PERFECT EDGE ROMAN DOMINATION IN FUZZY SIMPLE GRAPHS

Abstract

A perfect edge Roman dominating function (PERDF) of a graph $G = (V, E)$ is a function $f: E(G) \to \{0,1,2\}$ which satisfies the rule that every edge x with $f(x) = 0$ is adjacent to exactly one edge y with $f(y) = 2$ so that $\sum_{e \in N(x)} f(e) = 2$. The weight of a PERDF is $\sum_{x \in E(G)} f(x)$. The minimum $\sum_{x \in E(G)} f(x)$ is the perfect edge Roman domination number (PERDN). The symbol $\gamma_R'^P(G)$ is used to denote PERDN. In this paper, we introduce and investigate perfect edge Roman domination in graphs. We obtain strict bounds for PERDN and determine PERDN for some standard graphs.

Keywords: Perfect edge Roman dominating function, Perfect edge Roman domination number.

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I. INTRODUCTION

Let $G=(V,E)$ be a simple connected graph. Let K be a subset of $V(G)$. If each vertex that is not in K is adjacent to a vertex of K, then K is said to be a dominating set. The minimum cardinality of a dominating set of G is the domination number $\gamma(G)$. Mitchell and Hedetniemi [8] introduced edge domination in graphs. A collection M of edges of $E(G)$ form an edge dominating set if every edge of G is either in M or is adjacent to an edge in M. The edge domination number (EDN) is the minimum number of elements in an edge dominating set of G. The symbol $\gamma'(G)$ is used to denote the EDN. The collection M is called a perfect edge dominating set if each edge that is not in M is adjacent to one and only one edge in M.

Motivated by Stewart's article "Defend the Roman Empire" [3], Cockayne et al. [1] introduced Roman dominating function (RDF). The edge version of Roman domination was introduced by Roushini Leely Pushpam et al. [7]. A function $f: E\ (G) \rightarrow \{0,1,2\}$ having the property that each edge e with $f(e)=0$ is adjacent to an edge e_1 with$f(e_1)=2$ is called an edge Roman dominating function (ERDF). The weight of an ERDF is $\sum_{e\ \in E(G)} f(e)$. The minimum weight of an ERDF of G is called the edge Roman domination number $\gamma'_R(G)$. The perfect Roman domination was introduced by Henning et al. [5]. A function $f: V(G) \rightarrow \{0,1,2\}$ is called a perfect Roman dominating function (PRDF) if it satisfies the rule that each vertex v with $f(v)=0$ is adjacent to exactly one vertex u with$f(u)=2$. The weight of a PRDF is $\sum_{u\ \in V(G)} f(u)$. The minimum $\sum_{u\ \in V(G)} f(u)$ is called the perfect Roman domination number γ_R^p(G).

Chellali et al. [6] introduced Roman {2}-domination. Henning and Klostermeyer [4] renamed it as Italian domination. We have introduced the edge version of Italian domination in graphs in [9] and its perfect version in [10]. A function $f: E\ (G) \rightarrow \{0,1,2\}$ which has the property that every edge x with $f(x)=0$ is adjacent to an edge y with $f(y)=2$ or is adjacent to at least two edges z_1 and $z_2$with $f(z_1)=f(z_2)=1$ is called an edge Italian domination function (EIDF). The weight of an EIDF is $\sum_{x\ \in E(G)} f(x)$. The minimum $\sum_{x\ \in E(G)} f(x)$ is the edge Italian domination number γ_I'(G). If the function $f: E\ (G) \rightarrow \{0,1,2\}$ satisfies the rule that every edge x with $f(x)=0$ is adjacent to exactly one edge y with $f(y)=2$ or is adjacent exactly two edges z_1 and z_2 with $f(z_1)=f(z_2)=1$, then f is called perfect edge Italian domination function (PEIDF). The weight of a PEIDF is $\sum_{x\ \in E(G)} f(x)$. The minimum $\sum_{x\ \in E(G)} f(x)$ is the perfect edge Italian domination number $\gamma_I'^P(G)$.

We now review some results which are used in the sequel.

Theorem 1.1: [9] For the path graph P_n, $\gamma'_R(P_n)=\left\lceil \frac{2n}{3} \right\rceil$ and for the cycle graph C_n, $\gamma'_R(C_n)=\left\lceil \frac{2n}{3} \right\rceil$.

Theorem 1.2: [7] For every graph G, $\gamma'(G) \leq \gamma_I'(G) \leq \gamma_R'(G)$.

