APPLICATIONS OF SOHAMINTEGRAL TRANSFORM TO SOLVE VOLTERRA INTEGRAL EQUATIONS OF SECOND KIND

Abstract

Many advanced scientific and engineering problems can be theoretically represented as linear Volterra integral equations. In this work, we use Soham transform for obtaining solution of Volterra integral equations of 2^{nd} kind. To demonstrate the applicability of the Soham transform, some numerical problems were presented and solved by the Sohamtransform. Numerical results show that the Soham transform is very much useful to obtain the accurate solution of the 2^{nd} kind Linear Volterra integral equations.

Keywords: Volterra integral equations of 2^{nd} kind, Soham transform, Soham inverse transform, Convolution theorem.

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I. INTRODUCTION

The 2^{nd} kind of linear Volterra integral equation is defined as follows

$$\phi(\mathrm{y}) = \mathrm{f(y)} + \lambda \int_0^y k(y,t)\,\phi(t)dt$$

Here the function f(y) and kernel k(y, t) are known real- valued functions. The unknown Function is denoted by$\phi(\lambda)$ and λ is a non-zero real parameter.

We solve linear Volterra integral equations of 2^{nd} kind. For that purpos we use the Soham transform To obtain the solutions of advanced problems which occurs in fields like Science, Engineering, Health Sciences, Pharmacy, Technology, Commerce and Economics, the most useful and simple mathematical technique is use of integral transform. It is very useful and effective technique for solving a linear differential equations and system of such equations and integral equations under the given initial conditions.Integral transforms provides exact solution of the problem without doing lengthy calculations is the important feature of integral transforms .

As integral transforms has this important feature of obtaining solution of the problem using small number of calculations, many researchers are interested to use this field. They make themselves busy in formation of various new integral transforms with various Kernels on various domains. Recently, the integral transform called as Kushare transform is developed by S. R. Kushare and D. P. Patil [1] in October 2021and used it to solve differential equations and system of differential equations. Further new integral transform named as Soham transform is introduced by Savita Khakale and Dinkar Patil [2] in November 2021. Further, Patil and some authors [5, 6, 7] used Kushare transform for solving different problems.

Anuj integral transform is used by Patil et al [4] to solvefirst kind Volterra integral equations . Patil together with Suryawanshi [3, 8] solved volterra integral equations and solve system of differential equations in the mathematical models occurring in health sciences and biotechnology by using Soham transform. Various applications of Soham transform are studied by Patil et al [9,10,11].

We organize this paper as follows. First section contains introduction . Second section is devoted for preliminary concepts. In third section we state and prove Convolution theorem of Soham transform. Soham transform is applied to obtain the solution of Volterra integral equations in fourth section.

II. PRELIMINARIES

We state some basic required concepts here. Now we state some definitions, properties and formulae which are required .

1. **Soham Transform:** Soham Transform is denoted by the operator S(.) and it is defined by the integral equation $\mathrm{S[f(t)]} = \mathrm{q(v)} = \frac{1}{v}\int_0^\infty f(t)e^{-v^\alpha t}\mathrm{dt}$
   where α is a non zero real number, $t \geq 0$, $k_1 \leq \mathrm{v} \leq k_2$

2. **Inverse Soham Transform:** If we dwnote the Soham transform of the function f(t) by q(v) then we define inverse Soham transform as $s^{-1}[q(v)] = f(t)$

3. **Properties of Soham Transform:** We state some useful properties of Soham integral transform in this subsection [1].

   - **Linearity property**: If f_1 (t) and f_2(t) are two functions of t andc_1 and $c_2$are any two constants then
     $s\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 s\{f_1(t)\} + c_2 s\{f_2(t)\}$

   - **Transform of derivative**: Let P(v) be the Soham transform of the function f(t) i.e . [S[f(t)] = P(v)] then
     $S[f'(t)] = v^{\alpha} q(v) - \frac{1}{v} f(0)$

4. **Soham Transform of Elementary Functions:** "For t ≥ 0 the function f(t) is piecewise continuous and is of exponential order" , these are the sufficient conditions for the existence of Soham transform. If these conditions are not satisfied then Soham transform of that function may or may not exist. In this section we state Soham transform of some elementary functions.

