non-homogeneous, If $Q(x) = 0$, then equation (1) becomes,
$a_n D^n y + a_{n-1} D^{n-1} y + \cdots + a_0 y = 0$ ... .(2)

This equation (2) is called a homogeneous equation.
In short equation (1) and (2) are given by *f (D) =Q(x)* and *f (D) =0*

2. Solution: The solution of equation (1) is given by $y = y_c + y_p$, where y_c and y_p are called complementary solution and particular solution.

In case *Q(x) =0* the solution of the equation will be *y = C.F.*

Theorem 1: If *y=y(x)* is the solution of equation (1) then *u(x) =c.y(x)* is also a solution of equation (1), where *c* is any real number. [3]

Theorem 2: If $y = y_1(x)$ and $y = y_2(x)$ are any two solutions of equation (1) and if c_1 and c_2 are any two real numbers, then the solution of equation (1) is also given by [3].
$y = c_1\, y_1(x) + c_2\, y_2(x)$

Theorem 3: If $y = y_1(x)$ and $y = y_2(x)$ ... $y = y_n(x)$ are any *n* solutions of equation
$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = 0$ such that every solution can be written as
$y = c_1\, y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x), x \in (a, b)$
Where $c_1, c_2, c_3 \ldots . c_n$ are constants. [4]

Proof: Given

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = 0$$

Or
$a_0 y^n(x) + a_1 y^{n-1}(x) + \cdots + a_n y = 0$………. (1)

Let x_0 is any point in the interval (a, b) and $y_1(x)$, $y_2(x)$ ... $y_n(x)$ be the solution of (1) satisfying

$y_1(x_0) = 1,\ y_{1'}(x_0) = 0, {y_1}'^{'}(x_0) = 0,\ y_2(x_0) = 0,\ y_{2'}(x_0) = 1, {y_2}'^{'}(x_0) =$
$0 \ldots\ldots {y_n}^{(n)}(x_0) = 1,$……….. (2)

To prove that solution set is linearly independent, we assume that the solution set is linearly dependent. Then, by definition, all the constants $c_1, c_2, c_3 \ldots. c_n$ are not zero, such that.

$c_1\, y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0$ for each $x \in (a,b)$.... (3)
$c_1\, y_{1'}(x) + c_2 y_{2'}(x) + \cdots + c_n {y'}_n(x) = 0$ for each $x \in (a,b)$.... *(4)*
$c_1\, {y_1}''(x) + c_2 {y_2}''(x) + \cdots + c_n {y''}_n(x) = 0$ for each $x \in (a,b)$……(5)
………………………………………………………………
…………………………………………………up to n^{th} order derivative

Using the above equations, we get, $c_1 = 0, c_2 = 0, c_3 = 0 \ldots. c_n = 0$. Which is a contradiction of the fact that all the constants are not zero? Hence, our assumption that solution sets y=$y_1(x)$ and y=$y_2(x)$ ... y=$y_n(x)$ are linearly dependent is wrong, and so the above set of solution is linearly independent.

II. METHOD TO FIND COMPLEMENTARY SOLUTION

1. In equation $f(D) = Q(x)$, we put $D=m$ to obtain the auxiliary equation, which is $f(m) = 0$.
$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_0 = 0$

On solving we get the roots of the equation. Which will provide complementary solution?

2. If the roots of the equation are m_1, m_2, m_3...
Then $C.F = c_1 e^{m1x} + c_2 e^{m2x} + \cdots + c_n e^{mnx}$ [3]
For two values m_1, m_2 it can also represented as
$C.F. = c_1 coshx + c_2 sinhx$, where $coshx = \frac{e^{m1x} + e^{m2x}}{2}$, $sinhx = \frac{e^{m1x} - e^{m2x}}{2}$ [2], [4].