II. PERFECT EDGE ROMAN DOMINATION

In this paper we introduce and investigate the edge variant of the perfect Roman dominating function. A perfect edge Roman dominating function (PERDF) of a graph $G = (V, E)$ is a function $f: E(G) \rightarrow \{0,1,2\}$ which satisfies the rule that every edge e with $f(e) = 0$ is adjacent to exactly one edge e' with $f(e') = 2$ so that $\sum_{x \in N(e)} f(x) = 2$. The minimum weight of a PERDF is the perfect edge roman domination number (PERDN) $\gamma_R'^P(G)$.

Let E_0, E_1, E_2 be the partitions of the edge set E, such that $E_i = \{x \in E: f(x) = i\}$ for $i = 0, 1, 2$. Then,

- none of the edges of E_0 is adjacent to an edge of E_1
- every edge of E_0 is adjacent to exactly one edge of E_2

Proposition 2.1: For every graph G, $\gamma_I'^P(G) \leq \gamma_R'^P(G)$.
Proof: It is immediate from the definition that every perfect edge Roman dominating function is a perfect edge Italian dominating function. Hence, $\gamma_I'^P(G) \leq \gamma_R'^P(G)$.

Proposition 2.2: For any graph G, $\gamma'(G) \leq \gamma_I'(G) \leq \gamma_R'(G) \leq \gamma_R'^P(G)$.
Proof: Every perfect edge Roman dominating function is an edge Roman dominating function and hence $\gamma_R'(G) \leq \gamma_R'^P(G)$. Also, by Theorem 1.4, $\gamma'(G) \leq \gamma_I'(G) \leq \gamma_R'(G)$. Thus, we get $\gamma'(G) \leq \gamma_I'(G) \leq \gamma_R'(G) \leq \gamma_R'^P(G)$.

Proposition 2.3: For a connected graph G on n vertices, $1 \leq \gamma_R'^P(G) \leq \frac{n(n-1)}{2}$.
Proof. If G has only 1 edge, in any PERDF on G this edge gets the weight 1. So, $\gamma_R'^P(G) \geq 1$. A connected graph G on n vertices can have at most $\frac{n(n-1)}{2}$ edges. In a PERDF on G each edge can get the weight 1. In that case $\gamma_R'^P(G) \leq \frac{n(n-1)}{2}$.

Theorem 2.4: For a graph G on n vertices with $n \geq 3$, $\gamma_R'^P(G) = 2$ if and only if $\gamma'(G) = 1$.
Proof: Suppose $\gamma_R'^P(G) = 2$. Then three cases arise.

Case 1: If G has exactly two edges, then G is isomorphic to P_3 and so $\gamma'(G) = 1$.

Case 2: If G has exactly three edges, then G is isomorphic to P_4, $K_{1,3}$ or C_3. In all cases, $\gamma'(G) = 1$.

Case 3: If G has more than three edges, since $\gamma_R'^P(G) = 2$ there exists an edge $e = uv$ with $f(e) = 2$. Then all other edges are incident at u or v and hence get the weight 0. So e is the only edge in the minimum dominating set. Hence, $\gamma'(G) = 1$.

Conversely let us assume that $\gamma'(G) = 1$. So, the minimum edge dominating set of G has exactly one edge say $e = uv$ and all other edges are incident at u or v. If G has only 2 edges e and e_1 then we can get a $\gamma_R'^P$-function either by assigning the weight 1 to both the edges or by assigning the weights 2 and 0 to the edges e and e_1 respectively. In any case $\gamma_R'^P(G) = 2$. If G has more than 2 edges we can get a $\gamma_R'^P$-function by assigning the weight 2 to e and the weight 0 to all other edges incident at u or v. Therefore, $\gamma_R'^P(G) = 2$.

The following results are immediate from Theorem 2.4

Proposition 2.5: For $n \geq 2$, $\gamma_R'^P(K_{1,n}) = 2$.

Proposition 2.6: For the bistar $B_{m,n}$ we have $\gamma_R'^P(B_{m,n}) = 2$.

Theorem 2.7: For a path graph, $P_n, n \geq 3$, $\gamma_R'^P(P_n) = \begin{cases} \frac{2n}{3}, & if\ n \equiv 0(mod3) \\ \frac{2(n-1)}{3}, & if\ n \equiv 1(mod3) \\ \frac{2(n+1)}{3}, & if\ n \equiv 2(mod3) \end{cases}$

Proof: Let $P_n = (v_1, e_1, v_2, e_2, \dots\dots.., v_{n-1}, e_{n-1}, v_n)$; $e_i = \{v_i, v_{i+1}\}$ be a path on n vertices.

