PRODUCT SIGNED DOMINATING FUNCTION

Abstract

Let $G = (V, E)$ be a simple graph. A function $f: V \rightarrow \{-1,1\}$ is called a product signed dominating function, if $f[v] = 1\ \forall v \in V$ where $f[v] = \prod_{u \in N[v]} f(u)$ and $N[v]$ denotes the closed neighborhood of v.The weight of a graph G with respect to the function f which is hereafter denoted by $w_f(G) = \sum_{v \in V} f(v)$. The minimum positive weight of a product signed dominating function is called product signed domination number of a graph G and is denoted by $\gamma_{sign}^*(G)$. In this paper, we discuss product signed dominating functions for some special graphs.

Keywords: Fan graph, wheel graph, helm graph, flower graph, product signed dominating function, product signed domination number.

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I. INTRODUCTION

The domination problem was studied from 1950s onwards. Richard Karp proved the set cover problem to be NP-complete which had implications for the dominating set problem. Dunbar et al. introduced signed domination number [2],[3],[4],[5]. The concept of product signed domination was introduced in [11].Hereafter, we denote the weight of a graph G with respect to the function f as $w_f(G)$.Definitions of fan graph, wheel graph and helm graph are from [1]. Seoud and Youssef defined flower graph in [1]. In this paper, we find product signed domination number for fan graph, wheel graph, helm graph and flower graph.

II. MAIN RESULTS

1. Theorem

For $n \geq 3, \gamma^*_{sign}(F_{1,n-1}) = \begin{cases} \frac{n-8}{3} \; if \; n \equiv 2(mod\ 6) \; and \; n > 8 \\ \frac{n-6}{3} \; if \; n \equiv 3(mod\ 6) \; and \; n > 8 \\ \frac{n-4}{3} \; if \; n \equiv 4(mod\ 6) \; and \; n > 8 \\ n \; otherwise \end{cases}$

Proof:
Let $F_{1,n-1}$ be a fan graph on n vertices.
Let $V = \{v_1, v_2, \dots, v_{n-1}, v\}$ and $E = \{v_b v_{b+1} | 1 \leq b \leq n-2\} \cup \{v v_b | 1 \leq b \leq n-1\}$

Case 1:$f(v) = 1$
Subcase 1.1:
If $f(v_1) = -1$, to get $f[v_1] = 1$, set $f(v_2) = -1$

Again to get $f[v_2] as\ 1$, set $f(v_3) = 1$

Proceeding like this, we define $f: V(F_{1,n-1}) \rightarrow \{-1, +1\}$ as

For $1 \leq b \leq n, f(v_b) = \begin{cases} 1 \; if b \equiv 0(mod\ 3) \\ -1 \; otherwise \end{cases}$

This f may be a product signed dominating function. If it is, the weight will be negative since $N_f[-1] > N_f[1]$.**[11]**

Subcase 1.2:
If $f(v_1) = 1$, to get $f[v_1] = 1$, set $f(v_2) = 1$. Again to get $f[v_2] as\ 1$, set $f(v_3) = 1$
Proceeding like this, we have $f(v_4) = f(v_5) = \cdots = f(v_{n-1}) = 1$
In this case the weight is n, the total number of vertices,

Case 2:$f(v) = -1$
For $2 \leq b \leq n-2$, it is observed that
If $f(v_b) = -1$ then 2 cases arise

(i) if $f(v_{b-1}) = -1$, then $f(v_{b+1}) = -1$
(ii) if $f(v_{b-1}) = 1$, then $f(v_{b+1}) = 1$
And if $f(v_b) = 1$ then 2 cases arise
(i) if $f(v_{b-1}) = -1$, then $f(v_{b+1}) = 1$
(ii) if $f(v_{b-1}) = 1$, then $f(v_{b+1}) = -1$

Subcase 2.1:
If $f(v_1) = 1$, to get $f[v_1] = 1$, set $f(v_2) = -1$
Again to get $f[v_2] as$ 1, set $f(v_3) = 1$
Again to get $f[v_3] as$ 1, set $f(v_4) = 1$
Proceeding like this, we define $f: V(F_{1,n-1}) \to \{-1, +1\}$ as
For $1 \le b \le n, f(v_b) = \begin{cases} -1 \; if b \equiv 2 (mod\; 3) \\ 1 \; otherwise \end{cases}$

Subcase 2.2:
If$f(v_1) = -1$, to get $f[v_1] = 1$, set $f(v_2) = 1$
Again to get $f[v_2] as$ 1, set $f(v_3) = 1$
Again to get $f[v_3] as$ 1, set $f(v_4) = -1$
Again to get $f[v_4] as$ 1, set $f(v_5) = 1$
Proceeding like this, we define $f: V(F_{1,n-1}) \to \{-1, +1\}$ as
For $1 \le b \le n, f(v_b) = \begin{cases} -1 \; if b \equiv 1 (mod\; 3) \\ 1 \; otherwise \end{cases}$

When $n = 3$
By subcase 1.2, $w_f(F_{1,n-1}) = n = 3$.By subcase 2.1, $w_f(F_{1,n-1}) = -1$, a negative integer.By subcase 2.2, $w_f(F_{1,n-1}) = -1$, a negative integer.Therefore, $\gamma^*_{sign}(F_{1,2}) = 3$.

When $n = 4$
By subcase 1.2, $w_f(F_{1,n-1}) = n = 4$. By subcase 2.1, $w_f(F_{1,n-1}) = 0$. By subcase 2.2, $f[v_3] = -1 \neq 1$. Therefore, $\gamma^*_{sign}(F_{1,3}) = 4$.

