SEMI ANALYTIC-NUMERICAL SOLUTION OF FINGERO - IMBIBITION PHENOMENON IN HETEROGENEOUS POROUS MEDIUM WITH MAGNETIC FIELD EFFECT

Abstract

The present paper studies the magnetic field effect on Fingero- Imbibition phenomenon in a double phase flow in a heterogeneous porous media. Fingero-Imbibition Phenomenon is due to the occurrence of two key phenomena, namely imbibition and fingering. Imbibition is the displacement of a non-wetting fluid by a wetting fluid in a porous medium, where the non-wetting fluid is oil and the wetting fluid is water. When a fluid with a higher viscosity is displaced by a fluid with a lower viscosity, protuberances occur instead of the normal displacement of the entire front and these protuberances shoot through the porous medium at a relatively rapid speed, leading to the development of fingers and this is termed as fingering phenomenon. Fingero-Imbibition phenomenon has tremendous importance in secondary oil recovery process. This phenomenon is mathematically formulated as a highly nonlinear partial differential equation, which is solved using the Multistep Hybrid Differential Transform Finite Difference Method. A solution is obtained using this Semi Analytic-Numerical Multistep Hybrid Differential Transform Finite Difference Method. The solution to this equation determines the saturation of the injected water in a double phase flow at various distances and times, as well as the effect of the magnetic field on the saturation can be obtained.

Combining the Multistep Differential Transform Method and Finite Difference Method results in the Multistep Hybrid Differential Transform Finite

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Difference Method. The effectiveness of the Finite Difference Method is combined with the adaptability of the Differential Transform Method. The numerical solution is acquired using an easy iterative method, which speeds up computation in comparison to the traditional Differential Transform Method. This approach has been discovered to be reliable and effective. By making all calculations simpler, it lessens computing work. In this method the Multistep Differential Transform Method (MDTM) and Finite Difference Method (FDM) have been used to achieve the solution for large values of time. Using MATLAB, the numerical solution and graphical representation were obtained. The results obtained were compared with the existing results and found to be in close agreement.

Keywords: Fingering, Imbibition, Hybrid Differential transform Method, Double phase flow, Heterogeneous porous media,magenetic field

I. INTRODUCTION

The primary aim here is to study the effect of magnetic field on Fingero- Imbibition phenomenon in heterogeneous porous media. If a porous medium saturated with a nonwetting fluid (oil) comes into contact with a wetting fluid (water), the wetting fluid spontaneously flows into the medium and the nonwetting fluid spontaneously flows out of the medium. This is referred to as the imbibition phenomena that arises due to differences in fluid wetting abilities. When a porous media filled with one fluid is displaced by another fluid of lower viscosity, protuberances occur instead of the normal displacement of the entire front and these protuberances shoot through the porous medium at a relatively rapid speed, leading to the development of fingers. Fingero-Imbibition Phenomenon is caused due to the simultaneous occurrence of fingering and imbibition. This phenomenon is observed in many engineering disciplines, including petroleum technology, soil mechanics, agricultural engineering, hydrology, etc. Many researchers are also investigating this phenomenon.

This phenomenon was investigated by Shah and Verma [5] in a heterogeneous porous media with magnetic field effect. The similarity transformation was utilized by Desai [21]. Mehta and Verma [3] investigated the composite expansion of finger imbibition in heterogeneous porous media. Patel, K. R., Mehta, M. N., and Patel, T. R. [4] investigated fingero-imbibition in double phase flow across heterogeneous porous media. The fingero-imbibition phenomenon in homogeneous porous material with magnetic field influence in vertical downward direction was discussed by Parikh, A. K., Mehta, M. N., and Pradhan, V. H. [10].

This paper aims to measure the saturation of injected water at a distance of 'x' and a time of 't' and to examine how the magnetic field affects saturation for the Fingero-Imbibition phenomenon in heterogeneous porous media. With the proper initial and boundary conditions, the Multistep Hybrid Differential Transform Finite Difference Method was used to solve the nonlinear partial differential equation that results from the mathematical formulation.

II. STATEMENT OF PROBLEM

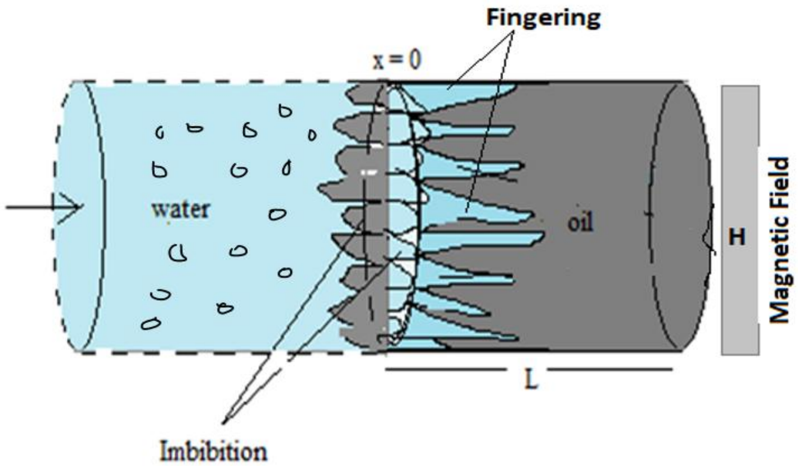

Figure 1

Consider a heterogeneous porous medium that is fully saturated with native fluid oil(o) and uniformly injected with water with magnetic particles. Assume that oil is nonconductive and the injected water is conductive. This Fingero-imbibition Phenomenon is caused by the simultaneous occurrence of the imbibition and fingering phenomena for a preferentially wetting phase and a less viscous phase in the secondary oil recovery process. In a heterogeneous porous medium, rock characteristics like porosity and permeability vary from place to place. Porosity in a heterogeneous porous material is regarded as a function of space 'x'. The fingers vary in size and shape, but we assume that they are rectangles and only take into account the average cross-sectional area that is filled by the fingers, omitting the specific size and shape of each finger. The variable magnetic field has the effect of increasing the velocity of injected water by a gradient $\frac{\omega H^2}{8\pi}$, where ω is the permeability of the magnetic field. The average cross-sectional area occupied by the injected water at a distance of 'x' and a time of 't' is defined as the saturation $S_w(x,t)$ of injected water. The average cross-sectional area filled by fingers is taken into consideration, regardless of size or shape