Case (i): If $n \equiv 0(mod3)$

Define $f: E(G) \to \{0,1,2\}$ such that

$$f(e_i) = \begin{cases} 2, & for\ i = 3_j - 1,\ j = 1,2,3\dots, \frac{n}{3} \\ 0, & otherwise \end{cases}$$

Hence, $\sum f(e) \leq \frac{2n}{3}$.

To obtain the lower bound, consider an edge $e \in E(P_n)$. Then e has at most two neighbours. Let f be a $\gamma_R'^P$-function on P_n and let $f(e) = 0$. Then e must be adjacent to exactly one edge of weight 2. Also, the other edge adjacent to e (if it exists) must be given the weight 0 otherwise it contradicts the definition of PERDF. Thus, every three consecutive edges of the path contribute weight 2 to $\sum f(e)$. Since P_n has $n - 1$ edges and $n \equiv 0(mod3)$, $\frac{n-3}{3}$ edges can get the weight 2 and the remaining two edges together contribute at least 2 to $\sum f(e)$.

So, $\sum f(e) \geq \left(\frac{n-3}{3}\right)2 + 2 = \frac{2n}{3}$. Therefore, $\gamma_R'^P(P_n) = \frac{2n}{3}$.

Case (ii): If $n \equiv 1(mod3)$

Define $f: E(G) \to \{0,1,2\}$ such that $f(e_i) = \begin{cases} 2, & for\ i = 3_j - 1,\ j = 1,2,3\dots, \frac{n-1}{3} \\ 0, & otherwise \end{cases}$.

Then, $\sum f(e) \leq \frac{2(n-1)}{3}$.

Now consider a $\gamma_R^{\prime P}$-function f on P_n. Then an edge with weight 2 can have at most two adjacent edges with weight 0. Since, $n \equiv 1(mod3)$ and P_n has $(n-1)$ edges, at least $\frac{(n-1)}{3}$ edges must have weight 2. So, $\sum f(e) \geq (\frac{n-1}{3})2$. Hence, we get $\gamma_R^{\prime P}(P_n) = \frac{2(n-1)}{3}$.

Case (iii): If $n \equiv 2(mod3)$

Define $f: E(G) \rightarrow \{0,1,2\}$ such that $f(e_i) = \begin{cases} 2, for\ i = 3_j - 1, j = 1,2,3 \ldots, \frac{n-2}{3} \\ 1, \quad for\ i = n-1\ and\ n-2 \\ 0, \quad otherwise \end{cases}$

So, $\sum f(e) \leq \frac{2(n-2)}{3} + 2 = \frac{2(n+1)}{3}$.

Let f be a $\gamma_R^{\prime P}$-function on P_n. Then an edge with weight 2 can be adjacent to at most two edges with weight 0. Since P_n has $(n-1)$ edges and $n \equiv 2(mod3)$, $\frac{(n-2)}{3}$ edges must have weight 2 and they contribute $\left(\frac{n-2}{3}\right)2$ to $\sum f(e)$ and the remaining one edge can contribute a maximum weight of 2. So, $\sum f(e) \geq \left(\frac{n-2}{3}\right)2 + 2 = \frac{2(n+1)}{3}$. Thus, $\gamma_R^{\prime P}(P_n) = \frac{2(n+1)}{3}$.

The next result follows directly from Theorem 1.1 and Theorem 2.7.

Corollary 2.8: $If\ n \equiv 0\ (mod3)$ and $n \equiv 1\ (mod3)$, then $\gamma_R^{\prime P}(P_n) = \gamma'_R(P_n)$.

Corollary 2.9: For $n \geq 4$, the $\gamma_R^{\prime P}$-function of the path P_n has $E_1 = \emptyset$.

Proof: In a path an edge has at most two neighbours. Consider a $\gamma_R^{\prime P}$-function f on P_n. It follows from the definition of PERDF that an edge having the weight 0 must be adjacent to exactly one edge of weight 2 and no edge of weight 1. We claim that the $\gamma_R^{\prime P}$-function of the path P_n has $E_1 = \emptyset$.

If possible, assume that there exist an edge e in P_n having weight 1.

Case(i) $f(e) = 1$ and e has two neighbours e_1 and e_2.