When $n = 5$
By subcase 1.2, $w_f(F_{1,n-1}) = n = 5$. By subcase 2.1, $f[v_4] = -1 \neq 1$. By subcase 2.2, $f[v] = -1 \neq 1$. Therefore, $\gamma^*_{sign}(F_{1,4}) = 5$.

When $n = 6$
By subcase 1.2, $w_f(F_{1,n-1}) = n = 6$. By subcases 2.1 and 2.2, $f[v] = -1 \neq 1$. Therefore, $\gamma^*_{sign}(F_{1,5}) = 6$.

When $n = 7$
By subcase 1.2, $w_f(F_{1,n-1}) = n = 7$. By subcase 2.1, $f[v] = -1 \neq 1$. By subcase 2.2, $f[v] = f[v_6] = -1 \neq 1$. Therefore, $\gamma^*_{sign}(F_{1,6}) = 7$.

When $n = 8$
By subcase 1.2, $w_f(F_{1,n-1}) = n = 8$. By subcase 2.1, $f[v] = f[v_7] = -1 \neq 1$. By subcase 2.2, $w_f(F_{1,n-1}) = 0$. Therefore, $\gamma^*_{sign}(F_{1,7}) = 8$.

Consider $n > 8$

For $n \equiv 0(mod6)$
By subcase 1.2, $w_f(F_{1,n-1}) = n$. By subcases 2.1 and 2.2, $f[v] = -1 \neq 1$. Therefore, $\gamma^*_{sign}(F_{1,n-1}) = n$.

For $n \equiv 1(mod6)$
By subcase 1.2, $w_f(F_{1,n-1}) = n$. By subcase 2.1, $f[v] = -1 \neq 1$. By subcase 2.2, $f[v] = f[v_{n-1}] = -1 \neq 1$. Therefore, $\gamma^*_{sign}(F_{1,n-1}) = n$.

For $n \equiv 2(mod6)$
By subcase 1.2, $w_f(F_{1,n-1}) = n$. By subcase 2.1, $f[v] = f[v_{n-1}] = -1 \neq 1$. By subcase 2.2, $w_f(F_{1,n-1}) = \frac{n-8}{3}$. Therefore, $\gamma^*_{sign}(F_{1,n-1}) = min\left\{n, \frac{n-8}{3}\right\} = \frac{n-8}{3}$.

For $n \equiv 3(mod6)$
By subcase 1.2, $w_f(F_{1,n-1}) = n$. By subcase 2.1, $w_f(F_{1,n-1}) = \frac{n-6}{3}$. By subcase 2.2, $w_f(F_{1,n-1}) = \frac{n-6}{3}$. Therefore, $\gamma^*_{sign}(F_{1,n-1}) = min\left\{n, \frac{n-6}{3}, \frac{n-6}{3}\right\} = \frac{n-6}{3}$.

For $n \equiv 4(mod6)$
By subcase 1.2, $w_f(F_{1,n-1}) = n$. By subcase 2.1, $w_f(F_{1,n-1}) = \frac{n-4}{3}$. By subcase 2.2, $f[v_{n-1}] = -1 \neq 1$.Therefore, $\gamma^*_{sign}(F_{1,n-1}) = min\left\{n, \frac{n-4}{3}\right\} = \frac{n-4}{3}$.

For $n \equiv 5(mod\ 6)$
By subcase 1.2, $w_f(F_{1,n-1}) = n$. By subcase 2.1, $f[v_{n-1}] = -1 \neq 1$. By subcase 2.2, $f[v] = -1 \neq 1$. Therefore, $\gamma^*_{sign}(F_{1,n-1}) = n$.

Also from the above discussion, it is clear that, by subcase 2.1, f is not a product signed dominating function when $n \equiv 0,1,2,5(mod\ 6)$ and by subcase 2.2, f is not a product signed dominating function when $n \equiv 0,1,4,5(mod\ 6)$

$$\text{Therefore, } \gamma^*_{sign}(F_{1,n-1}) = \begin{cases} \frac{n-8}{3}\ if\ n \equiv 2(mod\ 6)\ and\ n > 8 \\ \frac{n-6}{3}\ if\ n \equiv 3(mod\ 6)\ and\ n > 8 \\ \frac{n-4}{3}\ if\ n \equiv 4(mod\ 6)\ and\ n > 8 \\ n\ otherwise \end{cases}$$

2. Illustration

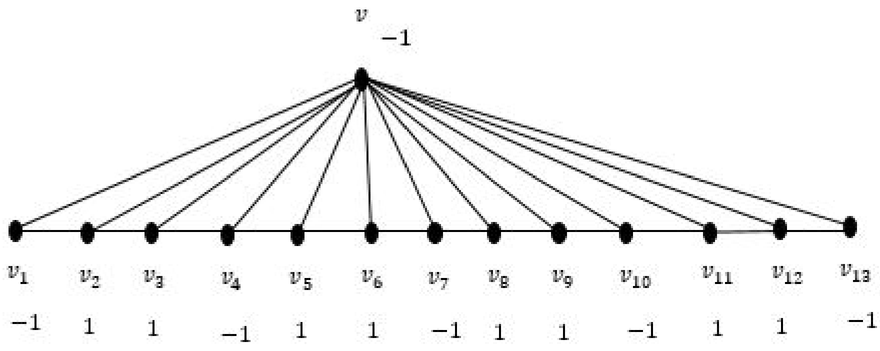

Figure 1

Product signed dominating function for fan graph on $n = 14 \equiv 2(mod\ 6)$ vertices. $\gamma^*_{sign}(F_{1,13}) = \frac{14-8}{3} = 2$.

