RADIAL RADIO NUMBER AND SOME OTHER LABELING PARAMETERS

Abstract

Let G(V, E) be a simple, connected, and undirected graph. A radial radio labeling ψ of G is an assignment of positive integers to the vertices, such that for any two distinct vertices w, z ∈ V, the inequality d(w, z) + | ψ (w)- ψ (z)| ≥ 1 + r holds, where d(w, z) represents the distance between vertices w and z, and r is the radius of the graph G. The span of a radial radio labeling ψ is defined as the highest integer value in the range of ψ and is denoted as span ψ. In this paper, we establish the relationships among the radial radio number, the radio number, and the L(2,1)-labeling number. Furthermore, we construct specific graphs where the radio number equals the algebraic sum of the radial radio number and a given nonnegative integer. Similarly, we provide a proof of the existence of graphs for which the L(2,1)-labeling number is the algebraic sum of the radial radio number and a given nonnegative integer.

Keywords: radial radio number, radio number, $L(2,1) -$ labeling number.

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I. INTRODUCTION

In this paper, we only consider a simple, connected, finite and undirected graph. The radius of G is denoted by r or rad(G) and the diameter of G is denoted by d or diam(G). For further details, one can refer [3].

Graph labeling is an assignment of nonnegative integers, sometimes called colors, to the vertices or edges or both. Motivated by the Frequency Assignment Problem[7], numerous mathematicians introduced various graph labeling concepts. Some of them are discussed in this paper, namely, L(2,1) – labeling[6], radio labeling[4] and radial radio labeling[8].

The concept of **L(2,1)-labeling** was introduced by Griggs and Yeh[6]. It is defined as a function φ: V → {1, 2, 3, ...} that adheres to the following conditions for any two distinct vertices w and z in graph G:

i. $|\varphi(w) - \varphi(z)| \geq 2$ if d(w, z) = 1
ii. $|\varphi(w) - \varphi(z)| \geq 1$ if d(w, z) equals 2.

An L(2,1)-labeling with the additional constraint that no label exceeds the value of k is called as the **k-L(2,1) labeling**. The **L(2,1)-labeling number** of G is denoted by λ(G) and represents the smallest integer value k for which G possesses a k-L(2,1) labeling.

The notion of radio labeling was originally introduced by Chartrand et al[4]. A function ξ: V→ N is considered a **radio labeling** if it adheres to the condition:

$$d(w,z) + |\xi(w) - \xi(z)| \geq 1 + d \tag{1}$$

for any distinct vertices w and z in graph G. This condition is referred to as the radio condition. The span of a radio labeling is the largest integer in the range of ξ and is denoted as span(ξ). The **radio number**, denoted as rn(G), is defined as the minimum span taken over all possible radio labelings of G.

Motivated by the frequency assignment problem[7] and the concept of radio labeling[4], KM. Kathiresan and S. Vimalajenifer introduced a novel graph labeling known as radial radio labeling. A **radial radio labeling**, ψ, is an assignment of positive integers to all vertices in such a way that it satisfies the condition:

$$d(w,z) + |\psi(w) - \psi(z)| \geq 1 + r \tag{2}$$

for any distinct vertices w and z in G. The span of a radial radio labeling, ψ, is the largest integer in the range of ψ and is denoted as span ψ. The **radial radio number**, rr(G), is defined as the minimum span taken over all possible radial radio labelings G. Mathematically, this can be expressed as: $rr(G) = \min_{\psi} span\ \psi$.