Then $f(e_1) \neq 0$ and $f(e_2) \neq 0$. Since f is minimum, both e_1 and e_2 must get the minimum positive weight 1. That is $f(e_1) = 1$ and $f(e_2) = 1$. Again, using similar arguments edges adjacent to e_1 and e_2 must be given the minimum positive weight 1 and so on.

Hence in a $\gamma_R^{\prime P}$-function on P_n, if an edge is given the weight 1, then all the edges of the path gets the weight 1. Therefore $\gamma_R^{\prime P}(P_n) = n - 1$, which contradicts Theorem 2.7.

Case (ii) $f(e) = 1$ and e is a pendant edge.

In this case e has only one neighbour say e_1 and $f(e_1) \neq 0$. Since f is minimum, e_1 must get the minimum positive weight 1. Now the edge adjacent to e_1 must also get the minimum positive weight 1 and so on. In this case also a $\gamma_R^{\prime P}$-function on P_n in which an edge is given the weight 1, will have all its edges with weight 1 and hence $\gamma_R^{\prime P}(P_n) = n - 1$, which is again a contradiction to Theorem 2.7. Therefore, a $\gamma_R^{\prime P}$-function on P_n has $E_1 = \emptyset$.

Theorem 2.10: For a cycle, C_n, on n vertices, $\gamma_R'^P(C_n) = \begin{cases} \frac{2n}{3}, & if\ n \equiv 0(mod3) \\ \frac{2n+4}{3}, & if\ n \equiv 1(mod3) \\ \frac{2n+5}{3}, & if\ n \equiv 2(mod3) \end{cases}$

Proof: Let $C_n = (v_1, e_1, v_2, e_2, \dots \dots, v_{n-1}, e_{n-1}, v_n, e_n, v_1)$; $e_i = \{v_i, v_{i+1}\}$ be a cycle on n vertices.

Case (i): If $n \equiv 0(mod3)$

Define $f: E(G) \to \{0,1,2\}$ such that,

$$f(e_i) = \begin{cases} 2, & for\ i = 3_j - 2, j = 1,2,3 \dots \frac{n}{3} \\ 0, & otherwise \end{cases}$$

Hence $\sum f(e) \leq \frac{2n}{3}$.

In C_n an edge has exactly two neighbours. So, in a $\gamma_R'^P$-function f on C_n an edge having the weight 0 must be adjacent to one edge with weight 2 and another with weight 0. So, every three consecutive edges contribute 2 to $\sum f(e)$. Now, since $n \equiv 0(mod3)$ and C_n has n edges we get $\sum f(e) \geq \left(\frac{n}{3}\right) 2 = \frac{2n}{3}$. Thus, $\gamma_R'^P(C_n) = \frac{2n}{3}$.

Case (ii): If $n \equiv 1(mod3)$

Define $f: E(G) \to \{0,1,2\}$ such that

$$f(e_i) = \begin{cases} 2, & for\ i = 3_j - 2, j = 1,2,3, \dots, \frac{n+2}{3} \\ 0, & otherwise \end{cases}$$

So, $\sum f(e) \leq \frac{2(n+2)}{3} = \frac{2n+4}{3}$.

Let f be a $\gamma_R'^P$-function on C_n. Here, C_n has n edges and $n \equiv 1(mod3)$. Applying a similar argument as in the second part of case(i) $(\frac{n-1}{3})$ edges contribute $2(\frac{n-1}{3})$ to $\sum f(e)$. The maximum weight that can be given to the remaining one edge is 2.

So, $\sum f(e) \geq 2\left(\frac{n-1}{3}\right) + 2 = \frac{2n+4}{3}$. Therefore, $\gamma_R'^P(C_n) = \frac{2n+4}{3}$.

Case (iii): If $n \equiv 2(mod3)$

Define $f: E(G) \to \{0,1,2\}$ such that

$$f(e_i) = \begin{cases} 2, & for\ i = 3_j - 2, j = 1,2,3, \dots, \frac{n+1}{3} \\ 1, & for\ i = n \\ 0, & otherwise \end{cases}$$

Then $\sum f(e) \leq \frac{2(n+1)}{3} + 1 = \frac{2n+5}{3}$.

Consider a $\gamma_R'^P$-function f on C_n. Since $n \equiv 2(mod3)$, applying a similar argument as in the second part of the above two cases $(\frac{n-2}{3})$ edges contribute $2(\frac{n-2}{3})$ to $\sum f(e)$. Since f is minimum, the remaining two edges can get at most the weights 2 and 1.