3. Illustration

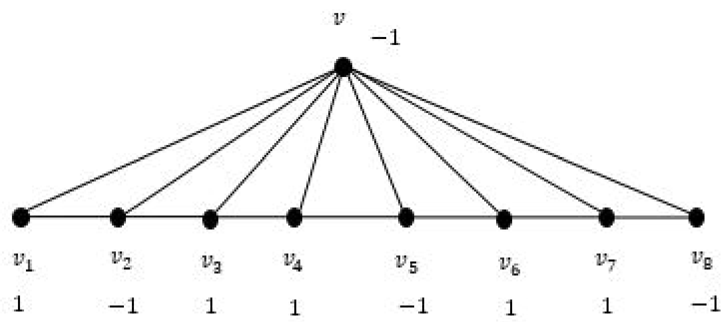

Figure 2

Product signed dominating function for fan graph on $n = 9 \equiv 3(mod\ 6)$ vertices by subcase 2.1 of 1

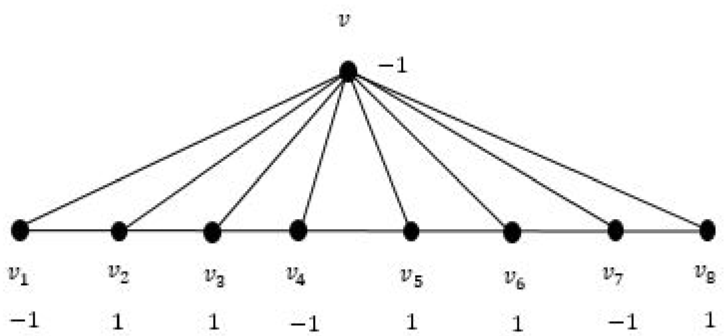

Figure 3

Product signed dominating function for fan graph on $n = 9 \equiv 3(mod\ 6)$ vertices by subcase 2.2 of 1

By subcase 2.1 of 1, $w_f(F_{1,8}) = \frac{9-6}{3} = 1$. By subcase 2.2 of 1, $w_f(F_{1,8}) = \frac{9-6}{3} = 1$. Therefore, $\gamma^*_{sign}(F_{1,8}) = 1$.

4. Illustration

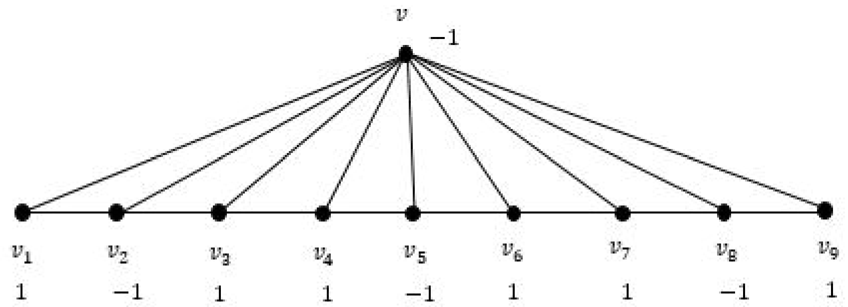

Figure 4

Product signed dominating function for fan graph on $n = 10 \equiv 4(mod\ 6)$ vertices.
$\gamma^*_{sign}(F_{1,9}) = \frac{10-4}{3} = 2$.

5. Theorem

For $n \geq 4, \gamma^*_{sign}(W_n) = \begin{cases} \frac{n-4}{3}\ if\ n \equiv 4(mod\ 6)\ and\ n > 4 \\ n\ otherwise \end{cases}$

Proof:

Let W_n represent a wheel graph on n vertices.

Let $V = \{v_1, v_2, \ldots, v_{n-1}, v\}$ and $E = \{v_b v_{b+1} | 1 \leq b \leq n-2\} \cup \{v v_b | 1 \leq b \leq n-1\} \cup \{v_1 v_{n-1}\}$

Case 1:

$f(v) = 1$

Subcase 1.1:

If $f(v_1) = -1$, to get $f[v_1] = 1$, set $f(v_2) = -1$
Again to get $f[v_2] as$ 1, set $f(v_3) = 1$
Proceeding like this, we define $f: V(W_n) \rightarrow \{-1, +1\}$ as
For $1 \leq b \leq n, f(v_i) = \begin{cases} 1\ if\ b \equiv 0(mod\ 3) \\ -1\ otherwise \end{cases}$
This f may be a product signed dominating function. If it is, the weight will be negative since $N_f[-1] > N_f[1]$.**[11]**

Subcase 1.2:

If $f(v_1) = 1$, to get $f[v_1] = 1$, set $f(v_2) = 1$
Again to get $f[v_2]\ as$ 1, set $f(v_3) = 1$
Proceeding like this, we have $f(v_4) = f(v_5) = \cdots = f(v_{n-1}) = 1$
In this case the weight is n, the total number of vertices,

Case 2:

