ON CONVERGENCE OF 2 – DIMENSIONAL q – ANALOGUESOF JAFARI’S INTEGRAL TRANSFORMATION

Abstract

In this paper, we have extended the newly defined q – analogues of the Jafari’s integral transformation towards its 2 – dimensional integral transformation. The q – analogues of the Jafari’s integral transformation has simple relationship with other two dimensional q – integral transformations. As an application we have found the conditions under which 2 – dimensional q – analogues of Jafari’s integral transformation were convergent.

Keywords: Integral Transform, q – Calculus, Convergence

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I. INTRODUCTION

The theory of quantum calculus i.e. q – calculus [6] which also be defined ascalculus without limits now becoming the important topic in the field of Mathematics and Physicsmainly dealing with the field of Number theory especially in Cryptography, Combinatory,Mechanics , Theory of Relativity and other sciences quantum theory.

In this paper, we have extend the definition of 1 – dimensional q – analoguesJafari's integral transformation towards 2 – dimensional q – analogues and find out its relationshipwith other 2 - dimensional Laplace type q – analogues integral transformations [11,12]. The paperwere arranged as follows.

The paper mainly divided into three parts, in the first part the generalized definition of one dimensional q – analoguesof Jafari's Integral Transformation and some other basic integral transformation definition given, in the second part the generalized definition of 2 – dimensional q – analoguesof Jafari's Integral Transformation and its relationship with some basic integral transformation were explained. In the last part we have proved the conditions for convergence and uniform convergence of 2 – dimensional q – analoguesof Jafari's Integral Transformation.

In the following, we present some basic definitions needed in proving the main results.

II. BASIC DEFINITIONS

1. **Jafari's Integral Transformation**: If a function $f(t)$ which is to be integrable and defined for t ≥ 0 and $p(s) \neq 0$ and $q(s)$ are positive real valued function then its Jafari's integral transformation [5] is given by

$$J\{f(t);s\} = \mathcal{F}(s) = p(s)\int_0^\infty f(t)e^{-q(s)t}dt \tag{1}$$

Provided that the integral exist for $q(s)$

2. **q – analogues of Exponential function:** The q – analogues of exponential function e^t is denoted by $\hat{e}_q(t)$ and $e_q(t)$ and is given by [6]

$$\hat{e}_q(t) = \prod_{i=1}^{\infty}\left(1+(1-q)q^{i-1}t\right) = \sum_{k=0}^{\infty} q^{\binom{k}{2}}\frac{t^k}{[k]_q!} \tag{2}$$

$$e_q(t) = \prod_{i=1}^{\infty}\left(1-(1-q)q^{i-1}t\right)^{-1} = \sum_{k=0}^{\infty}\frac{t^k}{[k]_q!} \tag{3}$$

3. **q – Derivative:** The q – derivative of a function $f(t)$ is denoted by $D_q f(t)$ and is given by [6],

$$D_q f(t) = \frac{d_q f}{d_q t} = \frac{f(qt)-f(t)}{(q-1)t} \tag{4}$$

4. **Laplace type Integral Transformation:** If function $f(t)$ is continuous piecewise and is of exponential order then its Laplace – type integral transformation [14, 15] is given by:

$$\mathcal{L}_\varepsilon\{f(t);s\} = \int_0^\infty \varepsilon'(t)e^{-\Phi(s)\varepsilon(t)}f(t)dt \quad (5)$$

In the above definition,$\Phi(s)$ is a function which is invertible such that$\varepsilon(t) = \int e^{-a(t)}dt$ is exponential function and $a(t)$ is a function which also invertible.

III. TWO – DIMENSIONAL q – ANALOGUES OF JAFARI'S INTEGRAL TRANSFORMATION

In this section, we introduce the extension of q – analogues of Jafari's integral transformation [13] towards 2 – dimensional q – analogues of Javari's integral transformation of along with some properties;

1. **Definition:** We consider the definition of 2 – dimensional q – analogues of Jafari's integral transformation using the definition [13] as;

$$\widehat{\boldsymbol{JJ}}_q[f(x,t)](u,v) = P(u,v)\int_0^\infty\int_0^\infty e_q[-\varepsilon(u,v,x,t)]\,f(x,t)d_qxd_qt \quad \text{--} \quad [A]$$