III. MATHEMATICAL FORMULATION

Assuming Darcy's Law [14,15] to be valid, the velocity V_w of injected water and velocity V_o of oil can be expressed as

$$V_w = -\frac{k_w}{\delta_w} k \left(\frac{\partial P_w}{\partial x} + \frac{\omega H}{4\pi} \frac{\partial H}{\partial x} \right) \tag{1}$$

$$V_o = -\frac{k_o}{\delta_o} k \frac{\partial P_o}{\partial x} \tag{2}$$

The equation of continuity of injected water is

$$\emptyset \frac{\partial S_w}{\partial t} + \frac{\partial V_w}{\partial x} = 0 \tag{3}$$

Where,

$\emptyset = \emptyset(x)$	represents the porosity of the heterogeneous medium
k	Variable permeability in m^2
k_w	relative permeability of water
k_o	relative permeability of oil
P_w	pressure of water (injected fluid)
P_o	pressure of oil (native fluid)
V_w	seepage velocity of water (m /s)
V_o	seepage velocity of oil (m /s)
δ_w	constant kinematic viscosity of water
δ_o	constant kinematic viscosity of oil
$\emptyset$	porosity of a medium

In the counter current imbibition phenomenon, the combined velocities of the injected water and native oil are zero[25].Therefore,

$$V_w + V_o = 0 \tag{4}$$

From (1) and(2) and (4) in (5) we get,

$$-\frac{k_w}{\delta_w} k \left(\frac{\partial P_w}{\partial x} + \frac{\omega H}{4\pi}\frac{\partial H}{\partial x}\right) = \frac{k_o}{\delta_o} k \frac{\partial P_o}{\partial x}$$

Substitute for P_o,we get,

$$-\frac{k_w}{\delta_w} k \left(\frac{\partial P_w}{\partial x} + \frac{\omega H}{4\pi}\frac{\partial H}{\partial x}\right) = \frac{k_o}{\delta_o} k \frac{\partial}{\partial x}(P_c + P_w)$$

Simplifying we get,

$$-\frac{k_w}{\delta_w} k \frac{\partial P_w}{\partial x} - \frac{k_w}{\delta_w} k \frac{\omega H}{4\pi}\frac{\partial H}{\partial x} = \frac{k_o}{\delta_o} k \frac{\partial}{\partial x}(P_c + P_w) \tag{5}$$

Simplifying ,

$$-\frac{k_w}{\delta_w} k \frac{\partial P_w}{\partial x} - \frac{k_o}{\delta_o} k \frac{\partial P_w}{\partial x} = \frac{k_w}{\delta_w} k \frac{\omega H}{4\pi}\frac{\partial H}{\partial x} + \frac{k_o}{\delta_o} k \frac{\partial P_c}{\partial x}$$

$$\frac{\partial P_w}{\partial x} = -\frac{\left(\frac{k_w}{\delta_w} k \frac{\omega H}{4\pi}\frac{\partial H}{\partial x} + \frac{k_o}{\delta_o} k \frac{\partial P_c}{\partial x}\right)}{k\left(\frac{k_w}{\delta_w} + \frac{k_o}{\delta_o}\right)} \tag{6}$$

According to Scheidegger [14] it is assumed ,

$$\frac{\frac{k_w}{\delta_w}\frac{k_o}{\delta_o}}{\left(\frac{k_w}{\delta_w} + \frac{k_o}{\delta_o}\right)} \approx \frac{k_o}{\delta_o} \tag{7}$$

Substitute equation (6) and (7) in equation (1), we get,

$$V_w = -\frac{k_w}{\delta_w} k \left(-\frac{\left(\frac{k_w}{\delta_w} k \frac{\omega H}{4\pi}\frac{\partial H}{\partial x} + \frac{k_o}{\delta_o} k \frac{\partial P_c}{\partial x}\right)}{\left(\frac{k_w}{\delta_w} + \frac{k_o}{\delta_o}\right)} + \frac{\omega H}{4\pi}\frac{\partial H}{\partial x}\right)$$

$$V_w = \frac{k_w}{\delta_w} k \frac{\left(\frac{k_o}{\delta_o} k \frac{\partial P_c}{\partial x}\right)}{\left(\frac{k_w}{\delta_w} + \frac{k_o}{\delta_o}\right)} + \frac{k_w}{\delta_w} k \frac{\left(\frac{k_w}{\delta_w} k \frac{\omega H}{4\pi}\frac{\partial H}{\partial x}\right)}{\left(\frac{k_w}{\delta_w} + \frac{k_o}{\delta_o}\right)} - \frac{k_w}{\delta_w} k \frac{\omega H}{4\pi}\frac{\partial H}{\partial x}$$

Therefore,

$$V_w = \frac{k_o}{\delta_o} k \frac{\partial P_c}{\partial x} - \frac{k_o}{\delta_o} k \frac{\omega H}{4\pi} \frac{\partial H}{\partial x}$$

$$V_w = \frac{k_o}{\delta_o} k \left(\frac{\partial P_c}{\partial x} - \frac{\omega H}{4\pi} \frac{\partial H}{\partial x} \right) \tag{9}$$

Substituting in equation (3) we get,

$$\emptyset \frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x} \left(\frac{k_o}{\delta_o} k \left(\frac{\partial P_c}{\partial x} - \frac{\omega H}{4\pi} \frac{\partial H}{\partial x} \right) \right) = 0 \tag{9}$$

Assuming A The capillary pressure is considered as a linear function [19]

$$P_c = -\beta S_w \tag{10}$$

where, β is a constant.

The standard relationship between phase saturation and relative permeability [1] is considered as,

$$k_w = S_w$$

$$k_0 = 1 - \alpha S_w \tag{11}$$

Where, α is a constant.

Assume porosity and permeability are functions of $'x'$ only, as we are considering a uniform heterogeneous porous medium [2]

$$\emptyset = \emptyset(x) = \frac{1}{a_1 - a_2 x}$$

$$k = k(x) = k_0(1 + bx) \tag{12}$$

Where a_1, a_2, k_0 $and\ b\ are\ positive\ constants.$
Since, $\emptyset(x)$ cannot exceed unity we assume, $a_1 - a_2 x \geq 1$ and $0 \leq x \leq L$

For simplicity we consider [12]

$$k \propto \emptyset$$

$$k = k_c \emptyset \tag{13}$$

Where k_c is a constant.