Table: Soham Transform of Some Functions

Sr.No.	f(t)	$S[f(t)] = q(v)$
1	1	$\frac{1}{v^{\alpha+1}}$
2	t	$\frac{1}{v^{2\alpha+1}}$
3	t^n	$\frac{\Gamma(n+1)}{v^{\alpha n+\alpha+1}}$
4	e^{at}	$\frac{1}{v(v^{\alpha} - a)}$
5	e^{-at}	$\frac{1}{v(v^{\alpha} + a)}$
6	$sinat$	$\frac{a}{v(v^{2\alpha} + a^2)}$
7	$cosat$	$\frac{v^{\alpha}}{v(v^{2\alpha} + a^2)}$
8	$sin\square at$	$\frac{a}{v(v^{2\alpha} - a^2)}$
9	$cos\square at$	$\frac{v^{\alpha}}{v(v^{2\alpha} - a^2)}$

III. CONVOLUTION THEOREM

Now we state and prove convolution theorem.
The convolution of two function f(t) and g(t) is defined as. $(f * g)(t) = \int f(t-\tau)\, g(\tau) d\tau$

Theorem: If f(t) and g(t) are two functions such that $s\{f(t)\} = p(v)$ and $s\{g(t)\} = q(v)$,

then

$$S(f * g)(t) = v\{p(v)q(v)\}$$

Proof: Applying Soham transform.

$$S(f * g)(t) = S[\int_0^t f(t-\tau)g(\tau)d\tau] = \frac{1}{v}\int_0^\infty e^{-v^\alpha t}(\int_0^t f(t-\tau)g(\tau)d\tau$$

Change order of integration and put $t - \tau = b$

$$S(f * g)(t) = \frac{1}{v}\int_0^\infty f(\tau)\int_0^\infty e^{-v^{\alpha(\tau-b)}} g(b)dbd\tau$$

$$= v\{\frac{1}{v}\int_0^\infty f(\tau)\int_0^\infty e^{-v^{\alpha(\tau-b)}} g(b)dbd\tau\}$$

$$S(f * g)(t) = v\{p(v)q(v)\}$$

Thus the convolution theorem forintegral transform (Soham transform);

$$S(f * g)(t) = v\{p(v)q(v)\}$$

is proved.

IV. SOHAM TRANSFORM FOR CONVOLUTION TYPE 2ND KIND LINEAR VOLTERRA INTEGRAL EQUATIONS

In this study, the kernel $k(y,t)$ will be assumed to be a difference kernel, as described by the difference(y-t).The 2nd kind of Volterra integral equation can thus expressed as:

$$\phi(y) = f(y) + \lambda \int_0^y k(y-t)\phi(t)dt \qquad (1)$$

The Soham integral transform method is applied to both sides of (1), yielding

$$S\{\phi(y)\} = s\{f(y)\} + \lambda s\{\int_0^y k(y-t)\phi(t)dt\}$$

$$\Rightarrow s\{\phi(y)\} = s\{f(y)\} + \lambda\, s\{k(y) * \phi(y)\} \qquad (2)$$

Now we use convolution theorem of Soham integral transform on both sides of the equation (2), we have

$$S\{\phi(y)\} = s\{f(y)\} + \lambda v.s\{k(y)\}.s\{\phi(y)\}$$
$$\therefore s\{\phi(y)\} - \lambda v.s\{k(y)\}.s\{\phi(y)\} = s\{f(y)\}$$
$$\therefore s\{\phi(y)\}[\,1 - \lambda v.s\{k(y)\}] = s\{f(y)\}$$
$$\therefore \ \mathrm{s}\{\phi(\mathrm{y})\} = \left[\frac{s\{f(y)\}}{1-\lambda v s\{k(y)\}}\right] \tag{3}$$

Inverting the Soham integral transform on equation (3), We obtain:

$$\phi(y) \ = \ \mathrm{s}-1\left[\frac{s\{(y)}{1-\lambda.v.s\{k(y)\}}\right] \tag{4}$$

Equation (4) is the required solution of equation (1)

V. NUMERICAL APPLICATIONS

Some applications are presented in this part to show the unity of Soham integral transform for solving Volterra integral equation of the 2nd kind. Now we solve following Volterra integral equations of second kind.