Where, $\varepsilon(u,v,x,t) = Q(u)x + Q(v)t$are invertible functions with the property that $f(x,t) \in \mathrm{S} = \left\{f(x,t): \exists\, k_1,k_2 > 0, |f(x,t)| < Me^{\frac{|x|}{k_j}}, x \in (-1)^j \times [0,\infty), a.e.\,'t', M > 0\right\}$.and $P(u,v) = P(u)P(v)$

2. **Relationship with Some q – Analogues of Some Integral Transformations**

- **2 – dimensional q – analogues of Laplace transformation:** The two dimensional q – analogues of Laplace transformation [13] of a function $f(x,t)$ can be obtained by taking$Q(u) = u \; and \; Q(v) = v$, $P(u,v) = 1$ in equation [A] gives;

$$\widehat{\boldsymbol{LL}}_q[f(x,t)](u,v) = \int_0^\infty\int_0^\infty e_q[-(ux+vt)]\,f(x,t)d_qxd_qt$$

- **2 – dimensional q – analogues of Elzaki transformation:** The two dimensional q – analogues of Elzaki transformation [13] of a function $f(x,t)$ can be obtained by taking$Q(u) = \frac{1}{u} \; and \; Q(v) = \frac{1}{v}$, $P(u,v) = uv$ in equation [A] gives;

$$\widehat{\boldsymbol{TT}}_q[f(x,t)](u,v) = uv\int_0^\infty\int_0^\infty e_q\left[-(\frac{x}{u}+\frac{t}{v})\right]f(x,t)d_qxd_qt$$

- **2 – dimensional q – analogues of Sumudu transformation:** The two dimensional q – analogues of Sumudu transformation [13] of a function $f(x,t)$ can be obtained by taking $Q(u)=\frac{1}{u}\ and\ Q(v)=\frac{1}{v}\ , P(u,v)=\frac{1}{uv}$ in equation [A] gives;

$$\widehat{\boldsymbol{SuSu}}_q[f(x,t)](u,v)=\frac{1}{uv}\int_0^\infty\int_0^\infty e_q\left[-(\frac{x}{u}+\frac{t}{v})\right]f(x,t)d_qxd_qt$$

- **2 – dimensional q – analogues of Aboodh transformation:** The two dimensional q – analogues of Aboodh transformation of a function $f(x,t)$ can be obtained by taking $Q(u)=u\ and\ Q(v)=v\ , P(u,v)=\frac{1}{uv}$ in equation [A] gives;

$$\widehat{\boldsymbol{AA}}_q[f(x,t)](u,v)=\frac{1}{uv}\int_0^\infty\int_0^\infty e_q[-(ux+vt)]\,f(x,t)d_qxd_qt$$

- **2 – dimensional q – analogues of Pourreza transformation:** The two dimensional q – analogues of Pourreza transformation of a function $f(x,t)$ can be obtained by taking $Q(u)=u^2\ and\ Q(v)=v^2\ , P(u,v)=uv$ in equation [A] gives;

$$\widehat{\boldsymbol{PP}}_q[f(x,t)](u,v)=uv\int_0^\infty\int_0^\infty e_q[-(xu^2+tv^2)]\,f(x,t)d_qxd_qt$$

- **2 – dimensional q – analogues of Mohand transformation:** The two dimensional q – analogues of Mohand transformation of a function $f(x,t)$ can be obtained by taking $Q(u)=u\ and\ Q(v)=v\ , P(u,v)=u^2v^2$ in equation [A] gives;

$$\widehat{\boldsymbol{MM}}_q[f(x,t)](u,v)=u^2v^2\int_0^\infty\int_0^\infty e_q[-(xu+tv)]\,f(x,t)d_qxd_qt$$

- **2 – dimensional q – analogues of Sawi transformation:** The two dimensional q – analogues of Sawi transformation of a function $f(x,t)$ can be obtained by taking $Q(u)=\frac{1}{u}\ and\ Q(v)=\frac{1}{v}\ , P(u,v)=\frac{1}{u^2v^2}$ in equation [A] gives;

$$\widehat{\boldsymbol{SS}}_q[f(x,t)](u,v)=\frac{1}{u^2v^2}\int_0^\infty\int_0^\infty e_q\left[-(\frac{x}{u}+\frac{t}{v})\right]f(x,t)d_qxd_qt$$

In the similar manner, by substitution of various values of $Q(u), Q(v)\ and\ P(u,v)$ one can obtain the relationship with q – analogues of Natural Transformation, and q – analogues of G_Transformation of order α.