We express H as [4,22] , considering the magnetic field in the 'x' direction.

H=λx^n

Here 'n' is an integer and λ is a constant parameter. Using the value of H for n=1 in equation (9) with equations (10), (11), and (13), we get,

$$\emptyset\frac{\partial S_w}{\partial t}+\frac{\partial}{\partial x}\left(\frac{k_o}{\delta_o}k_c\emptyset\left(\frac{\partial(-\beta S_w)}{\partial x}-\frac{\omega H\lambda}{4\pi}\right)\right)=0 \tag{14}$$

$$\emptyset\frac{\partial S_w}{\partial t}+\frac{\partial}{\partial x}\left(\frac{(1-\alpha S_w)}{\delta_o}k_c\emptyset\left(\frac{\partial(-\beta S_w)}{\partial x}-\frac{\omega H\lambda}{4\pi}\right)\right)=0$$

$$\emptyset\frac{\partial S_w}{\partial t}=\frac{\beta k_c}{\delta_o}\frac{\partial}{\partial x}\left(\left(\emptyset(1-\alpha S_w)\frac{\partial(S_w)}{\partial x}\right)+\frac{k_c}{\delta_o}\frac{\omega\lambda^2}{4\pi}\frac{\partial}{\partial x}(\emptyset(1-\alpha S_w)x)\right)=0 \tag{15}$$

Using dimensionless variables,

$$X=\frac{x}{l}\quad,T=\frac{k_c\beta}{\delta_o L^2}t,$$

$$\frac{\partial S_w}{\partial T}=\frac{1}{\emptyset}\frac{\partial}{\partial X}\left(\emptyset(1-\alpha S_w)\frac{\partial(S_w)}{\partial X}\right)+\frac{\omega\lambda^2 L^2}{4\pi\beta\emptyset}\frac{\partial}{\partial X}(\emptyset(1-\alpha S_w)X) \tag{16}$$

Since,

$$\frac{1}{\emptyset}\frac{\partial\emptyset}{\partial X}=\frac{\partial(\log\emptyset)}{\partial X}$$

$$=\frac{\partial(-\log a_1+\frac{a_2 LX}{a_1})}{\partial X}$$

(Neglecting higher order terms in X)

Therefore, $\frac{1}{\emptyset}\frac{\partial\emptyset}{\partial X}=\frac{a_2}{a_1}L$

Substituting equation (16) simplifies to,

$$\frac{\partial S_w}{\partial T}=\frac{\partial}{\partial X}\left((1-\alpha S_w)\frac{\partial(S_w)}{\partial X}\right)+A\left((1-\alpha S_w)\frac{\partial(S_w)}{\partial X}\right)+B\frac{\partial}{\partial X}\big((1-\alpha S_w)X\big) + AB(1-\alpha S_w)X \tag{17}$$

where A=$\frac{a_2}{a_1}L$, B=$\frac{\omega\lambda^2 L^2}{4\pi\beta}$ and $S_w(x,t)=S_w(X,T)$

(17) is the non-linear partial differential equation for the finger-imbibition phenomenon in the heterogeneous porous medium and its solution $S_w(X,T)$ gives the saturation of injected water at time 'T' and distance'X'.

Assuming initial and boundary conditions as follows [9]:

Initial conditions: $S_w(X,0) = \frac{X^2}{5}$, $0 \leq X \leq 1$

Boundary conditions: $S_w(0,T) = \frac{T}{5}$, $T \geq 0$

$S_w(1,T) = \frac{1+3T}{5}$, $T \geq 0$

IV. PROBLEM SOLUTION

Consider Fingero-Imbibition phenomenon equation

$$\frac{\partial S_w}{\partial T} = \frac{\partial}{\partial X}\left((1-\alpha S_w)\frac{\partial(S_w)}{\partial X}\right) + A\left((1-\alpha S_w)\frac{\partial(S_w)}{\partial X}\right) + B\frac{\partial}{\partial X}\big((1-\alpha S_w)X\big) + AB(1-\alpha S_w)X$$

The governing equation is a nonlinear partial differential equation.

Therefore, the initial condition is ,

$$S_w(X,0) = \frac{X^2}{5}, \quad , \qquad 0 \leq X \leq 1 \tag{18}$$

Taking appropriate boundary conditions as

$$S_w(0,T) = \frac{T}{5} \quad , \quad T \geq 0$$
$$S_w(1,T) = \frac{1+3T}{5} \quad , \quad T \geq 0 \tag{19}$$

(17) is solved using the Hybrid Differential Transform and the Finite Difference Method (HDTFDM) for small values of “T” and Multistep Hybrid Differential Transform Finite Difference Method for large values of ‘T’.

V. SOLUTION BY HYBRID DIFFERENTIAL TRANSFORM FINITE DIFFERENCE METHOD (HDTFDM)

1. **Methodology:** The nonlinear partial differential equation [17] is solved using the Hybrid Differential Transform and Finite Difference Method (HDTFDM). This method combines the Differential Transform Method (DTM) and the Finite Difference Method (FDM). The Finite Difference Method is used to approximate the spatial variables, and the Differential Transform Method is used to approximate the time variables.

   Zhou [25] introduced the DTM for the solution of linear and nonlinear differential equations in electrical circuit analysis. The hybrid approach was employed by Yu, L.T., and Chen, C.K. [20].

   The finite difference approach is applied to the 'X variable and the differential transform is employed for the 'T' variable. The hybrid approach is effective for solving both linear and nonlinear partial differential equations because it has been shown to converge quickly with few iterations

2. **Preliminaries:** The Differential Transform of the kth derivative of $u(x,t)$ applied to the 't' variable is given as

$$U(i,k) = \frac{1}{k!}\left[\frac{d^k u(x,t)}{dt^k}\right]_{t=0} \qquad k = 0,1,2,\ldots \text{ and i } = 0,1,2 \tag{20}$$

The inverse Transformation of $U(i,k)$ is given as

$$u(x,t) = \sum_{k=0}^{\infty} U(i,k)\, t^k \tag{21}$$

Where $u(x,t)$ represents the original function and $U(i,k)$ represents transformed function.