$f(v) = -1$

For $2 \leq b \leq n-2$, it is observed that
If $f(v_b) = -1$ then 2 cases arise
(i) if $f(v_{b-1}) = -1$, then $f(v_{b+1}) = -1$
(ii) if $f(v_{b-1}) = 1$, then $f(v_{b+1}) = 1$
And if $f(v_b) = 1$ then 2 cases arise
(i) if $f(v_{b-1}) = -1$, then $f(v_{b+1}) = 1$
(ii) if $f(v_{b-1}) = 1$, then $f(v_{b+1}) = -1$

Subcase 2.1:

If $f(v_1) = 1$, to get $f[v_1] = 1$, set $f(v_2) = -1$
Again to get $f[v_2]\ as\ 1$, set $f(v_3) = 1$
Again to get $f[v_3]\ as\ 1$, set $f(v_4) = 1$
Proceeding like this, we define $f: V(W_n) \to \{-1, +1\}$ as
For $1 \leq b \leq n, f(v_b) = \begin{cases} -1\ if\ b \equiv 2(mod\ 3) \\ 1\ otherwise \end{cases}$

Subcase 2.2:

If $f(v_1) = -1$, to get $f[v_1] = 1$, set $f(v_2) = 1$
Again to get $f[v_2]\ as\ 1$, set $f(v_3) = 1$
Again to get $f[v_3]\ as\ 1$, set $f(v_4) = -1$
Again to get $f[v_4]\ as\ 1$, set $f(v_5) = 1$
Proceeding like this, we define $f: V(W_n) \to \{-1, +1\}$ as
For $1 \leq b \leq n, f(v_b) = \begin{cases} -1\ if\ b \equiv 1(mod\ 3) \\ 1\ otherwise \end{cases}$

When $n = 4$

By subcase 1.2, $w_f(W_n) = n = 4$. By subcases 2.1 and 2.2, $w_f(W_n) = 0$. Therefore, $\gamma^*_{sign}(W_4) = 4$.

When $n = 5$

By subcase 1.2, $w_f(W_n) = n = 5$. By subcase 2.1, $f[v] = f[v_1] = -1 \neq 1$. By subcase 2.2, $f[v] = f[v_1] = f[v_4] = -1 \neq 1$. Therefore, $\gamma^*_{sign}(W_5) = 5$.

Consider $n \geq 6$

For $n \equiv 0(mod\ 6)$

By subcase 1.2, $w_f(W_n) = n$. By subcase 2.1, $f[v] = f[v_1] = -1 \neq 1$. By subcase 2.2, $f[v] = f[v_{n-1}] = -1 \neq 1$. Therefore, $\gamma^*_{sign}(W_n) = n$.

For $n \equiv 1(mod\ 6)$

By subcase 1.2, $w_f(W_n) = n$. By subcases 2.1 and 2.2, $f[v] = -1 \neq 1$. Therefore, $\gamma^*_{sign}(W_n) = n$.

For $n \equiv 2(mod\ 6)$

By subcase 1.2, $w_f(W_n) = n$. By subcase 2.1, $f[v] = f[v_{n-1}] = -1 \neq 1$. By subcase 2.2, $f[v_1] = f[v_{n-1}] = -1 \neq 1$. Therefore, $\gamma^*_{sign}(W_n) = n$.

For $n \equiv 3(mod\ 6)$

By subcase 1.2, $w_f(W_n) = n$. By subcase 2.1, $f[v_1] = -1 \neq 1$. By subcase 2.2, $f[v_{n-1}] = -1 \neq 1$. Therefore, $\gamma^*_{sign}(W_n) = n$.

For $n \equiv 4(mod\ 6)$

By subcase 1.2, $w_f(W_n) = n$. By subcases 2.1 and 2.2, $w_f(W_n) = \frac{n-4}{3}$. Therefore, $\gamma^*_{sign}(W_n) = min\left\{n, \frac{n-4}{3}, \frac{n-4}{3}\right\} = \frac{n-4}{3}$.

For $n \equiv 5(mod\ 6)$

By subcase 1.2, $w_f(W_n) = n$. By subcase 2.1, $f[v_{n-1}] = -1 \neq 1$. By subcase 2.2, $f[v] = f[v_1] = f[v_{n-1}] = -1 \neq 1$. Therefore, $\gamma^*_{sign}(W_n) = n$.
Also from the above discussion, it is clear that the functions defined in subcases 2.1 and 2.2 are not product signed dominating functions when $n \equiv 0,1,2,3,5(mod\ 6)$

Therefore, $\gamma^*_{sign}(W_n) = \begin{cases} \frac{n-4}{3}\ if\ n \equiv 4(mod\ 6)\ and\ n > 4 \\ n\ otherwise \end{cases}$

6. Illustration

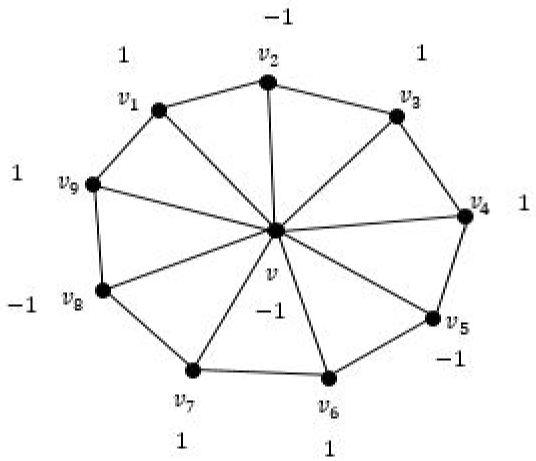

Figure 5

Product signed dominating function for wheel graph on $n = 10 \equiv 4(mod\ 6)$ vertices by subcase 2.1 of 5