1. Consider the Volterra integral equation of the 2nd kind :

$$\phi(y) = cosy + \int_0^y sin(y-t)\,\phi(t)dt \tag{5}$$

The Soham transform is applied to both sides of equation (5), yielding,

$$\mathrm{S}\{\phi(\mathrm{y})\} = \mathrm{s}\{\mathrm{cosy}\} + s\{\int_0^y \sin(\mathrm{y}-\mathrm{t})\phi(t)dt \ \Rightarrow \{\phi(y)\} \ = \ s\,\{\cos y\} + s\,\{\sin y\ *\phi(y)\} \tag{6}$$

Now using the Soham transform convolution theorem on equation (6), we have,

$$\mathrm{S}\{\phi(\mathrm{y})\}=\frac{v^{\alpha}}{v(v^{2\alpha}+1)}+\mathrm{vs}\{\mathrm{siny}\}.\mathrm{s}\{\phi(\mathrm{y})\}$$
$$\Rightarrow\mathrm{s}\{\phi(\mathrm{y})\}[1-\frac{v}{v(v^{2\alpha}+1)}] =\frac{v^{\alpha}}{v(v^{2\alpha}+1)}$$
$$\Rightarrow\mathrm{s}\{\phi(\mathrm{y})\}=\frac{1}{v^{\alpha+1}} \tag{7}$$

We obtained required solution (5) by applying the inverse Soham transformation of both sides of (7)

$$\therefore \ \phi(y) = \ s^{-1}\left[\frac{1}{v^{\alpha+1}}\right]$$

$\therefore \phi(\mathrm{y}) = 1$. It is required solution.

2. Consider Volterra integral equation

$$\phi(y)=y+\int_0^y \sin(y-t)\phi(t)dt \tag{8}$$

We apply Soham transform on both sides of equation (8), and obttain,

$$s\{\phi(y)\}= s\{y\}+s\{\int_0^y \sin(y-t)\phi(t)dt\}$$

$$\Rightarrow s\{\phi(y)\}=s\{y\}+s\{\sin y*\phi(y)\} \tag{9}$$

By using convolution theorem of Soham transformation,

$$s\{\phi(y)\} = \frac{1}{v^{2\alpha+1}} + v.s\{\sin y\} * s\{\phi(y)\}$$

$$\Rightarrow s\{\phi(y)\}\left[1 - v.\frac{1}{v(v^{2\alpha}+1)}\right] = \frac{1}{v^{2\alpha+1}}$$

$$\Rightarrow s\{\phi(y)\}\frac{v^{2\alpha}}{v^{2\alpha}+1} = \frac{1}{v^{2\alpha+1}}$$

$$\Rightarrow s\{\phi(y)\} = \frac{1}{v^{2\alpha+1}} \times \frac{v^{2\alpha}+1}{v^{2\alpha}}$$

$$\therefore s\{\phi(y)\} = \frac{1}{v^{2\alpha+1}} + \frac{1}{v^{4\alpha+1}} \tag{10}$$

By applying inverse Soham transformation to both sides of equation (10),

ϕ(y)=$s^{-1}\{\frac{1}{v^{2\alpha+1}}\}$ +$s^{-1}\{\frac{1}{v^{4\alpha+1}}\}$

ϕ(y)= t +$\frac{t^3}{6}$

It is the required solution.

3. Now consider the Volterra integral equation of the 2nd kind:

$$\phi(y) = y + \int_0^y e^{-(y-t)}\phi(t)dt \tag{11}$$

Applying Soham transform on both sides,

$$S\{\phi(y) = s\{y\} + s\{\int_0^y e^{-(y-t)}\phi(t)dt\}$$

$$S\{\phi(y)\} = s\{y\} + v.s\{e^{-y} * \phi(y)\}$$

By using convolution theorem of Soham transform,

S{ϕ(y)}=$\frac{1}{v^{2\alpha+1}}$+ v.s{e^{-y}}. S{ϕ(y)}

$$\therefore S\{\phi(y)\}[1 - \frac{v}{v(v^{\alpha}+1)}] = \frac{1}{v^{2\alpha+1}}$$

$\therefore$ S{ϕ(y)} = $\frac{v^{\alpha}+1}{v^{\alpha}(v^{2\alpha+1})} = \frac{1}{v^{2\alpha+1}} + \frac{1}{v^{3\alpha+1}}$

Applying inverse Soham transform,

$$\phi(y) = s^{-1}\{\frac{1}{v^{2\alpha+1}}\} + s^{-1}\{\frac{1}{v^{3\alpha+1}}\}$$

$$\phi(y) = t + \frac{t^2}{2}$$

It is the required solution.