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{d^k u(x,t)}{dt^k}\right]_{t=0} t^k \tag{22}$$

where $U(i,k) = U(x_i, k)$, $x_i = ih$, $i = 0,1,2,3,\ldots..$
where h is the finite difference step interval.

The theorems are as follows [6] and [7]

Theorem 1: $If\ f(x,t) = \frac{\partial m}{\partial t}, then\ F(i,k) = (k+1)M(i,k+1)$
Theorem 2: $If\ f(x,t) = \frac{\partial^2 m}{\partial t^2}\ , then\ F(i,k) = (k+1)(k+2)M(i,k+2)$
Theorem 3: $If\ f(x,t) = xe^{-t}\ . then\ F(i,k) = x\frac{(-1)^k}{k!}$
Theorem 4: $If\ f(x,t) = \sin x\ , then\ F(i,k) = \sin x$
Theorem 5: $If\ f(x,t) = \sin t\ , then\ F(i,k) = \sin\left(\frac{\pi k}{2}\right)\frac{1^k}{k!}$
Theorem 6: $If\ f(x,t) = \frac{\partial m}{\partial x}(x,t), then$
$$F(i,k) = \frac{M(i+1,m) - M(i-1,m)}{2h}$$
Theorem 7: $If\ f(x,t) = m(x,t)\frac{\partial m}{\partial x}(x,t), then$
$$F(i,k) = \sum_{m=0}^{k} M\,(i, k-m)\frac{M(i+1,m) - M(i-1,m)}{2h}$$

3. **Numerical Solution:** Applying HDTFDM (17),we get

$$\frac{\partial S_w}{\partial T} = \frac{\partial}{\partial X}\left((1-\alpha S_w)\frac{\partial(S_w)}{\partial X}\right) + A\left((1-\alpha s_w)\frac{\partial(S_w)}{\partial X}\right) + B\frac{\partial}{\partial X}\big((1-\alpha S_w)X\big) + AB(1-\alpha S_w)X$$

Subject to

Initial Conditions: $S_w(X,0) = \frac{X^2}{5}$, , $0 \leq X \leq 1$

Boundary Conditions: $S_w(0,T) = \frac{T}{5}$, $T \geq 0$

$S_w(1,T) = \frac{1+3T}{5}$, $T \geq 0$

Taking $\propto = 1.11$

Applying, Differential Transformation to the 'T' variable and finite difference to the 'X' variable ,

$$\frac{\partial S_w}{\partial T} = (k+1)\mathbf{S}(i,k+1)$$

$$\frac{\partial^2 S_w}{\partial X^2} = \frac{\mathbf{S}(i+1,k) - 2\mathbf{S}(i,k) + \mathbf{S}(i-1,k)}{h^2}$$

$$S_w \frac{\partial^2 S_w}{\partial X^2} = \sum_{r=0}^{k} \mathbf{S}(i,k-r) \frac{\mathbf{S}(i+1,r) - 2\mathbf{S}(i,r) + \mathbf{S}(i-1,r)}{h^2}$$

$$\left(\frac{\partial S_w}{\partial X}\right)^2 = \sum_{r=0}^{k} \frac{\mathbf{S}(i+1,k-r) - \mathbf{S}(i-1,k-r)}{2h} \frac{\mathbf{S}(i+1,r) - \mathbf{S}(i-1,r)}{2h}$$

$$S_w \frac{\partial S_w}{\partial X} = \sum_{r=0}^{k} S(i,k-r) \frac{\mathbf{S}(i+1,r) - \mathbf{S}(i-1,r)}{2h}$$

Where $S_w(X,T)$ is the original function and, $\mathbf{S}(\boldsymbol{i},\mathrm{k}) = \mathbf{S}(X_{\mathrm{i}}, k)$, $X_{\mathrm{i}} = ih$, $i = 0,1,2,3,\dots..$ is transformed function

Initial Conditions: $S(i,0) = \frac{(X_i)^2}{5}$, $x_i = ih$, $i = 0,1,2,3,\dots$

Boundary Conditions: $\mathbf{S}(0,k) = \frac{\delta(k-1)}{5}$

$= \frac{1}{5}, k = 1$

$= 0, k \neq 1$

$$\mathbf{S}(N,k) = \frac{\left(\delta(k) + 3\delta(k-1)\right)}{5}$$

$= \frac{1}{5}, k = 0$

$= \frac{3}{5}, k = 1$

$= 0, k = 2,3,4,\dots..$

Where N is the no. of spatial segments.
Substituting in (17) we get,
relation,

$$(k+1)\mathbf{S}(i,k+1) = \frac{\mathbf{S}(i+1.k) - 2\mathbf{S}(i.k) + \mathbf{S}(i-1,k)}{h^2} - \alpha \sum_{r=0}^{k} \mathbf{S}(i,k-r)) \frac{\mathbf{S}(i+1.r) - 2\mathbf{S}(i.r) + \mathbf{S}(i-1,r)}{h^2} - \alpha \sum_{r=0}^{k} \frac{\mathbf{S}(i+1,r) - \mathbf{S}(i-1,r)}{2h} \frac{\mathbf{S}(i+1,k-r) - \mathbf{S}(i-1,k-r)}{2h} + A\frac{\mathbf{S}(i+1,k) - \mathbf{S}(i-1,k)}{2h} - A\alpha \sum_{r=0}^{k} \mathbf{S}(i,k-r)) \frac{\mathbf{S}(i+1.r) - \mathbf{S}(i-1,r)}{2h} + B - \alpha BS(i,k) - \alpha BX \frac{\mathbf{S}(i+1.k) - \mathbf{S}(i-1,k)}{2h} + ABX - \alpha ABXS(i,k) \tag{23}$$

Where for $k = 0,1,2,3, \ldots$ $\mathbf{S}(i,0), \mathbf{S}(i,1), \mathbf{S}(i,2), \ldots$ are obtained. The approximate solutions for various values ‘X’ and ‘T’ are found using the inverse transformation,

$$S_w(X,T) = \sum_{k=0}^{\infty} \mathbf{S}(i,k)\, T^k \tag{24}$$

$when,$ x_i=0,

$$S_w(0,T) = \sum_{k=0}^{\infty} \mathbf{S}(0,k)\, T^k \tag{25}$$

$$= \mathbf{S}(0,0) + \mathbf{S}(0,1)T + \mathbf{S}(0,2)T^2 + \mathbf{S}(0,3)T^3 \ldots \ldots \ldots$$
$$= \qquad = \mathbf{0.2T}$$

where $X_I = ih$ mesh points for $h = 0.1,\ i = 0,1,2, \ldots..$

VI. RESULTS AND DISCUSSION

The numerical saturation values obtained from equation (24) for various distances *'X'* at fixed time T=0.001,0.002,0.003,0.004,0.005,0.006,0.007,0.008,0.009,.01 are shown in Table 1 below.