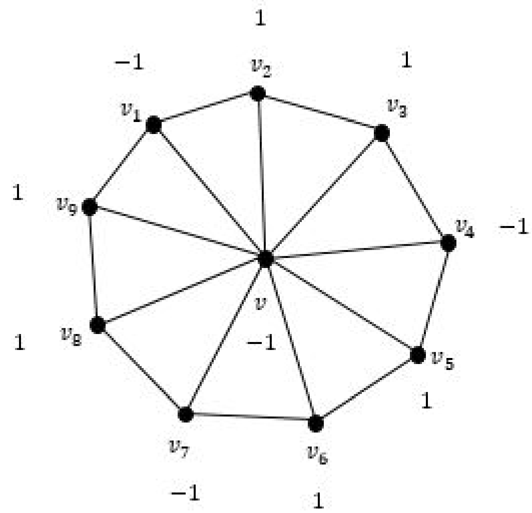

Figure 6

Product signed dominating function for wheel graph on $n = 10 \equiv 4(mod\ 6)$ vertices by subcase 2.2 of 5

By subcase 2.1 of 5, $w_f(W_{10}) = \frac{10-4}{3} = 2$. By subcase 2.2 of 5, $w_f(W_{10}) = \frac{10-4}{3} = 2$. Therefore, $\gamma^*_{sign}(W_{10}) = 2$.

7. Theorem:

Let $n > 3$ be any integer and $G \cong H_n$, a helm graph on $2n-1$ vertices. Then

$$\gamma^*_{sign}(G) = \begin{cases} 1 \ when \ n \equiv 1(mod4) \\ 2n-1 \ otherwise \end{cases}$$

Proof:

Let $V(G) = \{v, v_b, u_b | 1 \le b \le n-1\}$ with $u_b, 1 \le b \le n-1$ as the pendant vertices and $E(G) = \{vv_b | 1 \le b \le n-1\} \cup \{v_b v_{b+1} | 1 \le b \le n-2\} \cup \{v_1 v_{n-1}\} \cup \{v_b u_b | 1 \le b \le n-1\}$

Here v_b and u_b where $1 \le b \le n-1$ must be assigned the same functional value **[11]**.

Let $f(v) = -1$.

To get $f[v]$as 1, odd number of v_b's where $1 \le b \le n-1$ must be assigned -1.

Suppose$n-1$ is even,

Assign -1 to $v_1, v_2, \dots, v_{n-2}$ and take $f(v_{n-1}) = 1$. Correspondingly, $f(u_b) = -1$ for $1 \le b \le n-2$and $f(u_{n-1}) = 1$.

Now $f[v] = 1$ obviously.

$$f[v_{n-1}] = f(v)f(v_{n-1})f(u_{n-1})f(v_{n-2})f(v_1)$$
$$= (-1)(1)(1)(-1)(-1) = -1$$

Hence f is not a product signed dominating function.

Assign -1 to $v_1, v_2, \dots, v_{n-4}$ and 1 to $v_{n-3}, v_{n-2}, v_{n-1}$

Correspondingly, $f(u_b) = \begin{cases} -1, & 1 \le b \le n-4 \\ 1 \ otherwise \end{cases}$

Here also $f[v] = 1$ obviously.

$$f[v_{n-1}] = f(v)f(v_{n-1})f(u_{n-1})f(v_{n-2})f(v_1)$$
$$= (-1)(1)(1)(1)(-1) = 1$$
$$f[v_{n-2}] = f(v)f(v_{n-1})f(v_{n-2})f(u_{n-2})f(v_{n-3})$$
$$= (-1)(1)(1)(1)(1) = -1$$

Hence f is not a valid product signed dominating function.

Assign -1 to $v_1, v_2, \dots, v_{n-6}$ and 1 to $v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}$

Correspondingly, $f(u_b) = \begin{cases} -1, & 1 \le b \le n-6 \\ 1 \ otherwise \end{cases}$

Clearly, here also $f[v] = 1$.

$$f[v_{n-1}] = f(v)f(v_{n-1})f(u_{n-1})f(v_{n-2})f(v_1)$$
$$= (-1)(1)(1)(1)(-1) = 1$$
$$f[v_{n-2}] = f(v)f(v_{n-1})f(v_{n-2})f(u_{n-2})f(v_{n-3})$$
$$= (-1)(1)(1)(1)(1) = -1$$

Hence f is not a valid product signed dominating function.

Continuing like this,
Assign -1 to v_1 and 1 to v_b where $2 \leq b \leq n-1$
Correspondingly, $f(u_b) = \begin{cases} -1 \; if \; b = 1 \\ 1 \; otherwise \end{cases}$
Clearly, $f[v] = 1$.
$f[v_{n-1}] = f(v)f(v_{n-1})f(u_{n-1})f(v_{n-2})f(v_1)$
$= (-1)(1)(1)(1)(-1) = 1$
$f[v_{n-2}] = f(v)f(v_{n-1})f(v_{n-2})f(u_{n-2})f(v_{n-3})$
$= (-1)(1)(1)(1)(1) = -1$