Below are listed a few fundamental outcomes that aid in the subsequent advancement of this paper:

Theorem A: $rn(K_n) = n, n \geq 1$.[4]
Theorem B: $\lambda(K_n) = 2n - 1, n \geq 1$.[6]
Theorem C: For any self – centered graph G, $rr(G) \geq |V(G)|$.[1]
Theorem D: If $rad(G) = diam(G)$, then $rr(G) = rn(G)$.[1]
Theorem E: For any simple connected graph G, $rr(G) \geq \omega(G)$, where $\omega(G)$ is the clique number.[1]
Theorem F: For any simple connected graph G, $rn(G) \geq \Delta(d-1)+2$ and $rr(G) \geq \Delta(r-1)+2$, where Δ is the maximum degree in G.[1]

This paper focuses on establishing the correlations among the radial radio number, radio number, and the L(2,1) – labeling number. Moreover, we substantiate the existence of graphs where the radio number is the algebraic sum of its radial radio number and any non-negative integer. Furthermore, we construct graphs wherein the L(2,1) – labeling number is the algebraic sum of its radial radio number and any non-negative integer.

II. RELATIONS CONNECTING RADIAL RADIO NUMBER, RADIO NUMBER AND L (2,1) – LABELING NUMBER

Throughout this section, we solely contemplate simple connected graphs, denoted as G. The first two theorems provide the relationship between $rr(G)$ and $\lambda(G)$. Also, assume that w and z are two distinct vertices of graph G.

Theorem 2.1: If radius of G is 1, then $\boldsymbol{rr(G)} < \lambda(G)$.

Proof: Assume that $r = 1$ and ψ is one of the radial radio labelings of G such that $span\,\psi = rr(G)$. Then by definition, ψ satisfies:

$$d(w,z) + |\psi(w) - \psi(z)| \geq 1 + r = 2 \tag{3}$$

Inequality (3) implies that,
i. if $d(w,z) = 1$, then $|\psi(w) - \psi(z)| \geq 1$
ii. if $d(w,z) = 2$, then $|\psi(w) - \psi(z)| \geq 0$

From i) and ii), we observe that ψ does not satisfy the L(2,1) – labeling condition so that the set $\{1,2,3 \ldots, rr(G)\}$ of positive integers is not sufficient to label the vertices of G under $L(2,1)$ – labeling condition. This forces that, $rr(G) < \lambda(G)$.

Theorem 2.2: If $\boldsymbol{r} \geq 3$, then $\boldsymbol{rr(G)} > \lambda(G)$.

Proof: Let ψ be a radial radio labeling of G such that $span\,\psi = rr(G)$. By (**) ψ satisfies $|\psi(w) - \psi(z)| \geq 4 - d(w,z)$. If w and z are adjacent, then $|\psi(w) - \psi(z)| \geq 3$ and if w and z are non adjacent, then $|\psi(w) - \psi(z)| \geq 2$. This implies that, $rr(G) = \min_{\psi} \max\{\psi(w): w \in V\} > \lambda(G)$.

Theorem 2.3: If diameter of G is 1, then $\boldsymbol{rn(G)} < \lambda(G)$.

Proof: If d=1, then G must be isomorphic to the complete graph K_n. By Theorems A and B, we obtain that, $rn(G) < \lambda(G)$.

From this theorem, we deduce that:

Corollary 2.4: If diameter of G is 1, then $\boldsymbol{rr(G)} = \boldsymbol{rn(G)} < \lambda(G)$.

Proof: If d=1, then G is self – centered with radius 1 and so $rr(G) = rn(G)$[D]. By Theorem 2.1, we get $rr(G) = rn(G) < \lambda(G)$.

Theorem 2.5: If $\boldsymbol{d} \geq \mathbf{2}$, then $\boldsymbol{rn(G)} \geq \boldsymbol{\lambda(G)}$.

Proof: Assume that, φ and ξ are $L(2,1)$ and radio labeling of G, respectively, such that $\max\{\varphi(w): w \in V\} = \lambda(G)$ and $span(\xi) = rn(G)$. Let $w, z \in V$. Then ξ satisfies the radio condition

$$d(w,z) + |\xi(w) - \xi(z)| \geq 1 + d \tag{4}$$

Case 1: when d=2

i. if $d(w,z) = 1$, then (4) becomes $|\xi(w) - \xi(z)| \geq 2$ and
ii. if $d(w,z) = 2$, then (4) becomes $|\xi(w) - \xi(z)| \geq 1$

Here the statements in i) and ii) are as same as the $L(2,1)$ – labeling conditions and hence $\varphi = \xi$. Thus, in this case, $rn(G) = \lambda(G)$.