IV. CONVERGENCE OF TWO – DIMENSIONAL q – ANALOGUES OF JAFARI'S INTEGRAL TRANSFORMATION

Theorem 1:

If $f(x,t)$ is continuous on $[0,\infty)\times[0,\infty)$ and integral converges at $\mathrm{Q}(u_0)$ and $\mathrm{Q}(v_0)$. Then the two – dimensional q – analogues of Jafari's Integral transform of $f(x,t)$ converges on for $\mathrm{Q}(u) > Q(u_0)$ and $\mathrm{Q}(v) > Q(v_0)$ where $\varepsilon(u,v,x,t) \geq 0$ in the positive quadrant.

To prove the proof we will use the following lemmas.

Lemma: If $\widehat{\boldsymbol{JJ}}_q[f(x,t)];(u_0) = \int_0^t P(u_o,v)\,f(x,t)e_q[-\mathrm{Q}(u_o)\varepsilon(x,t)]d_qx$ is bounded on $[0,\infty)$ then the two – dimensional q – analogues of Jafari's Integral transform w.r.t u converges for $\mathrm{Q}(u) > Q(u_o)$ and $\varepsilon(x,t) = x \geq 0$ in the positive quadrant such that $\varepsilon(x,t) = x$ bounded in first variable.

Proof: Consider the set

S_1 = $\left\{(x,t){:}\ g(x,t) = P(u_o,v)\int_0^t f(x,t)e_q[-\mathrm{Q}(u_o)\varepsilon(x,t)]d_qx < \infty\right\}$ for $0 < t < \infty$. Then by property of S_1 we have,

$g(x,0) = 0$ and $\lim_{t\to\infty} g(x,t)$ will exist and bounded this is because integral is bounded on $[0,\infty)$

Then by fundamental theorem of calculus, we get

$$g_t(x,t) = P(u_o,v)f(x,t)e_q[-\mathrm{Q}(u_o)\varepsilon(x,t)]$$

Where $P(u_o,v) \neq 0$

Now, weChooseδ_1 and R_1 with $0 < \delta_1 < R_1$, Then the integral

$$I = \int_{\delta_1}^{R_1} P(u_o,v)\,f(x,t)e_q[-\mathrm{Q}(u)\varepsilon(x,t)]d_qx$$

$$= \int_{\delta_1}^{R_1} g_t(x,t)e_q[-[Q(u)-\mathrm{Q}(u_o)]\varepsilon(x,t)]d_qx \text{ with } P(u_o,v) \neq 0$$

Applying integration by parts then the integral turns out to be

$$I = [\left[e_q[-[Q(u)-\mathrm{Q}(u_o)]\varepsilon(x,t)]g(x,t)\right]_{\delta_1}^{R_1}$$

$$+ \int_{\delta_1}^{R_1} [Q(u)-\mathrm{Q}(u_o)]e_q[-[Q(u)-\mathrm{Q}(u_o)]\varepsilon(x,t)]g_t(x,t)]$$

Now let, $\delta_1 \to 0$

$$\Rightarrow I = \left[e_q[-[Q(u) - \mathrm{Q}(u_o)]\varepsilon(x, R_1)]g(x, R_1) + \int_0^{R_1} [Q(u) - \mathrm{Q}(u_o)]e_q[-[Q(u) - \mathrm{Q}(u_o)]\varepsilon(x, R_1)]\, g(x,t)d_qx \right]]$$

Now let $R_1 \longrightarrow \infty$ then $e_q[-[Q(u) - \mathrm{Q}(u_o)]\varepsilon(x, R_1)] \longrightarrow 0$ as $Q(u) > Q(u_o)$ and $\varepsilon(x,t) \geq 0$ in the positive quadrant and bounded in first variable.
Which exist as the integral is bounded $Q(u) > Q(u_o)$.

In the similar manner we can prove that if the integral
$I_1 = \int_0^\infty P(u, v_0)f(x,t)e_q[-\mathrm{Q}(v)\varepsilon(x,t)]d_qt$ Converges at $\mathrm{Q}(v_0)$ then the integral converges for $\mathrm{Q}(v) > \mathrm{Q}(v_0), \varepsilon(x,t) = t \geq 0 \; and \; 0 < x < \infty$. Hence the theorem hold.