Table 1: Saturation S_w (X, T) for values of distance’ X’ and time ‘T’ (HDTFDM)

X\T	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009	0.01
0	0.0002	0.0004	0.0006	0.0008	0.001	0.0012	0.0014	0.0016	0.0018	0.002
0.1	0.0024	0.0028	0.0032	0.0035	0.0039	0.0042	0.0046	0.0049	0.0052	0.0056
0.2	0.0084	0.0089	0.0093	0.0097	0.0101	0.0105	0.0109	0.0113	0.0117	0.0121
0.3	0.0184	0.0189	0.0193	0.0197	0.0202	0.0206	0.021	0.0214	0.0219	0.0223
0.4	0.0324	0.0329	0.0333	0.0337	0.0342	0.0346	0.035	0.0355	0.0359	0.0363
0.5	0.0504	0.0509	0.0513	0.0517	0.0521	0.0526	0.053	0.0534	0.0538	0.0542
0.6	0.0724	0.0728	0.0732	0.0737	0.0741	0.0745	0.0749	0.0753	0.0757	0.0761

0.7	0.0984	0.0988	0.0992	0.0996	0.1	0.1004	0.1008	0.1012	0.1015	0.1019
0.8	0.1284	0.1287	0.1291	0.1295	0.1299	0.1302	0.1306	0.131	0.1314	0.1318
0.9	0.1623	0.1627	0.1631	0.1635	0.1639	0.1643	0.1648	0.1652	0.1656	0.1661
1	0.2006	0.2012	0.2018	0.2024	0.203	0.2036	0.2042	0.2048	0.2054	0.206

1. Graphical Representation

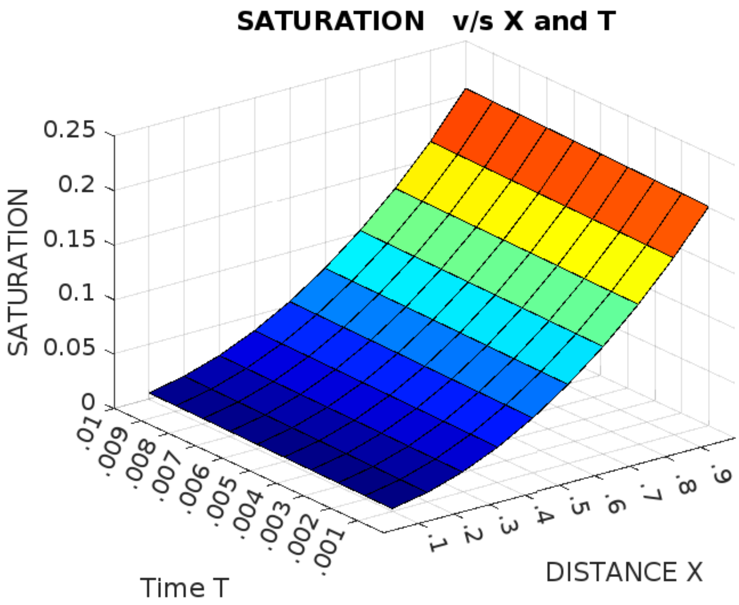

Figure 3: Saturation $S_w\ (X, T)$ for values of distance' X' and time 'T' (HDTFDM)

From the graph in Figure 3, the saturation $S_w(X, T)$ of the injected water increases when the distance 'X' increases for a fixed time T' and also increases when the time 'T' increases for a fixed distance 'X'.

2. **Limitation of Hybrid Differential transform Method:** The HDFDTM is used to approximate solutions to a wide range of nonlinear problems with convergent series and easily computed components; nonetheless, it has certain limitations: In a small region, the series solution always converges quickly, but in a wider region, it converges slowly. In a very small region, this series solution provides a good approximation to the true solution. The Multi-step DTM (MDTM) and finite Difference Method (FDM) are improved methods that accelerate series solution convergence over a large region while also improving DTM accuracy.

VII. MULTISTEP HYBRID DIFFERENTIAL TRANSFORM FINITE DIFFERENCE METHOD (MHDTFDM)

The computation interval [0, T] is not small in some modelling settings, hence the full domain T is partitioned into 'N' subdomains to speed convergence and increase calculation accuracy. The key advantage of partitioning the domain is that the solution can be constructed in a small period of time using a minimal number of Taylor series terms. It is important to remember that, if necessary, the time interval might be arbitrarily tiny. The differential equations in each subdomain can be solved as a result.

1. **Methodolgy:** DTM solutions have a small convergence interval, but multi-step DTM solutions have a wide convergence interval. This depicts how the MDTM increases the interval of convergence. Of course, by reducing the step size h and raising the number of terms in each subinterval, the accuracy can be enhanced.

   Definition 1: The Differential Transform is defined as follows

$$X(k) = \frac{1}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_0} \tag{26}$$

   Where $x(t)$ and $X(k)$ are the original function and transformed function respectively.

   The inverse transformation is defined as,

$$x(t) = \sum_{k=0}^{k=\infty} X(k)(t-t_0)^k$$

$$x(t) = \sum_{k=0}^{k=\infty} \frac{1}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_0} (t-t_0)^k \tag{27}$$

   and the function $x(t)$ is considered as a series with finite terms

$$x(t) = \sum_{k=0}^{k=m} \frac{1}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_0} (t-t_0)^k \tag{28}$$

   Where m represents no. of Taylor series components. Increasing the no. of terms can improve the accuracy of the solution.

2. **Solution of Partial Differential equation in $\boldsymbol{u(x,t)}$ in domain $[\mathbf{0},\boldsymbol{T}]$:** We divide the domain $[0,T]$ into N sections where H= T/N is the length of each subdomain.