3. If the roots of the equation are the same, i.e., $m_1 = m_2 = m$
$C.F = (c_1 + c_2 x)e^{mx}$ [3]

4. If the roots of the equation are imaginary, i.e. $m = \alpha \pm i\beta$ then
$C.F = e^{\alpha x}\ [A cos\beta x + B sin\beta x]$ [3]

III. METHOD OF FINDING PARTICULAR SOLUTION

1. When $Q(x) = e^{ax}$ [3]
P.I. $= \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}, f(a) \neq 0$

$= x\frac{1}{f'(a)}e^{ax}, f(a) = 0$
$= x^2\frac{1}{f''(a)}e^{ax}, f'(a) = 0$

2. If $Q(x) = sinaxorcosax$[3]

P.I. = $\frac{1}{f(D^2)}sinaxorcosax = \frac{1}{f(-a^2)}sinaxorcosax, f(-a^2) \neq 0$
$= x\frac{1}{f'(-a^2)}sinaxorcosax, f(-a^2) = 0$
$= x^2\frac{1}{f''(-a^2)}e^{ax}, f'(-a^2) = 0$

3. If Q(x)= xm[3]

P.I. = $\frac{1}{f(D)}x^m = [f(D)]^{-1}x^m$ now convert $[f(D)]^{-1}$ in binomial expansion

$(1+D)^{-1} = 1- D+ D^2 - D^3+$…….
$(1 – D)^{-1} = 1+ D+ D^2+ D^3+$…..
$(1+D)^{-2} = 1- 2D + 3D^2-$…….

4. If *Q(x)* = $e^{ax}.F(x)$ [3], then

P.I. = $\frac{1}{f(D)}e^{ax}.F(x) = e^{ax}\frac{1}{f(D+a)}.F(x)$

5. Particular integral of *F(x)* by General method [3]

P.I. = $\frac{1}{f(D-a)}F(x) = e^{ax}\int e^{-ax} F(x)dx$
P.I. = $\frac{1}{f(D+a)}F(x) = e^{-ax}\int e^{ax} F(x)dx$

IV. BASIC PROBLEMS

1. Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx}$+2y = 0 [3]

Solution: Given Differential Equation is,
$\frac{d^2y}{dx^2} - 3\frac{dy}{dx}$+2y = 0

Putting *D*=*m*, we get auxiliary equation $m^2 -3m+2 =0$,
On solving we get *m= -1* and *m =-2* then

Solution of the equation will be y = *C.F.* = $c_1e^{m1x}+c_2e^{m2x}$
$y = c_1e^{-x}+c_2e^{-2x}$

2. Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx}$+ 3y = $e^{2x} + cosx$ [3]

Solution: Given Differential Equation is,

$\frac{d^2y}{dx^2} - 4\frac{dy}{dx}$+3y = $e^{2x} + cosx$

Or Let *d/dx* =*D* then $(D^2 - 4D + 3)y = e^{2x} + cosx$

Putting *D*=*m*, we get auxiliary equation m^2 *-4m+3 =0*,

On solving we get *m= -1* and *m* =-*3*then

Solution of the equation will be $C.F. = c_1e^{m1x}+c_2e^{m2x}$

$C.F. = c_1e^{-x}+c_2e^{-3x}$

Now

$P.I. = \frac{1}{f(D)}Q(x)$

$= \frac{1}{(D^2-4D+3)}e^{2x} + cosx$

$= \frac{1}{(D^2-4D+3)}e^{2x} + \frac{1}{(D^2-4D+3)}cosx$

$= \frac{1}{(2^2-4.2+3)}e^{2x} + \frac{1}{[-(1)^2-4D+3)}cosx$

$=\frac{1}{(4-8+3)}e^{2x}+\frac{1}{(-4D+2)}cosx$

$=\frac{1}{(-1)}e^{2x}+\frac{1}{(-4D+2)}\frac{(-4D-2)}{(-4D-2)}cosx$

$= -e^{2x} + \frac{(-4D-2)}{(16D^2-4)}cosx$

$= -e^{2x} + \frac{(-4D-2)cosx}{(16(-1)-4)}$

$= -e^{2x} + \frac{(-4Dcosx -2cosx)}{-20}$

$= -e^{2x} + \frac{(4sinx -2cosx)}{-20}$

Solution of the equation is given by *y =C.F.+P.I.*

$y = c_1e^{-x}+c_2e^{-3x} + \{-e^{2x} + \frac{(4sinx -2cosx)}{-20}\}$

$y = c_1e^{-x}+c_2e^{-3x} - \left\{e^{2x} + \frac{(4sinx -2cosx)}{20}\right\}$

3. Solve $\frac{d^2y}{dt^2} + \frac{g}{l}y=\frac{g}{l}L$, subject to the condition *y =a*, $\frac{dy}{dt} = 0$ at *t=0,* Where *g, l, L* are constants. [3]