Theorem 2:

If $f(x,t)$ is continuous and bounded on $[0,\infty) \times [0,\infty)$ and integral converges at $\mathrm{Q}(u_o) \; and \; \mathrm{Q}(v_o)$. Then the 2 – dimensional q – analogues of Jafari's Integral transform of $f(x,t)$converges uniformly on $[u,\infty) \times [\mathrm{v},\infty)$ if $\mathrm{Q}(u) > Q(u_o) \; and \; \mathrm{Q(v)} > Q(v_o)$ where $\varepsilon(x,t) \geq 0$ in the positive quadrant.

To prove the proof we will use the following lemmas.

Lemma: If $\widehat{JJ}_q[f(x,t)];(u_0) = \int_0^t P(u_o,v)\, f(x,t)e_q[-\mathrm{Q}(u_o)\varepsilon(x,t)]d_qx$ is bounded on $[0,\infty)$ then the 2 – dimensional q – analogues of Jafari's Integral transformconverges uniformly on $[u,\infty)$. If $\mathrm{Q}(u) > Q(u_o)$ and $\varepsilon(x,t) \geq 0$ in the positive quadrant and bounded in first variable.

$S_1 = \left\{(x,t)\colon g(x,t) = P(u_o,v) \int_0^t f(x,t)e_q[-\mathrm{Q}(u_o)\varepsilon(x,t)]d_qx < \infty\right\}$ for $0 < t < \infty$. Then by property of S_1 we have,

$g(x,0) = 0 \; and \; \lim_{t\to\infty} g(x,t)$ will exist and bounded this is because integral is bounded on $[0,\infty)$

So by fundamental theorem of calculus, we get
$g_t(x,t) = P(u_o,v)f(x,t)e_q[-\mathrm{Q}(u_o)\varepsilon(x,t)]$, where $P(u_o,v) \neq 0$ ----- I

Now, we chooseδ_1 and δ such that $0 < \delta < \delta_1$, then the integral

$$I = \int_\delta^{\delta_1} P(u_o,v)\, f(x,t)e_q[-\mathrm{Q}(u)\varepsilon(x,t)]d_qx$$

$$= \int_{\delta_1}^{\delta_1} e_q[-[Q(u) - Q(u_o)]\varepsilon(x,t)]g_t(x,t)\, d_q x \qquad \text{with } P(u_o, v) \neq 0$$

Applying integration by parts then the integral turns out to be

$$I = \left[\left[e_q[-[Q(u) - Q(u_o)]\varepsilon(x,t)]g(x,t)\right]_{\delta_1}^{R_1} + \int_{\delta_1}^{R_1} [Q(u) - Q(u_o)]e_q[-[Q(u) - Q(u_o)]\varepsilon(x,t)]g(x,t)d_q x \right]$$

$$= [e_q[-[Q(u) - Q(u_o)]\varepsilon(R_1,t)]g(R_1,t) - e_q[-[Q(u) - Q(u_o)]\varepsilon(\delta_1,t)]g(\delta_1,t) + \int_{\delta_1}^{R_1} [Q(u) - Q(u_o)]e_q[-[Q(u) - Q(u_o)]\varepsilon(x,t)]g(x,t)d_q x\,]$$

By property of bounded ness $\exists\, M > 0$ such that

$|g(x,t)| \le M$it gives us;

$$|I| \le \{Me_q[-[Q(u) - Q(u_o)]\varepsilon(R_1,t)] + Me_q[-[Q(u) - Q(u_o)]\varepsilon(\delta_1,t)] + M[Q(u) - Q(u_o)]e_q[-[Q(u) - Q(u_o)]\varepsilon(R_1,t)] + M\,[Q(u) - Q(u_o)]e_q[-[Q(u) - Q(u_o)]\varepsilon(\delta_1,t)]\}$$

Hence by Cauchy's criteria for uniform convergence for the given integral converges uniformly on $[u, \infty)$ under the condition that ; $Q(u) > Q(u_o)$.

In the similar manner we can prove that if the integral $I_1 = \int_0^\infty P(u,v_0)f(x,t)e_q[-Q(v)\varepsilon(x,t)]d_q t$ Converges at $Q(v_0)$ then the integral converges uniformly for $Q(v) > Q(v_0), \varepsilon(x,t) = t \ge 0$ and $0 < x < \infty$. Hence the theorem hold.

V. CONCLUSION

The paper gives the conditions about convergence and uniform convergence of the 2 – D q – analogues of Jafari's Integral Transformation along with its relationship with some other q – Integral transformation.

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