Case 2: when d>2

iii. If $d(w,z) = 1$, then (4) becomes $|\xi(w) - \xi(z)| > 2$ and
iv. if $d(w,z) = 2$, then (4) becomes $|\xi(w) - \xi(z)| > 1$

In this case also, ξ satisfies the $L(2,1)$ – labeling conditions. But the strict inequalities in iii) and iv) forces that, $\max\{\xi(w): w \in V\} > \max\{\varphi(w): w \in V\}$. Thus $rn(G) > \lambda(G)$. This completes the proof.

Corollary 2.6: If G is self – centered with $d \geq 2$, then $rr(G) = rn(G) \geq \lambda(G)$.
Overall, from this section, we can assert that:

Theorem 2.7: For any simple connected graph G,

i. if $r = 1$ and $d = 1$, then $rr(G) = rn(G) < \lambda(G)$.
ii. if $r = 1$ and $d = 2$, then $rr(G) < rn(G) = \lambda(G)$.
iii. if G is self – centered with $d = 2$, then $rr(G) = rn(G) = \lambda(G)$.
iv. if G is not self – centered and $r \geq 3$, then $\lambda(G) < rr(G) < rn(G)$.

III. RADIO NUMBER AND RADIAL RADIO NUMBER

Within this section, we demonstrate the existence of graphs in which the radio number equals the algebraic sum of the radial radio number and a specified nonnegative integer.

Theorem 3.1: There is a graph satisfying the condition that $rn(G) = rr(G) + n$, where n=0.

Proof: Let us take $G = K_m, m \geq 2$. Since $r = d = 1$, $rn(G) = rr(G)$.

Theorem 3.2: There is no graph G exists, for which $rn(G) = rr(G) + 1$.

Proof: Since for each self – centered graph G, $rr(G) = rn(G)$.[1] Assume that, G is not self – centered. That is, $d > r$, which implies that, $d \geq r + 1$. As per Theorem F, we have $rn(G) \geq \Delta(d - 1) + 2$ and so

$$rn(G) \geq \Delta r + 2 \quad (5)$$

Also, by Theorem F, we have

$$rr(G) \geq \Delta(r - 1) + 2 \quad (6)$$

$(5) - (6)$ implies that, $rn(G) - rr(G) \geq \Delta$, and so $rn(G) \geq rr(G) + \Delta \geq rr(G) + 2$, since G is connected. Thus, there exists no graph such that, $rn(G) = rr(G) + 1$.

Theorem 3.3: There exists a graph G, for which $rn(G) = rr(G) + n$, where $n \geq 2$.

Proof: Assume that G is constructed by using two copies of K_n, n≥ 2. Let $V = \{x\} \cup \{w_s, z_t; 1 \leq s, t \leq n - 1\}$ and let $E = \{xw_s, xz_t; 1 \leq s, t \leq n - 1\} \cup \{w_s w_j; 1 \leq s \neq j \leq n - 1\} \cup \{z_t z_j; 1 \leq t \neq j \leq n - 1\}$.

Then $r = 1$ and $d = 2$.
We now find the radial radio number for G.

Define ψ: $V \to \mathbb{N}$ such that $\psi(x) = 1; \psi(w_s) = \psi(z_s) = s + 1$, where $1 \leq s \leq n - 1$. We have to prove that, for every pair of vertices w and z of G, ψ satisfies,

$$d(w, z) + |\psi(w) - \psi(z)| \geq 2. \quad (7)$$

Case 1a: Consider the pair $(x, z_t), 1 \leq t \leq n - 1$.