Hence f is not a valid product signed dominating function.
Suppose $n-1$ is odd,
Assign $-1$to $v_1, v_2, \ldots, v_{n-1}$. Correspondingly, $f(u_b) = -1$for $1 \leq b \leq n-1$.
Now $f[v] = 1$ obviously.
$f[v_{n-1}] = f(v)f(v_{n-1})f(u_{n-1})f(v_{n-2})f(v_1)$
$= (-1)(-1)(-1)(-1)(-1)$
$= -1$
Hence f is not a product signed dominating function.
Assign -1 to $v_1, v_2, \ldots, v_{n-4}, v_{n-3}$ and 1 to v_{n-2}, v_{n-1}
Correspondingly, $f(u_b) = \begin{cases} -1, \; 1 \leq b \leq n-3 \\ 1 \; otherwise \end{cases}$
Here also $f[v] = 1$ obviously.
$f[v_{n-1}] = f(v)f(v_{n-1})f(u_{n-1})f(v_{n-2})f(v_1)$
$= (-1)(1)(1)(1)(-1)$
$= 1$
$f[v_{n-2}] = f(v)f(v_{n-1})f(v_{n-2})f(u_{n-2})f(v_{n-3})$
$= (-1)(1)(1)(1)(-1)$
$= 1$
$f[v_{n-3}] = f(v)f(v_{n-2})f(v_{n-3})f(u_{n-3})f(v_{n-4})$
$= (-1)(1)(-1)(-1)(-1)$
$= 1$
$f[v_{n-4}] = f(v)f(v_{n-3})f(v_{n-4})f(u_{n-4})f(v_{n-5})$
$= (-1)(-1)(-1)(-1)(-1)$
$= -1$
Hence f is not a valid product signed dominating function.
Assign -1 to $v_1, v_2, \ldots, v_{n-6}, v_{n-5}$ and 1 to $v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}$
Correspondingly, $f(u_b) = \begin{cases} -1, \; 1 \leq b \leq n-5 \\ 1 \; otherwise \end{cases}$
Clearly, here also $f[v] = 1$.
$f[v_{n-1}] = f(v)f(v_{n-1})f(u_{n-1})f(v_{n-2})f(v_1)$
$= (-1)(1)(1)(1)(-1)$
$= 1$
$f[v_{n-2}] = f(v)f(v_{n-1})f(v_{n-2})f(u_{n-2})f(v_{n-3})$
$= (-1)(1)(1)(1)(1)$
$= -1$
Hence f is not a valid product signed dominating function.
Continuing like this,
Assign -1 to v_1 and 1 to v_b where $2 \leq b \leq n-1$

Correspondingly, $f(u_b) = \begin{cases} -1 \ if \ b = 1 \\ 1 \ otherwise \end{cases}$
Clearly, $f[v] = 1$.
$f[v_{n-1}] = f(v)f(v_{n-1})f(u_{n-1})f(v_{n-2})f(v_1)$
$= (-1)(1)(1)(1)(-1)$
$= 1$
$f[v_{n-2}] = f(v)f(v_{n-1})f(v_{n-2})f(u_{n-2})f(v_{n-3})$
$= (-1)(1)(1)(1)(1)$
$= -1$
Hence f is not a valid product signed dominating function.
Therefore, assigning -1 or 1 to continuous v_b's fails to give a product signed dominating function.

Redefine f as $f(v) = -1$ and
$f(v_b) = \begin{cases} -1 \ if \ b \ is \ odd \\ 1 \ otherwise \end{cases}$
Correspondingly, $f(u_b) = \begin{cases} -1 \ if \ b \ is \ odd \\ 1 \ otherwise \end{cases}$

Now $f[v] = 1$ only when n is odd such that $\frac{n-1}{2}$ is odd.

But here, $f[v_1] = f(v)f(v_1)f(u_1)f(v_2)f(v_{n-1})$

$= (-1)(-1)(-1)(1)(1)$

$= -1$
Therefore this also does not lead to any product signed dominating function.
Assign $f(v) = f(v_1) = -1$. Then $f(u_1) = -1$.

Correspondingly, $f[v_1] = f(v)f(v_1)f(u_1)f(v_2)f(v_{n-1})$
$= (-1)(-1)(-1)f(v_2)f(v_{n-1})$

$= 1$ if and only if $f(v_2)$ and $f(v_{n-1})$ are of opposite sign.
Without loss of generality, assume $f(v_2) = -1$ and $f(v_{n-1}) = 1$

Then $f(u_2) = -1$ and $f(u_{n-1}) = 1$

Correspondingly, $f[v_2] = f(v)f(v_1)f(v_3)f(v_2)f(u_2)$

$= (-1)(-1)f(v_3)(-1)(-1)$
$= 1$ if and only if $f(v_3) = 1$

Let $f(v_3) = 1$. Then $f(u_3) = 1$.

Correspondingly, $f[v_3] = f(v)f(v_3)f(u_3)f(v_2)f(v_4)$

$= (-1)(1)(1)(-1)f(v_4)$

$= 1$ if and only if $f(v_4) = 1$.

Let $f(v_4) = 1$. Then $f(u_4) = 1$.

Repeating the above procedure, $f(v_5) = -1,\ f(v_6) = -1,\ f(v_7) = 1,\ f(v_8) = 1$ and so on.
(i.e) $f(v_b)$ where $1 \leq b \leq n-1$ follows the pattern $-1,-1,1,1$ for every four vertices starting from v_1Therefore, if $n-1 = 4k$, then the function is defined by $f(v) = -1$.
$f(v_{4k+1}) = f(v_{4k+2}) = -1andf(v_{4k+3}) = f\left(v_{4(k+1)}\right) = 1$ for all $k = 0$ to $\frac{n-5}{4}$Correspondingly, $f(u_{4k+1}) = f(u_{4k+2}) = -1$ and $f(u_{4k+3}) = f\left(u_{4(k+1)}\right) = 1$ for all $k = 0$ to $\frac{n-5}{4}$

Now by construction, $f[v_b] = 1\ \forall\ 1 \leq b \leq n-2$.