So, $\sum f(e) \geq 2\left(\frac{n-2}{3}\right)+2+1=\frac{2n+5}{3}$. Hence, $\gamma_R'^P(C_n)=\frac{2n+5}{3}$.

From Theorems 2.7 and 1.1, the next result follows.

Corollary 2.11: $\gamma_R'^P(C_n)=\gamma'_R(C_n)$ when $n \equiv 0(mod 3)$.

Proposition 2.12: For a complete bipartite graph $K_{m,n}$ with $m, n \geq 2$,

$$\gamma_R'^P(K_{m,n}) = 2m, if\ m \leq n\ .$$

Proof: Let $V(K_{m,n})$ be partitioned into two sets X and Y with $|X| = m$ and $|Y| = n$. Then, each edge of $K_{m,n}$ has its one end in X and other end in Y. There are n edges incident at each vertex of X and m edges incident at each vertex of Y. A minimum PERDF on $K_{m,n}$ can be obtained by assigning the weight 2 to all the m edges incident at one vertex v of Y and the weight 0 to all the remaining edges of $K_{m,n}$. Then exactly one edge of weight 2 is adjacent to each edge of weight 0. Hence $\gamma_R'^P(K_{m,n}) = 2m$.

Theorem 2.13: For a complete graph K_n with $n \geq 4$, $\gamma_R'^P(K_n)=\frac{n(n-1)}{2}$.

Proof: Let f be a minimum PERDF on K_n.

Claim: No edge of K_n can get the weight 0.

If possible, let $e = v_n v_1$ be an edge with $f(e) = 0$. Then e must be adjacent to an edge say $e_1 = v_1 v_2$ with $f(e_1) = 2$. Then no other edge incident at v can get a positive weight as in that case $\sum f(e)$ will be greater than 2, which contradicts the definition of PERDF. So, all the remaining $n-3$ edges $\{v_1 v_i, i = 3\ to\ n\}$ incident at v must be given the weight 0. Next consider the edges $F = E(G) - \{v_1 v_i\}, i = 2\ to\ n$. Then each $x \in F$ is adjacent to at least one edge of $\{v_1 v_i, i = 2\ to\ n\}$. Thus each $x \in F$ is adjacent to at least one edge having weight 0. So, these x edges cannot get a positive weight as it again contradicts the definition of PERDF. Thus, no edge of K_n can get the weight 0.

Hence the minimum positive weight 1 should be given to each edge of K_n to get a minimum PERDF. Therefore, $\gamma_R'^P(K_n)=\frac{n(n-1)}{2}$.

Remark 2.14: The bound obtained in Proposition 2.3 is sharp as $\gamma_R'^P(K_2) = 1$ and $\gamma_R'^P(K_n)=\frac{n(n-1)}{2}$, $n \geq 4$.

III. CONCLUSION

In this paper we initiate a study on PERDF and obtain PERDN of some simple Fuzzy graphs. We create an inequality chain involving PERDN and other edge domination parameters like EDN, EIDN and ERDN. We also establish sharp bounds for this parameter. Study of this parameter can be extended to other classes of fuzzy graphs.

REFERENCES

[1] E. J Cockayne, P.M. Dreyer Jr., S.M. Hedetniemi, and S.T. Hedetniemi, On Roman domination in graphs, Discrete Math. 278 (2004), 11–22.
[2] F. Harary, Graph Theory, Addison Wesley, Reading Mass, 1969.
[3] I. Stewart, Defend the Roman Empire, Sci. Amer. 281(6) (1999) 136-139
[4] S.Sangeetha,ERFORMANCE ANALYSIS OF GAME THEORY FOR NETWORK SECURITY IN WSN,Sangeetha S, JOURNAL OF CRITICAL REVIEW ,2637-2642
[5] S Sangeetha,Fuzzy graph theory and coloring ,material today proceedings
[6] S.Sangeetha,Turkish Journal of Computer and Mathematics Education Vol.12 No.7 (2021),1171-1176
[7] S.Sangeetha,Basic concepts of fuzzy matrices,IJERT
[8] S.Sangeetha Application of RAM in Dual Hesitant Fuzzy Transportation Problem,journal of algebraic statistics
[9] S.Sangeetha ,Bipolar valued fuzzy B-ideals on B-Algebra, European Journal of Molecular & Clinical Medicine.