4. Take the Volterra integral equation of 2^{nd} kind:

$$\emptyset(y)=e^{y}-\cos y-2\int_0^{y} e^{y-t}\,\emptyset(t).dt \tag{12}$$

Applying Soham transform on both sides, we get

$$S\{\emptyset(y)\}=s\{e^{y}\}-s\{\cos y\}-s\{2\int_0^{y} e^{y-t}\,\emptyset(t).dt\}$$

$$S\{\emptyset(y)\}=s\{e^{y}\}-s\{\cos y\}-s\{e^{y}*\emptyset(y)\}$$

Using convolution theorem,

$$s\{\phi(y)\}=\frac{1}{v(v^{\alpha}-1)}-\frac{v^{\alpha}}{v(v^{2\alpha}+1)}-2vs\{e^{y}\}.s\{\phi(y)\}$$

$$s\{\phi(y)\}[1+\frac{2v}{v(v^{\alpha}-1)}]=\frac{1}{v(v^{\alpha}-1)}-\frac{v^{\alpha}}{v(v^{2\alpha}+1)}$$

$$s\{\phi(y)\}=\frac{1}{v(v^{\alpha}+1)}-\frac{v^{\alpha}(v^{\alpha}-1)}{v(v^{2\alpha}+1)(v^{\alpha}+1)}$$

$$\therefore s\{\phi(y)\}=\frac{1}{v(v^{\alpha}+1)}\left\{1-\frac{v^{\alpha}(v^{\alpha}-1)}{(v^{2\alpha}+1)}\right\}$$

$$\therefore s\{\phi(y)\}=\frac{1}{v(v^{\alpha}+1)}\left\{\frac{v^{2\alpha}+1-v^{\alpha}(v^{\alpha}-1)}{(v^{2\alpha}+1)}\right\}$$

$$\therefore s\{\phi(y)\}=\frac{1}{v(v^{\alpha}+1)}\left\{\frac{v^{2\alpha}+1-v^{2\alpha}+v^{\alpha}}{(v^{2\alpha}+1)}\right\}$$

$$\therefore s\{\phi(y)\}=\frac{1}{v(v^{2\alpha}+1)}$$

We apply the inverse Soham transform to above equation.

$$\phi(y)=s^{-1}\{\frac{1}{v(v^{2\alpha}+1)}\}$$

$$\phi(y)=\sin t$$

It is the required solution.

5. Consider the Volterra integral equation of the 2^{nd} kind:

$$\phi(y)=\cos y-\sin y+\int_0^{y}\phi(t)dt$$

Applying Soham transform, on both side we get,

$$S\{\phi(y)\}=s\{\cos y\}-s\{\sin y\}+s\{\int_0^{y}\phi(t).dt$$

$$S\{\phi(y)\}=s\{\cos y\}-s\{\sin y\}+s\{1*\phi(y)\}$$

By using convolution theorem ofSoham transform

$S\{\phi(y)\}=\frac{v^{\alpha}}{v(v^{2\alpha}+1)}-\frac{1}{v(v^{2\alpha}+1)}+v\,s\{1\}.s\{\phi(g)\}$

$$S\{\phi(y)\}\left[1-\frac{v}{v^{\alpha+1}}\right]=\frac{v^{\alpha}-1}{v(v^{2\alpha}+1)}$$

$$\therefore S\{\phi(y)\}\left[1-\frac{1}{v^{\alpha}}\right]=\frac{v^{\alpha}-1}{v(v^{2\alpha}+1)}$$

$$\therefore S\{\phi(y)\}\left[\frac{v^{\alpha}-1}{v^{\alpha}}\right]=\frac{v^{\alpha}-1}{v(v^{2\alpha}+1)}$$

$$\therefore S\{\phi(y)\}=\frac{v^{\alpha}}{v(v^{2\alpha}+1)}$$

By using inverse Soham transform

$\phi(y)=s^{-1}\{\frac{v^{\alpha}}{v(v^{2\alpha}+1)}\}$

$\phi(y)=\cos y$ is the required solution.

VI. CONCLUSION

We applied Soham integral transform to solve the problems on Volterra integral transform successfully. Answers obtained are same as obtained by other methods.

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