LINEAR TRANSFORMATIONS AND MATRICES

Abstract

In this chapter, we study a special class of function known as linear transformation that map the vectors in one vector space to another. This chapter also gives a brief description of how linear transformation and matrices are connected and every matrix is associated with a linear transformation and vice versa. Some examples showing the connection between linear transformation and matrices are also given here.

Keywords: Linear Transformation, Matrices, Basis and Vector space.

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I. INTRODUCTION

Linear algebra plays an important role in mathematics that concerns with linear equations and their representations in the vector space using matrices. When information related to linear functions is presented in well organized form then it results in a matrix. Firstly, an English mathematician James Sylvester introduced the term matrix .Later on the mathematician Arthur Cayley developed the algebraic structure of matrices in 1850s in his two papers. Linear algebra facilitates modelling of many natural phenomena and hence an integral part of engineering and physics.

Definition: Linear Transformation

Let U and V be two vector spaces over the same field F. Then the mapping T from U to V i.e $T: U \to V$ is said to be a linear transformation (linear mapping or vector space homomorphism) if and only if it satisfies the following conditions:

1. $T(u + v) = T(u) + T(v) \forall u, v \in U$
2. $T(\alpha u) = \alpha T(u) \forall u \in U, \alpha \in F$

The conditions (1) and (2) can be combined as
$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v) \forall u, v \in U$ and $\alpha, \beta \in F$.

If T is a linear transformation from U into itself.Then it is known as linear operator. And if we replace V by F, then the mapping is called linear functional.

Example 1: Consider the mapping $T: R^3 \to R^2$ defined by $T(x, y, z) = (x - y, y + z)$.Prove that the T is a linear transformation.

Sol. Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2) \in R^3 = U_3(R)$ and α, β be any two real numbers. The $\alpha u + \beta v = \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2)$

$$= (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) \in R^3.$$

$$T(\alpha u + \beta v) = T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$$

$$= \left((\alpha x_1 + \beta x_2) - (\alpha y_1 + \beta y_2), (\alpha y_1 + \beta y_2) + (\alpha z_1 + \beta z_2)\right) \quad \text{(By definition of T)}$$

$$= (\alpha(x_1 - y_1) + \beta(x_2 - y_2), \alpha(y_1 + z_1) + \beta(y_2 + z_2))$$

$$= \alpha(x_1 - y_1, y_1 + z_1) + \beta(x_2 - y_2, y_2 + z_2)$$

$$= \alpha T(x_1, y_1, z_1) + \beta T(x_2, y_2, z_2)$$

$$= \alpha T(u) + \beta T(v).$$

Therefore $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$, it implies that the mapping T is a linear transformation.

There are many other examples of linear transformations some of which are given below:

1. $T: R^3 \to R$ defined by $T(x, y, z) = 2x + y - 3z$

2. $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x + y, x, x - y)$

But the mapping $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x + 5, x, x - y)$ is not linear transformation because it does not satisfy the conditions. Likewise, $T: \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x, y) = xy$ and $T(x, y) = |x - y|$ are not linear transformations.

Definition: Matrix.A matrix is the arrangement of numbers in rows and columns or when the numbers are arranged in rows and columns so as to form a rectangular array, the pattern so formed is called a matrix.The numbers are called entries or elements of the matrix. In common notation, matrix is denoted by a capital letter and the small letters with double subscript describes the entries of the matrix. If a matrix has m rows and n columns then $m \times n$ is defined as the order of the matrix.

II. MATRIX ASSOCIATED WITH LINEAR TRANSFORMATION

A matrix represents a linear transformation under a fixed basis and vice-versa. Let U and V are two finite dimensional vector spacesover the same field F and$T: U \to V$ is a linear transformation. Here $dimU = n$ and $dimV = m$. Let $B_1 = \{u_1, u_2, \dots, u_n\}$ and $B_2 = \{v_1, v_2, \dots, v_n\}$ be the ordered bases of U and V respectively.