Here, $d(x, z_t) = 1$, for all t, $1 \leq t \leq n - 1$. Since $d(x, z_t) + |\psi(x) - \psi(z_t)| \geq 1 + t \geq 2$, the pair $(x, z_t), 1 \leq t \leq n - 1$ satisfies (7). In a similar manner, we can show that the pair (x, w_t) also satisfies (7), for all $t, 1 \leq t \leq n - 1$.

Case 2a: Consider the pair $(w_s, w_t), 1 \leq s \neq t \leq n - 1$.

Since $d(w_s, w_t) = 1$ and $d(w_s, w_t) + |\psi(w_s) - \psi(w_t)| \geq 2$, the pair (w_s, w_t) satisfies (7) for all s, t, $1 \leq s \neq t \leq n - 1$. Similarly, we can prove that the pair (z_s, z_t) satisfies (3) for all s, t, $1 \leq s \neq t \leq n - 1$.

Case 3a: Consider the pair $(w_s, z_t), 1 \leq s, t \leq n-1$.

In this case, $d(w_s, z_t) = 2$. We have, $d(w_s, z_t) + |\psi(w_s) - \psi(z_t)| \geq 2$. Thus pair $(w_s, z_t), 1 \leq s, t \leq n-1$ also satisfies (7).

From the three cases, we can say that, ψ is a radial radio labeling of G and $span\ \psi = \max_i\{i+1; 1 \leq i \leq n-1\} = n$, which implies that $rr(G) \leq n$. Also, $\omega(G) = n$, by Theorem E, $rr(G) \geq n$. Thus $rr(G) = n$.

Next, we determine the radio number for G. Define $\xi: V \rightarrow \mathbb{N}$ such that $\xi(x) = 1; \xi(z_t) = 2t+2,\ 1 \leq t \leq n-1;\ \xi(w_s) = 2s+1,\ 1 \leq s \leq n-1$. We have to prove that, for every pair of vertices w and z of G, ξ satisfies,

$$d(w, z) + |\xi(w) - \xi(z)| \geq 3. \qquad (8)$$

Case 1b: Consider the pair (x, z_t), $1 \leq t \leq n-1$.

Since $d(x, z_t) = 1$, for all i, $d(x, z_t) + |\xi(x) - \xi(z_t)| = 1 + |1 - (2t+2)| \geq 3$, and hence the pair (x, z_t), satisfies (8), for all $t, 1 \leq t \leq n-1$. Proceeding like this, we can show that the pair (x, w_s), satisfies (8), for all s, $1 \leq s \leq n-1$.

Case 2b: Consider the pair $(w_s, w_t), 1 \leq s \neq t \leq n-1$.

Here $d(w_s, w_t) = 1$. Also, $d(w_s, w_t) + |\xi(w_s) - \xi(w_t)| = |1 - (2s+1-(2t-1))| \geq 3$, the pair (w_s, w_t) satisfies (4) for all s, t, $1 \leq s \neq t \leq n-1$. Similarly, we can prove that the pair (z_s, z_t) satisfies (8) for all s, t, $1 \leq s \neq t \leq n-1$.

Case 3b: Consider the pair $(z_s, w_t), 1 \leq s, t \leq n-1$.

We have, $d(z_s, w_t) = 2$ and $d(z_s, w_t) + |\xi(z_s) - \xi(w_t)| \geq 3$ and so $(z_s, w_t), 1 \leq s, t \leq n-1$ satisfies (8).

From cases 1b, 2b, 3b, we arrive at a conclusion that, ξ is a radio labeling for G. Also, we have $span(\xi) = 2n$, which forces that, $rn(G) \leq 2n$. Since $\Delta(G) = 2n-1$, by Theorem E, $rn(G) \geq (2n-1)(2-1) + 2 = 2n$. Thus $rn(G) = 2n$. Finally, for this graph $rn(G) = 2n$ and $rr(G) = n$, which implies that, $rn(G) = rr(G) + n$. For m=5 the graph G is illustrated in Figure 1. The corresponding radio labeling and radial radio labeling are illustrated in Figure 2 and Figure 3, respectively.