   So, a separate function is considered for each sub domain.

$$u(x,t) = \begin{cases} u_1(j,t)\,, & t \in [t_1, t_2] \\ u_2(j,t)\,, & t \in [t_2, t_3] \\ u_N(j,t), & t \in [t_N, t_{N+1}] \end{cases} \tag{29}$$

   Where $t_i = (i-1)H$

   Where (30)

$$u_i(j,t) = \sum_{k=0}^{m} U_i(j,k)\left(\frac{t-t_i}{H}\right)^k$$

Where

$$U_i(j,k) = \frac{H^k}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=t_0} \tag{31}$$
$$k = 0,1,2, \dots \ and\ i = 0,1,2$$

The solution of equation (21) is of the form

$$S_{wi}(j,t) = \sum_{k=0}^{m} \boldsymbol{S}_i(j,k)\left(\frac{t-t_i}{H}\right)^k$$

for $t \in [t_i, t_{i+1}]$

$$\boldsymbol{S}_i(j,k) = \frac{H^k}{k!}\left[\frac{\partial^k S_w(x,t)}{\partial t^k}\right]_{t=t_0}$$

Applying Central finite difference to the spatial variable *'X'* and MDTM to the variable T*'*

From (17), the following recurrence relation can be obtained

$$\begin{aligned}
&\frac{(k+1)\boldsymbol{S}_i(j,k+1)}{H} \\
&= \frac{\boldsymbol{S}_i(j+1,k) - 2\boldsymbol{S}_i(j,k) + \boldsymbol{S}_i(j-1,k)}{h^2} \\
&- \alpha \sum_{r=0}^{k} \boldsymbol{S}_i(j,k-r)\frac{\boldsymbol{S}_i(j+1,r) - 2\boldsymbol{S}_i(j,r) + \boldsymbol{S}_i(j-1,r)}{h^2} \\
&- \alpha \sum_{r=0}^{k} \frac{\boldsymbol{S}_i(j+1,r) - \boldsymbol{S}_i(j-1,r)}{2h}\,\frac{\boldsymbol{S}_i(j+1,k-r) - \boldsymbol{S}_i(j-1,k-r)}{2h} \\
&+ A\frac{\boldsymbol{S}_i(i+1,k) - \boldsymbol{S}_i(i-1,k)}{2h} - A\alpha \sum_{r=0}^{k} \boldsymbol{S}_i(i,k-r)\frac{\boldsymbol{S}_i(i+1.r) - \boldsymbol{S}_i(i-1,r)}{2h} \\
&+ B - \alpha B\boldsymbol{S}_i(i,k) - \alpha BX\frac{\boldsymbol{S}_i(i+1.k) - \boldsymbol{S}_i(i-1,k)}{2h} + ABX - \alpha ABX\boldsymbol{S}_i(i,k)
\end{aligned} \tag{32}$$

Where $\mathbf{S}_i(j,k)$ is the differential transform of $s_{wi}(j,t)$
Here, separate functions are considered in each sub domain i.e

$$S_w\ \mathbf{(x,t)} = \begin{cases} S_{w_1}(j,t), & t \in [t_1,t_2] \\ S_{w_2}(j,t), & t \in [t_2,t_3] \\ S_{w_N}(j,t), & t \in [t_N,t_{N+1}] \end{cases} \quad \text{where } t_i = (i-1)H \tag{33}$$

Where $S_w(X,T)$ is the original function and, $\mathbf{S}(\boldsymbol{i},\mathrm{k}) = \mathbf{S}(X_\mathrm{i}\,,k)$, $X_\mathrm{i} = ih$, $i = 0,1,2,3,\ldots..$ is transformed function

Applying Multistep Differential Transform Method (MDTM) to the 'T' variable and Finite Difference Method (FDM) on the initial conditions, we have

Initial Conditions: $\boldsymbol{S}_i\left(X_j,0\right) = \frac{(X_j)^2}{5}$,

Boundary Conditions: $\boldsymbol{S}_i(0,k) = \frac{\delta(k-1)}{5}$

$$= \frac{1}{5}\ ,k = 1$$

$$= 0\ ,k \neq 1$$

$$\boldsymbol{S}_i(N,k) = \frac{\left(\ \delta(k) + 3\delta(k-1)\right)}{5}$$

$$= \frac{1}{5}\ ,k = 0$$

$$= \frac{3}{5}\ ,k = 1$$

$$= 0\ ,k = 2,3,4,\ldots..$$

Where N is the no. of spatial segments.

Here $N = 10\ ,T = 1, H = .005$ and we use the continuity condition in each time subdomain.

Firstly, the DTM is applied to the given PDE over the interval [0,.005]. For the next time step, the value at T = .005 is used as an initial condition in the interval[.005,.01]

For 1$\leq j \leq N_x + 1$, 1$\leq i \leq N_t$

$$\boldsymbol{S}_i(j,0) = \sum_{k=0}^{m} \boldsymbol{S}_{i-1}(j,k)$$

MDTM is implemented by dividing the solution interval [0,1] into 200 subintervals with step sizes equal to H = 0.005.

VIII. RESULTS AND DISCUSSION

The numerical saturation values obtained from equation (32) and (33) for various distances $'X'$ at fixed time $T = T$=0.01,0.02,0.03,0.04,0.05,0.06,0.07,0.08,0.09and .1, using MATLAB are shown in Table **2** below.