Since $T: U \to V$ is a linear transformation so for every $u \in U$, we have $T(u) \in V$. Since B_2 is a basis of V, so each element $T(u) \in V$ can be expressed as the linear combination of elements of B_2. It implies each $T(u_j) \in V$ where $j = 1,2, \dots n$. Therefore, we have

$$T(u_1) = a_{11}v_1 + a_{21}v_2 + \cdots + a_{m1}v_m$$
$$T(u_2) = a_{12}v_1 + a_{22}v_2 + \cdots + a_{m2}v_m$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$T(u_n) = a_{1n}v_1 + a_{2n}v_2 + \cdots + a_{mn}v_m$$

i.e. $T(u_j) = \sum_{i=1}^{m} a_{ij}\, v_i$ where $a_{ij} \in F$for$i = 1,2, \dots, m$and$j = 1,2, \dots, n$. Then the above equations can be expressed in matrix form as follows:

$$\begin{bmatrix} T(u_1) \\ T(u_2) \\ \vdots \\ T(u_n) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}.$$

So, the coefficient matrix in above form is

$$\begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}.$$

The transpose of the above coefficient matrix is defined as the matrix associated with linear transformation T, relative to basis $B_1$and B_2. Conversely, for any matrix say $A = \left(a_{ij}\right)_{\mathrm{m\times n}}$, $a_{ij} \in R$ we define a mapping $\mathrm{T}: \mathrm{R^n} \to \mathrm{R^m}$ by $T(u) = v$ where $u = \{x_1, x_2, \dots, x_n\} \in \mathrm{R^n}$ and $v = \{y_1, y_2, \dots, y_m\} \in \mathrm{R^m}$ such that

$$
\begin{gathered}
y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
y_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
y_m = a_{m1}v_1 + a_{m2}x_2 + \cdots + a_{mn}x_n
\end{gathered}
$$

Or $Y = AX$. So it is clear that the mapping $\mathrm{T}: \mathrm{R^n} \to \mathrm{R^m}$defined by matrix A is a linear transformation. It implies that corresponds to any given matrix A, there exists a unique linear transformation T. If any linear transformation T maps $\mathrm{R^n}$ to $\mathrm{R^m}$, then there always exist an $m \times n$ matrix A such that $T(X) = AX$.

Example 1: Let $\mathrm{T}: \mathrm{R}^2 \to \mathrm{R}^3$ defined by $T(x, y) = (x + 5y, y, 2x - y)$ is a linear transformation. To find the matrix associated with T,

Let $T(x_1, x_2) = (y_1, y_2, y_3)$ where $(x_1, x_2) \in \mathrm{R}^2$ and $= (y_1, y_2, y_3) \in \mathrm{R}^3$. It gives $(y_1, y_2, y_3) = T(x_1, x_2) = (x_1 + 5x_2, x_2, 2x_1 - x_2)$
Therefore, we get $y_1 = x_1 + 5x_2$

$$
\begin{gathered}
y_2 = 0x_1 + x_2 \\
y_3 = 2x_1 - x_2
\end{gathered}
$$

The above system of equations can be expressed as

$$
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
$$

So the matrix associated with the linear transformation T is $\begin{bmatrix} 1 & 5 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}$. Conversely , one can easily find the linear transformation T associated with matrix A by using similar arguments.

Example 2: To find the linear transformation say $\mathrm{T}: \mathrm{R}^3 \to \mathrm{R}^3$ corresponding to the matrix

$$
A = \begin{bmatrix} 2 & 0 & -4 \\ -5 & 1 & -1 \\ 2 & -7 & 3 \end{bmatrix}.
$$

Let $Y = \mathrm{T(X)}$ where $\mathrm{X}=(x_1, x_2, x_3) \in \mathrm{R}^3$ and $Y = (y_1, y_2, y_3) \in \mathrm{R}^3$.
Then $Y = \mathrm{AX}$.

$$
\Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -4 \\ -5 & 1 & -1 \\ 2 & -7 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - 4x_3 \\ -5x_1 + x_2 - x_3 \\ 2x_1 - 7x_2 + 3x_3 \end{bmatrix}.
$$

$$
\therefore y_1 = 2x_1 - 4x_3, y_2 = -5x_1 + x_2 - x_3, y_3 = 2x_1 - 7x_2 + 3x_3.
$$

So , we get

$$T(x_1, x_2, x_3) = (2x_1 - 4x_3, -5x_1 + x_2 - x_3, 2x_1 - 7x_2 + 3x_3).$$

Example 3: Find a matrix representation of linear transformation T on R^3 defined as

$$T(x, y, z) = (7y + z, x - 5y, 3x - z).$$

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