Solution: Given equation is,

$\frac{d^2y}{dt^2} + \frac{g}{l}y= \frac{g}{l}L$

Let *dy/dt =D* then equation can be written as $(D^2+\frac{g}{l}) = \frac{g}{l}L$

Put $D=m$, auxiliary equation will be $m^2+\frac{g}{l}=0$

$m=\pm i\sqrt{\frac{g}{l}}$

$C.F = e^{\alpha x}\ [A\cos\beta x + B\sin\beta x]$

$C.F. = c_1\cos\sqrt{\frac{g}{l}}t + c_2\sin\sqrt{\frac{g}{l}}t$

$P.I = \frac{1}{f(D)}Q(x)$

$= \frac{1}{\left(D^2+\frac{g}{l}\right)}\frac{g}{l}L$

$= \frac{g}{l}L\frac{1}{\left(D^2+\frac{g}{l}\right)}e^{0t}$

$= \frac{g}{l}L\frac{1}{\left(0+\frac{g}{l}\right)}1$

$= L$

Therefore, general solution of the given equation is

$y = C.F. + P.I.$

$y = c_1\cos\sqrt{\frac{g}{l}}t + c_2\sin\sqrt{\frac{g}{l}}t + L$

On applying conditions, $y=a$, $\frac{dy}{dt}=0$ at $t=0$ $c_2=0$ and $c_1=a-L$ then

$y = (a-L)\cos\sqrt{\frac{g}{l}}t$

V. APPLICATIONS

In Mechanics:

1. A beam of length l and f uniform cross section has the differential equation of its elastic curve as $E.I\frac{d^2y}{dx^2} = \frac{w}{2}\left(\frac{l^2}{4}-x^2\right)$, where E is the modulus of elasticity, I is the moment of inertia of cross section , w is weight per unit length and x is measured from the Centre of span, subject to the condition, at $x=0$, $\frac{dy}{dx}=0$[3]

   Solution: Given differential equation is

   $E.I\frac{d^2y}{dx^2} = \frac{w}{2}\left(\frac{l^2}{4}-x^2\right)$

   This equation can be written as,

   $\frac{d^2y}{dx^2} = \frac{w}{2E.I}\left(\frac{l^2}{4}-x^2\right)$

   Let $d/dx = D$, then given equation can be written as $D^2 = \frac{w}{2E.I}\left(\frac{l^2}{4}-x^2\right)$

   Putting $D=m$, auxiliary equation will be $m^2=0$, $m=0, 0$ then

   $C.F. = (c_1+c_2x)e^{mx}$, $C.F. = c_1+c_2x$

$P.I=\frac{1}{f(D)}Q(x)$

$=\frac{1}{D^2}\frac{w}{2E.I}\ (\frac{l^2}{4}- x^2)$
$=\frac{w}{2E.I}\frac{1}{D^2}(\frac{l^2}{4}-x^2)$

On integrating two times we get, $P.I = \frac{w}{2E.I}(\frac{l^2}{4}\frac{x^2}{2} - \frac{x^4}{12})$

Solution of the equation will be,
$y=C.F.+P.I.$
$y= c_1+c_2x+\frac{w}{2E.I}(\frac{l^2x^2}{8} - \frac{x^4}{12})$

On applying conditions at $x=0$, $y=0$ and $dy/dx=0$ then
$c_1=0$, $c_2=0$ then depression in beam will be given by
$y = \frac{w}{2E.I}(\frac{l^2x^2}{8} - \frac{x^4}{12})$
And$y$ is maximum when $x=l$, $y_{max} = \frac{wl^4}{48E.I}$

2. A pendulum of mass 'm' is oscillating under damped position with damping constant 'r' and its equation of motion is given by
$m\frac{d^2y}{dt^2} + r\frac{dy}{dt} + ky = 0,$
Find displacement of an oscillator. [4]

Solution: Given differential equation of motion of oscillator,
$m\frac{d^2y}{dt^2} + r\frac{dy}{dt} + ky = 0,$
$\frac{d^2y}{dt^2} + \frac{r}{m}\frac{dy}{dt} + \frac{k}{m}y = 0,$
Now let $\frac{r}{m} = 2b$, $\frac{k}{m} = n^2$, let us now consider b is damping coefficient and $n = \sqrt{\frac{k}{m}}$ is frequency.