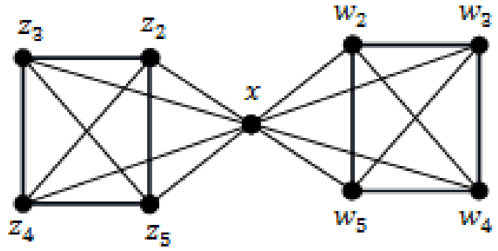

Figure 1

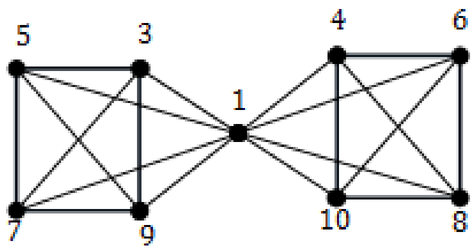

Figure 2

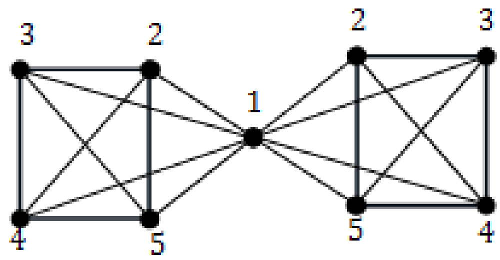

Figure 3

From Theorems 3.1, 3.2 and 3.3, we note that:

Theorem 3.4: For any given non negative integer n, there is a graph G such that $rn(G) = rr(G) + n$, where $n = 0$, $n \geq 2$ and there is no graph G such that $rn(G) = rr(G) + 1$.

IV. L(2,1) – LABELING NUMBER AND RADIAL RADIO NUMBER

In this section, we show the existence of graphs for which the L(2,1) – labeling number is the algebraic sum of the radial radio number and any given non negative integer.

Theorem 4.1: For $0 \leq n \leq 2$, there exists graph G such that $\lambda(G) = rr(G) + n$.

Proof

Case 1: n=0
Take $G = K_{m,m}$, where $m \geq 2$. We have $\lambda(K_{m,m}) = rr(K_{m,m}) = 2m + 1$. In this case, $K_{m,m}$ is the required graph.

Case 2: n=1
Consider G as K_2. We know that, $\lambda(K_2) = 3$ and $rr(K_2) = 2$.

Case 3: n=2
In this case, $K_{1,2}$ is the desired graph, since $\lambda(K_{1,2}) = 4$ and $rr(K_{1,2}) = 2$.

Theorem 4.2: For $n \geq 3$, there exists graph G such that $\lambda(G) = rr(G) + n$.

Proof: Let $V = \{z_1, z_2, z_3, z_4\} \cup \{w_1, w_2, \dots, w_{n-1}\}$ and $E = \{z_s z_t : 1 \leq s \neq t \leq 4\} \cup \{z_1 w_s : 1 \leq s \leq n-1\}$. Then r=1 and d=2.

Define $\varphi: V \to \{1,2,3,\dots\}$ such that $\varphi(z_s) = 2s - 1$, $1 \leq s \leq 4$; $\varphi(w_1) = 4$; $\varphi(w_2) = 6$; $\varphi(w_3) = 8$; $\varphi(w_s) = 5 + s$, $4 \leq s \leq n-1$. We now show that, φ is an L(2,1) – labeling of G.

i. for the pair (z_s, z_t), $1 \leq s \neq t \leq 4$, $|\varphi(z_s) - \varphi(z_t)| \geq 2$
ii. for the pair (z_1, w_s), $1 \leq s \leq n-1$, $|\varphi(z_1) - \varphi(w_s)| > 2$
iii. for the pair (w_s, w_t), $1 \leq s \neq t \leq n-1$, $|\varphi(w_s) - \varphi(w_t)| > 2$
iv. for the pair (z_s, w_t), $2 \leq s \leq 4$, $1 \leq t \leq n-1$, $|\varphi(z_s) - \varphi(w_t)| \geq 1$