$f[v_{n-1}] = f(v_{n-2})f(v_{n-1})f(v_1)f(u_{n-1})f(v)$

$= (1)(1)(-1)(1)(-1)$
$= 1$

Also by construction, $f[u_b] = 1\ \forall\ 1 \leq b \leq n-1$.

$$f[v] = f(v)\prod_{b=1}^{n-1} f(v_b)$$

$$= (-1)\prod_{b=1}^{n-1} f(v_b)$$

$= (-1)(1)$

$= -1$
Hence f is not a product signed dominating function.
Suppose for any odd n, if the above pattern of assignment of functional values is followed, then
$f[v] = 1 \Leftrightarrow \prod_{b=1}^{n-1} f(v_b) = -1$
$\Leftrightarrow n - 1 = 4k + 1$

$$\Leftrightarrow n = 4k + 2$$
$$\Leftrightarrow n \equiv 2(mod4)$$

but in this case, $f[v_1] = f(v_1)f(v_2)f(v)f(u_1)f(v_{n-1})$

$$= (-1)(-1)(-1)(-1)(-1)$$
$$= -1$$

Hence f fails to be a product signed dominating function.

Therefore, assigning -1 to v under $f$fails to give a product signed dominating function.

Let $f(v) = 1$.

Assign $f(v_1) = 1$. Then $f(u_1) = 1$.

Correspondingly, $f[v_1] = f(v)f(v_1)f(u_1)f(v_2)f(v_{n-1})$
$= (1)(1)(1)f(v_2)f(v_{n-1})$

$= 1$if and only if $f(v_2)$ and $f(v_{n-1})$ are of same sign.

Suppose $f(v_2) = f(v_{n-1}) = 1$. This procedure leads assigning 1 to all the vertices of G which gives a maximum weight.

So let us assign $f(v_2) = f(v_{n-1}) = -1$. Then $f(u_2) = f(u_{n-1}) = -1$.

Now, $f[v_2] = f(v)f(v_1)f(v_2)f(v_3)f(u_2)$

$= (1)(1)(-1)f(v_3)(-1)$

$= 1$if and only if $f(v_3) = 1$.

Let $f(v_3) = 1$. Then $f(u_3) = 1$.

Now, $f[v_3] = f(v)f(v_2)f(v_3)f(v_4)f(u_3)$

$= (1)(-1)(1)f(v_4)(1)$

$= 1$if and only if $f(v_4) = -1$.

Repeating the above procedure, $f(v_5) = 1,\ f(v_6) = -1,\ f(v_7) = 1,\ f(v_8) = -1$ and so on.

(i.e) $f(v_b)$ where $1 \le b \le n-1$ follows the pattern $1, -1$ for every two vertices starting from v_1Therefore, if $n-1 = 2k$, then the function is defined by $f(v) = 1.\, f(v_{2k+1}) = 1$ and $f\left(v_{2(k+1)}\right) = -1$ for all $k = 0$ to $\frac{n-3}{2}$.Correspondingly, $f(u_{2k+1}) = 1$ and $f\left(u_{2(k+1)}\right) = -1$ for all $k = 0$ to $\frac{n-3}{2}$

Now by construction, $f[v_b] = f[u_b] = 1\ \forall\ 1 \le b \le n-1$.

$$f[v] = f(v)\prod_{b=1}^{n-1} f(v_b)$$

$= (1)(1)(-1)^k$
$= 1$if and only if k is even

$= 1$if and only if $n-1$ is a multiple of 4

Therefore, f is a product signed dominating function when $n \equiv 1 (mod 4)$.

Now, $w_f(G) = \sum_{b=1}^{n-1}[f(u_b) + f(v_b)] + f(v)$

$= 0 + f(v)$
$= 1$

Therefore, $\gamma^*_{sign}(G) = \begin{cases} 1 \text{ when } n \equiv 1(mod4) \\ 2n-1 \text{ otherwise} \end{cases}$

8. Illustration:

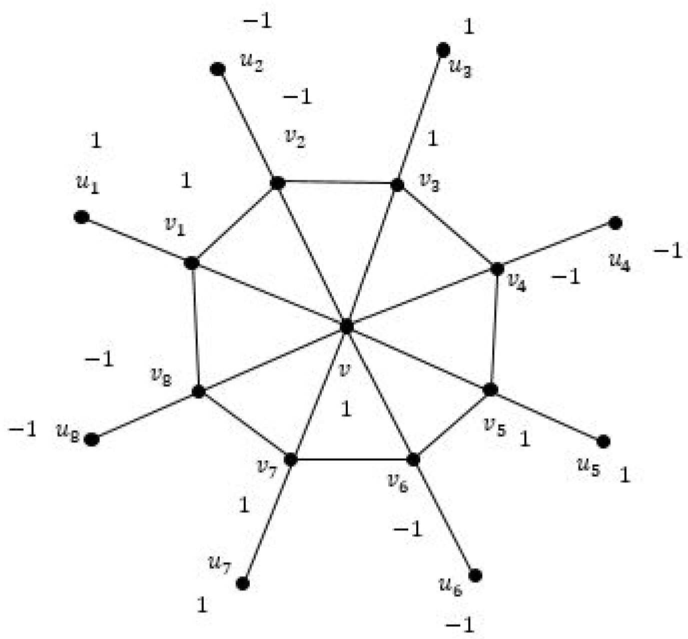

Figure 7

Product signed dominating function for graph $G \cong H_n$ on $n = 9 \equiv 1(mod4)$ vertices.
$\gamma^*_{sign}(G) = 1$.