Table 2: Saturation S_{wi} (X,T) for values of distance 'X' and time 'T' by Hybrid Multistep Differential Transform and Finite Difference method for $\boldsymbol{T} \in [0,.1]$

X\T	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
0	0.002	0.004	0.006	0.008	0.01	0.012	0.0140	0.016	0.018	0.02
0.1	0.0056	0.0086	0.0114	0.0141	0.0167	0.0193	0.0217	0.0242	0.0266	0.029
0.2	0.0121	0.0158	0.0193	0.0226	0.0257	0.0287	0.0317	0.0346	0.0374	0.0402
0.3	0.0223	0.0264	0.0302	0.0339	0.0374	0.0408	0.0442	0.0474	0.0507	0.0539
0.4	0.0363	0.0405	0.0446	0.0485	0.0523	0.056	0.0597	0.0633	0.0669	0.0704
0.5	0.0543	0.0585	0.0626	0.0666	0.0706	0.0746	0.0785	0.0825	0.0864	0.0902
0.6	0.0761	0.0803	0.0844	0.0886	0.0928	0.097	0.1012	0.1054	0.1096	0.1138
0.7	0.102	0.106	0.1102	0.1146	0.119	0.1235	0.128	0.1326	0.1371	0.1417
0.8	0.1318	0.1361	0.1405	0.1452	0.15	0.1548	0.1597	0.1647	0.1696	0.1746
0.9	0.1661	0.1709	0.176	0.1812	0.1865	0.1919	0.1973	0.2027	0.2081	0.2135
1	0.206	0.212	0.218	0.224	0.23	0.236	0.242	0.248	0.254	0.26

From Table 2, we observe that the values of saturation converge in this interval by using MDTM.

We continue finding $S_{w1}(x,t)$, $S_{w2}(x,t)$ … ..for the remaining intervals [.1,.2], [.2,.3] and so on and saturation at T=.1,.2,.3…… is shown in Table 3.

Table 3: Saturation S_{wi} (X,T) for values of distance 'X' and time 'T' by Multistep Hybrid Differential Transform and Finite Difference method for $\boldsymbol{T} \in [.1,1]$

X/T	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.02	0.04	0.06	0.08	0.1	0.1200	0.1400	0.16	0.18	0.2000
0.1	0.029	0.0528	0.0759	0.0988	0.1214	0.1436	0.1656	0.1871	0.2083	0.2292
0.2	0.0402	0.0675	0.0938	0.1197	0.1449	0.1694	0.1934	0.2166	0.2391	0.2609
0.3	0.0539	0.0847	0.1142	0.143	0.1709	0.1978	0.2239	0.2489	0.2729	0.2957
0.4	0.0704	0.1046	0.1374	0.1692	0.1999	0.2294	0.2577	0.2847	0.3103	0.3342
0.5	0.0902	0.1278	0.164	0.1988	0.2326	0.2649	0.2955	0.3247	0.352	0.3772
0.6	0.1138	0.1548	0.1945	0.2328	0.2697	0.305	0.3385	0.3701	0.3994	0.4262
0.7	0.1417	0.1864	0.2299	0.272	0.3126	0.3514	0.3883	0.4228	0.4545	0.4832
0.8	0.1746	0.2235	0.2713	0.3177	0.3628	0.4061	0.4473	0.4858	0.5211	0.5524
0.9	0.2135	0.2673	0.3204	0.3725	0.4234	0.4729	0.5205	0.5654	0.6069	0.6435
1	0.26	0.32	0.38	0.44	0.5	0.56	0.62	0.68	0.74	0.8

As we can see, the main advantage of the MDTM is that the obtained series solution converges for wide time region, which is not possible by the traditional DTM

From the table. we observe that the saturation $S_w(X,T)$ of injected water is increasing when distance ' X' is increasing for fixed time ' T' and saturation is also increasing when time 'T' is increasing for fixed distance 'X'.

Graphical Representation

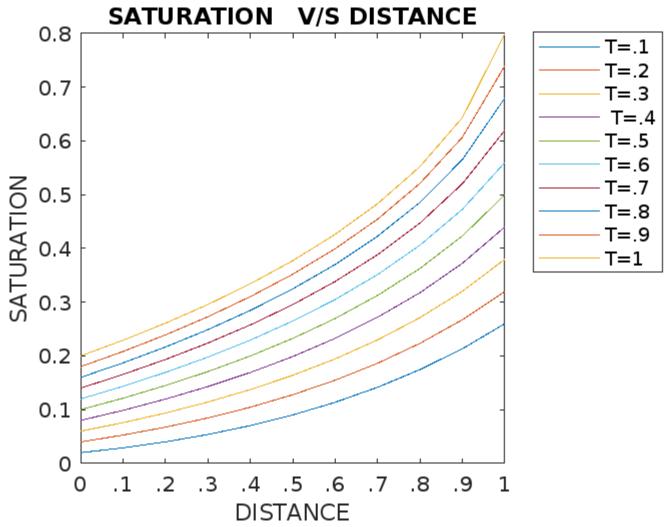

Figure 4: Saturation $S_w\,(X,T)$ for values of distance' X' and time 'T' by Multistep Hybrid Differential Transform Finite Difference Method

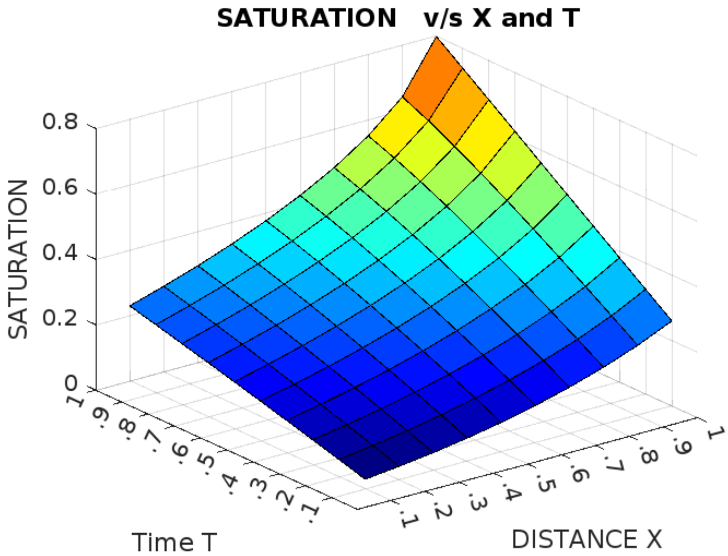

Figure 5: 3 D plot of Saturation $S_w\,(X,T)$ for different values of distance' X' and time 'T' by Multistep Hybrid Differential Transform Finite Difference Method

Figure 5 is a 3D plot of Saturation V/s Distance and Time. Here we can clearly observe that the saturation $S_w(X,T)$ of injected water is increasing when distance ‘ X’ is increasing for fixed time ‘ T’ and also increasing with time ‘T’ for fixed distance’X’.

The above solution was compared with Homotopy Analysis Method [9] and the results were found to be in close agreement.