Then equation will be
$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + n^2y = 0$
Let $d/dt=D$, then
$(D^2 + 2bD + n^2)y = 0$

Putting $D=m$, auxiliary equation will be
$m^2 + 2bm + n^2 = 0$

On solving we get,
$m = -b \pm \sqrt{b^2 - n^2}$

Therefore, displacement of an oscillator will be
$C.F. = c_1e^{m1x}+c_2e^{m2x}$
$y=C.F. = c_1e^{(-b+\sqrt{b^2-n^2})t}+c_2e^{(-b-\sqrt{b^2-n^2})t}$
$y=c_1e^{(-b+p)t}+c_2e^{(-b-p)t}$, where $p=\sqrt{b^2-n^2}$

3. A mass of 1 slug stretches a spring 2 ft and comes to rest at equilibrium. The system is attached to a dashpot that imparts a damping force equal to eight times the instantaneous velocity of the mass. Find the equation of motion if an external force equal to $f(t) =8sin4t$ is applied to the system beginning at time $t=0$. What is the transient and steady solution?

Solution: Here $mg=ks$, $1.(32)=2k$, $k=16$,
The differential equation of the above problem is given by,
$m x''+ r x'+ k x=f(t)$
$x''+8x'+16x=8sin4t.$

$$\frac{d^2x}{dt^2}+8\frac{dx}{dt}+16x=8sin4t$$

Let $d/dt=D$, then
$(D^2+8D+16)x=8sin4t$

Putting $D=m$, auxiliary equation is
$m^2+8m+16=0$, $(m+4)^2=0$, $m= -4,-4$
$C.F. = (c_1+c_2t)\, e^{mt}$
$C.F. = (c_1+c_2t)\, e^{-4t}$
$C.F. = c_1e^{-4t}+c_2t.e^{-4t}$
$P.I=\frac{1}{f(D)}f(t)$
$=\frac{1}{(D^2+8D+16)}8sin4t$
$=\frac{1}{(-16+8D+16)}8sin4t$
$=\frac{1}{-8D}8sin4t$
$=\frac{-cos4t}{4}$

The solution of the equation is
$x =C.F. +P.I.$
$x= c_1e^{-4t}+c_2t.e^{-4t}+\frac{-cos4t}{4}$

The first part $c_1e^{-4t}+c_2t.e^{-4t}$ Is called transient solution and second part $\frac{-cos4t}{4}$ is called steady solution.

4. A 16-pound weight is attached to a 6-foot-long spring. At equilibrium the spring measures 8 feet. If the weight is pushed up and released from at a point 4 feet above the equilibrium position. Find the displacement $y(t)$. if it is further known that surrounding medium offers a resistance numerically equal to the instantaneous velocity.

Solution: Here l = the elongation of the spring after the weight is attached =8 – 6=2ft. using Hooke's law we have 20 = *kl*, 20 = 2*k* so that *k=10 lb./ft.* again, *W =mg* then 16=*m* *32, m = 1/2 slug.* Also here damping factor *b=1*. Using the above information, for free damped motion the differential equation of the vibrations of the given mass on the spring is,[4]

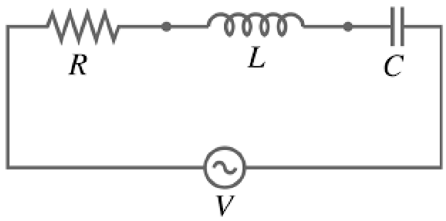
$m\frac{d^2y}{dt^2}+r\frac{dy}{dt}+ky$ =0,
$1/2\frac{d^2y}{dt^2}+\frac{dy}{dt}+10y$ =0,
$(D^2+2D+10)y=0$

Putting *D*=*m*, auxiliary equation is
$m^2+2m+10=0$
On solving we get,
$m=\frac{-2\pm\sqrt{4-40}}{2}=\frac{-2\pm 6i}{2}=-1\pm 3i$

Displacement of weight is given by
$y\,(t)=e^{\alpha t}(A\cos\beta t+B\sin\beta t)$
$y\,(t)=e^{-2t}(A\cos 3t+B\sin 3t)$

In Electric Circuit

1. In LCR circuit; Inductance *L*, Capacitor of capacity *C* and Resistance *R* are connected in series as shown in Figure. [A]



Let applied potential difference in LCR circuit is $V\,(t)=V_0\sin\omega t$, the equation of the charging of capacitor is given by

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=V_0\sin\omega t$$

Find the charge on capacitor and current in the circuit.[3]
Solution: Given differential equation of charging of capacitor is,