9. Theorem:

$\gamma^*_{sign}(Fl_n) = \begin{cases} 1 \text{ if } n \text{ is odd} \\ 2n-1 \text{ otherwise} \end{cases}$

Proof:

Let Fl_n represent a flower graph on $2n-1$ vertices.
Let $V = \{v, v_1, v_2, \ldots, v_{n-1}, u_1, u_2, \ldots, u_{n-1}\}$ and $E = \{v_b v_{b+1} | 1 \le b \le n-2\} \cup \{vv_b | 1 \le b \le n-1\} \cup \{vu_b | 1 \le b \le n-1\} \cup \{v_1 v_{n-1}\} \cup \{v_b u_b | 1 \le b \le n-1\}$

Case 1: $f(v) = -1$

Here to get any $f[u_b]$ $(1 \le b \le n-1)$as 1, one of $f(u_b)$, $f(v_b)$ must be equal to 1. But in this case, to get $f[v_b] = 1\ \forall b\ (1 \le b \le n-1)$, f should assign values to u_b and v_b for $1 \le b \le n-1$ such that $\sum_{b=1}^{n-1} f(u_b) + \sum_{b=1}^{n-1} f(v_b) = 0$. Finally, $w_f(Fl_n) = \sum_{v \in V} f(v) = -1$ which is negative.

Further to get $w_f(Fl_n)$ as positive among the remaining $2n-2$ vertices atleast n vertices must get 1under f.

But in this case, if one of v_b for $1 \leq b \leq n-1$gets 1, then $f(v_b) = 1\ \forall b\ (1 \leq b \leq n-1)$ and $f(u_b) = -1\ \forall b\ (1 \leq b \leq n-1)$ so that $f[u_b] = f[v_b] = 1\ \forall b\ (1 \leq b \leq n-1)$.

Subcase 1.1: n is even

Here $n-1$ is odd.

In this case f is a valid product signed dominating function with $w_f(Fl_n)$negative.

Subcase 1.2: n is odd

Then $n-1$ is even.

Here $f[v] = -1$ in which f fails to be a product signed dominating function.

Case 2: $f(v) = 1$

Here for every $b,\ 1 \leq b \leq n-1$, both u_b and v_b must have the same functional value. That is, $f(u_b) = f(v_b) = 1$ or $f(u_b) = f(v_b) = -1, 1 \leq b \leq n-1$.

Suppose $f(u_k) = f(v_k) = 1$ for some $k, 1 \leq k \leq n-1$. Then the neighbor vertices of v_k in the inner cycle $(v_1 v_2 \ldots v_{n-1})$ must get -1 to get minimum weight. --- (I)

At the same time the neighbors of v_{k-1} and v_{k+1} must get 1 (in the inner cycle) so that $f[v_{k-1}] = f[v_{k+1}] = 1$.

Repeating this procedure, the vertices of the inner and outer cycle get 1 and -1 alternately.

Subcase 2.1: n is even

Therefore $n-1$ is odd.

Here the above procedure fails to give a valid product signed dominating function.

Subcase 2.2: n is odd

Here $n-1$ is even.

In this case, the procedure yields a valid product signed dominating function and the corresponding

$$w_f(Fl_n) = f(v) + \sum_{b=1}^{n-1} f(u_b) + \sum_{b=1}^{n-1} f(v_b)$$

$$= 1 + 0$$

$$= 1$$

Hence this f is a product signed dominating function with a positive weight.

As the weight is 1, this is minimum and the corresponding $\gamma^*_{sign}(Fl_n) = 1$.

Further by statement (I) and subcase 2.1, the only product signed dominating function giving positive weight is $f(v) = 1\ \forall v \in V$ when $n-1$ is odd.

Hence $w_f(Fl_n) = |V| = 2n-1$ when n is even and the corresponding$\gamma^*_{sign}(Fl_n) = 2n-1$.

$$Therefore, \quad \gamma^*_{sign}(Fl_n) = \begin{cases} 1 \ \ if\ n\ is\ odd \\ 2n-1\ otherwise \end{cases}$$

10. Illustration:

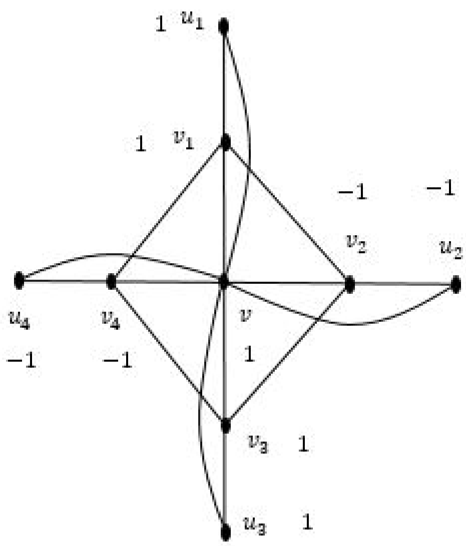

Figure 8

Product signed dominating function for flower graph Fl_5 on $2(5) - 1 = 9$ vertices.
$\gamma^*_{sign}(Fl_5) = 1$.

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