Form this discussion, if two vertices of G are adjacent, then the label difference between them is atleast 2 and if two vertices in G are non adjacent, then the label difference between them is atleast 1 and so φ is an L(2,1) – labeling. Also, $\lambda(G) \leq \max_{z \in V} \varphi(z) = n + 4$. We know that, $\lambda(G) \geq \Delta + 2$. Since $\Delta = n + 2$, $\lambda(G) \geq n + 4$. Thus, we conclude that, $\lambda(G) = n + 4$.

We now, determine the radial radio number of G. Define $\psi: V \to \mathbb{N}$ such that $\psi(z_s) = s, 1 \leq s \leq 4$ and $\psi(w_t) = 2, 1 \leq t \leq n - 1$. Here, we note the following:

$|\psi(z_s) - \psi(z_t)| \geq 1, 1 \leq s \neq t \leq 4;$
$|\psi(z_s) - \psi(w_t)| \geq 0, 2 \leq s \leq 4$ and $1 \leq t \leq n - 1;$
$|\psi(w_s) - \psi(w_t)| \geq 0, 1 \leq s \neq t \leq n - 1;$
$|\psi(z_1) - \psi(w_t)| \geq 1, 1 \leq t \leq n - 1.$

This implies that, every pair of vertices of G satisfies (**) and hence ψ is a radial radio labeling of G and $span\,\psi = 4$ and hence $rr(G) \leq 4$. Since $\omega = 4$, by Theorem E, $rr(G) \geq 4$. This gives that, $rr(G) = 4$. Hence G is the required graph.

For m=7, the constructed graph is drawn in Figure 4. The corresponding L(2,1) – labeling and radial radio labeling are shown in Figure 5 and Figure 6, respectively.

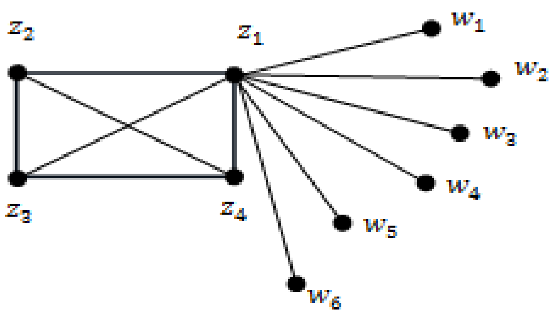

Figure 4

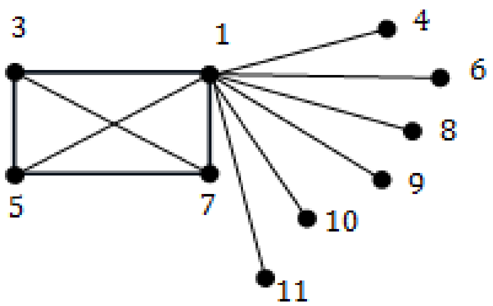

Figure 5

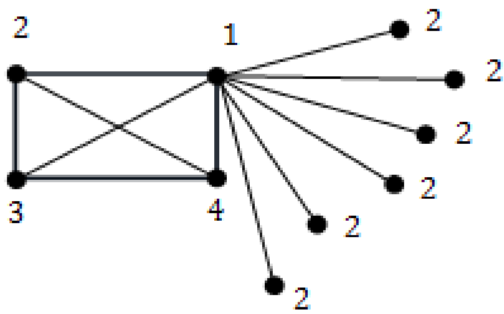

Figure 6

V. CONCLUSION

In this paper, we compared three labeling parameters, which are based on the distance between two vertices of a graph G. Also, we prove the existence of graphs whose radio number and L(2, 1) – labeling number as the algebraic sum of radial radio number and any given non negative integer. In a similar manner, we may compare other graph labeling parameters.

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