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=V_0\sin\omega t$$

It can be written as,

$$\frac{d^2q}{dt^2}+\frac{R}{L}\frac{dq}{dt}+\frac{1}{LC}q=\frac{V_0}{L}\sin\omega t$$

Let $\frac{R}{L}=2b,\frac{1}{LC}=n^2,n=\sqrt{\frac{1}{LC}}$

$$\frac{d^2q}{dt^2} + 2b\frac{dq}{dt} + n^2q = \frac{V_0}{L}\sin\omega t$$

Let $\frac{d}{dt} = D$

$$(D^2 + 2bD + n^2)q = \frac{V_0}{L}\sin\omega t$$

Putting $D=m$, auxiliary equation will be,

$m^2 + 2bm + n^2 = 0$

On solving we get

$m = -b \pm \sqrt{b^2 - n^2}$

$C.F. = c_1e^{m1x} + c_2e^{m2x}$

$C.F. = c_1\, e^{(-b+\sqrt{b^2-n^2})t} + c_2\, e^{(-b-\sqrt{b^2-n^2})t}$

$P.I. = \frac{1}{f(D)}Q(t)$

$= \frac{1}{(D^2+2bD+n^2)}\frac{V_0}{L}\sin\omega t$

$= \frac{V_0}{L}\frac{1}{(-\omega^2+2bD+n^2)} sin\omega t$

$= \frac{V_0}{L}\frac{1}{[(n^2-\omega^2)+2bD]} sin\omega t$

$$= \frac{V_0}{L}\frac{[(n^2-\omega^2)-2bD]sin\omega t}{[(n^2-\omega^2)^2-4b^2D^2]}$$

$$= \frac{V_0}{L}\frac{[(n^2-\omega^2)-2bD]sin\omega t}{[(n^2-\omega^2)^2+4b^2n^2]}$$

$= \frac{V_0}{L}\frac{[(n^2-\omega^2)sin\omega t - 2b\omega cos\omega t}{[(n^2-\omega^2)^2+4b^2n^2]}$

Charge on capacitor is given by,

$q = C.F. + P.I,$

$q = c_1\, e^{(-b+\sqrt{b^2-n^2})t} + c_2\, e^{(-b-\sqrt{b^2-n^2})t} + \frac{V_0}{L}\frac{[(n^2-\omega^2)sin\omega t - 2b\omega cos\omega t}{[(n^2-\omega^2)^2+4b^2n^2]}$

In Method of Variation of Parameter

1. Solve $\frac{d^2y}{dx^2} - 4y = e^{2x}$ [3]

   Solution: Given equation is,

   $\frac{d^2y}{dx^2} - 4y = e^{2x}$

   Let $d/dx = D,$

   $(D^2 - 4)y = e^{2x}$, Here, $R(x) = e^{2x}$

Let $D=m$, then auxiliary equation will be,
$m^2-4=0, m=\pm 2$

$C.F.=c_1e^{m1x}+c_2e^{m2x}$
$C.F.=c_1e^{2x}+c_2e^{-2x}$
$y_1=e^{2x}, y_2=e^{-2x}$
$y_1{}'=2e^{2x}, y_2{}'=-2e^{-2x}$

Now Wronskian determinate is

$$W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix}$$

$$=\begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix}=-4$$

Now $P.I$

$y_p=y_1u+y_2v$

$$u=\int \frac{-y_2R(x)}{W(y_1,y_2)}\,dx,\quad v=\int \frac{y_1R(x)}{W(y_1,y_2)}\,dx$$

$$u=\int \frac{-e^{-2x}e^{2x}}{-4}dx=\frac{x}{4}$$

$$v=\int \frac{e^{2x}e^{2x}}{-4}\,dx=-\frac{e^{4x}}{16}$$

$$y_p=e^{2x}\frac{x}{4}-e^{-2x}\frac{e^{4x}}{16}$$

$$y_p=e^{2x}\frac{x}{4}-\frac{e^{2x}}{16}=$$

$y=y_c+y_p$

$$y=c_1e^{2x}+c_2e^{-2x}+e^{2x}\frac{x}{4}-\frac{e^{2x}}{16}$$

VI. CONCLUSION

In this chapter we discuss brief summary related to the linear differential equation with constant coefficients and its applications. Solutions are given in a simple manner for better understanding of students. I hope this will be very useful to the students and they will also get the chance to work on such very interesting problems.

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