Table 4: Comparison of Multistep Hybrid Differential Transform Method and Homotopy Analysis Method [9]

X/T	0.1		0.3		0.5		0.7		0.9	
	HDTFD	**HAM**	**HDTFD**	**HAM**	**HDTFD**	**HAM**	**HDTFD**	**HAM**	**HDTFD**	**HAM**
0	**0.02**	**0.0200**	**0.06**	**0.06**	**0.1**	**0.1**	**0.1400**	**0.14**	**0.18**	**0.18**
0.1	0.029	0.0295	0.0759	0.0759	0.1214	0.1213	0.1656	0.1654	0.2083	0.2081
0.2	0.0402	0.0410	0.0938	0.0939	0.1449	0.1447	0.1934	0.1931	0.2391	0.2387
0.3	0.0539	0.0548	0.1142	0.1142	0.1709	0.1706	0.2239	0.2235	0.2729	0.2723
0.4	0.0704	0.0714	0.1374	0.1374	0.1999	0.1995	0.2577	0.2572	0.3103	0.3094
0.5	0.0902	0.0912	0.164	0.1639	0.2326	0.2322	0.2955	0.295	0.352	0.351
0.6	0.1138	0.1146	0.1945	0.1944	0.2697	0.2693	0.3385	0.3379	0.3994	0.3982
0.7	0.1417	0.1422	0.2299	0.2297	0.3126	0.3121	0.3883	0.3875	0.4545	0.453
0.8	0.1746	0.1749	0.2713	0.2711	0.3628	0.3624	0.4473	0.4464	0.5211	0.5191
0.9	0.2135	0.2136	0.3204	0.3202	0.4234	0.4231	0.5205	0.5197	0.6069	0.6045
1	0.26	0.2600	0.38	0.38	0.5	0.5	0.62	0.62	0.74	0.74

Table 4 shows the comparison of Multistep Hybrid Differential Transform Finite Difference Method with Homotopy Analysis Method and the results are found to be in close agreement

Table 5: Comparison of Saturation of Injected Water with Magnetic Effect and Without Magnetic Effect

	With Magnetic Field	**Without Magnetic field**	**With Magnetic Field**	**Without Magnetic field**	**With Magnetic Field**	**Without Magnetic field**	**With Magnetic Field**	**Without Magnetic field**	**With Magnetic Field**	**Without Magnetic field**
x/t	**0.1**	**0.1**	**0.3**	**0.3**	**0.5**	**0.5**	**0.7**	**0.7**	**0.9**	**0.9**
0.1	0.029	0.0289	0.0759	0.0758	0.1214	0.1212	0.1656	0.1654	0.2083	0.2082
0.2	0.0402	0.04	0.0938	0.0937	0.1449	0.1446	0.1934	0.1931	0.2391	0.2388
0.3	0.0539	0.0537	0.1142	0.114	0.1709	0.1706	0.2239	0.2235	0.2729	0.2726

0.4	0.0704	0.0702	0.1374	0.1371	0.1999	0.1995	0.2577	0.2572	0.3103	0.3098
0.5	0.0902	0.09	0.164	0.1637	0.2326	0.2322	0.2955	0.295	0.352	0.3515
0.6	0.1138	0.1136	0.1945	0.1942	0.2697	0.2693	0.3385	0.3381	0.3994	0.3989
0.7	0.1417	0.1415	0.2299	0.2296	0.3126	0.3122	0.3883	0.3879	0.4545	0.454
0.8	0.1746	0.1744	0.2713	0.2711	0.3628	0.3625	0.4473	0.4469	0.5211	0.5206
0.9	0.2135	0.2134	0.3204	0.3202	0.4234	0.4232	0.5205	0.5202	0.6069	0.6066
1	0.26	0.26	0.38	0.38	0.5	0.5	0.62	0.62	0.74	0.74

From table (5) we can observe that the Magnetic Field Effect increases the Saturation of injected water as compared to saturation of injected water without magnetic field effect

Convergence Criteria: From equation (21) we get the series solution for the nonlinear PDE as

$$u(x,t) = \sum_{k=0}^{\infty} U(i,\ k)\,(t)^k$$

then the convergence of the power series in ‘t’ can be found as per the following theorem [27]

Theorem: If $\varphi_k(x,t) = U(i,k)(t-t_0)^k$,then the series solution $\sum_{k=0}^{\infty}\varphi_k(x,t)$,stated in equation above,$\forall k \in N \cup \{0\}$ follows the following criteria.

It is convergent if $\exists\ 0 < \lambda < 1$,such that $\|\varphi_{k+1}\| \leq \lambda\|\varphi_k\|$
It is divergent if $\exists\ \lambda > 1$,such that $\|\varphi_{k+1}\| \geq \lambda\|\varphi_k\|$

IX. CONCLUSIONS

Numerical solution and graphs for obtaining the saturation of injected water in heterogeneous porous media during the Fingero- imbibition phenomenon considering magnetic field effect, with regard to distance and time are obtained using MATLAB.

The graph given by Figure. 2 shows that saturation of injected water increases as distance X increases for given time T. The saturation of water increases with distance as well as with time, which is consistent with physical phenomena. As indicated in Table 5, the saturation of injected water with magnetic field effect is greater than the saturation of injected water without the magnetic field effect. As a result, it is possible to deduce that the magnetic field aids in enhancing the saturation of injected water during the fingero-imbibition phenomena.

Multi-step DTM, a reliable modification of the DTM, was used in this work for the variable 'T,' which improves the convergence of the series solution. The method gives approximate numerical solutions to both linear and nonlinear differential equations. Table 1. shows that using DTM we get solutions which converge over a small interval of convergence, whereas Multi-step DTM solutions have a wide interval of convergence, as shown in Table 3. This demonstrates that the MDTM increases the interval of convergence for the series solution.

Comparative study of obtaining the result by the two methods, namely, HDTFDM and HAM shows that results closely agree with each other. Complex symbolic computation is not necessary because the Hybrid Differential Transform Finite Difference Method calculates numerical solutions through an iterative process. It has been shown that the suggested approach can produce very precise numerical approximations and that Multistep DTM improves the obtained solution. The last and most important advantage is that we do not use linearization.

Therefore, without the use of linearization or perturbation, these techniques can be utilised to solve a wide variety of challenging partial differential equations, both linear and nonlinear.

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