

EXPLORING FUZZINESS OF THE PARAMETERS IN PREDATOR-PREY MODEL WITH HARVESTING: AN APPLICATION OF HUKUHARA DERIVATIVES

Abstract

In today's ecological research, the uncertainty surrounding parameter values in mathematical models holds a crucial role. This research article introduces the application of the Hukuhara derivative technique for formulating predator-prey interactions with harvesting. We construct a two-dimensional predator-prey model that accounts for the uncertain behavior of both prey and predator populations. We formulate Fuzzy Differential Equations (FDE) for various scenarios: when both populations, prey $x(t)$ and predator $y(t)$, are i - gH differentiable; when $x(t)$ is i - gH differentiable while $y(t)$ is ii - gH differentiable; when $x(t)$ is ii - gH differentiable and $y(t)$ is i - gH differentiable; and finally, when both populations are in ii - gH differentiable states. We provide analytical descriptions of stability criteria for equilibrium points. Our findings highlight that the dynamic behavior of predator-prey interactions with harvesting hinges on parameter fuzziness. We conclude from this study that the mathematical model's dynamics are strongly influenced by the imprecise nature of these parameters, mirroring a more realistic representation of ecological systems. This research opens a novel dimension in ecological studies.

Keywords: Predator; Prey; Harvesting; Fuzzy; Hukuhara derivative.

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I. INTRODUCTION

In the field of prey-predator system, Harvesting has immense importance as a biological phenomenon to study the management of renewable resources. Fishery and Forestry uses harvesting of prey population [1, 2, 3]. In the recent years, researcher are more interested to work on prey-predator system with Harvesting to maintain renewable resources [4-10]. Fisheries and floristries has been exploiting renewable resources since the past few decades and has become a major concern in bio economic analysis [1, 11]. The dynamics of multy species harvesting models has demanded the time of many researchers [12, 13, 14, 15, 16]. Non-selective harvesting model is studied in [17, 18], though there is a problem on non-selective harvesting policy with logistic growth [1]. This type of model has maximum sustainable yield (MSY). The population will extinct if its harvesting rate exceeds its maximum sustainable yield, and the decreased harvested species can be recovered if the harvesting rate is smaller than its MSY [19].

Harvesting Policy are of two types – one is constant quota harvesting and the other one is constant-effort harvesting. In constant quota harvesting, a steady number of individuals from the population are harvested within a given time frame. Conversely, in constant-effort harvesting, a fixed effort is exerted per unit of time to capture animals from the population [20]. Although a particular rule is not followed by harvesting. For example, more number of fishes are harvested in warmer seasons than in cold seasons [21].

Fuzzy differential equations (FDEs) have gained widespread attention across various fields such as engineering, economics, biology, and physics. In real-world situations, complex problem can be modeled by fuzzy set theory. Individual behaviors within a system often exhibit inherent variations, which are confined to a relatively small set representing the collective characteristic behavior of the group. To effectively quantify the subjective qualities under study, it becomes essential to assign values or degrees that satisfactorily represent these qualities. This task cannot always be accomplished solely through objective measurements or statistics. Consequently, when the state variables of a given demographic system exhibit inherent uncertainty, it becomes imperative to incorporate fuzziness into the system, representing this uncertainty through fuzzy variables. In brief, when constructing models to depict real-world situations, it is common to encounter imprecise or uncertain data. Relying on precise real-life data in such instances may lead to errors. Therefore, it is prudent to consider parameters as imprecise in these situations. Naturally, a question arises: does the model's behavior remain consistent with that of the crisp (non-fuzzy) model? Clearly, the behavior differs, necessitating an investigation of this behavior within this context. FDEs emerge as a potent tool for characterizing the dynamics and characteristics of harvesting models in predator-prey systems.

Fuzzy logic's FL-generalization stands as a remarkable contribution, bridging the gap between crisp sets and those reliant on fuzzy sets. One prominent manifestation of this generalization is Fuzzy Differential Equations (FDEs), a cornerstone of fuzzy mathematics. When all coefficients or parameters within the boundary conditions of a differential equation adopt a fuzzy nature, it qualifies as a fuzzy differential equation [22]. The credit for proposing fuzzy set theory goes to Zadeh, as it encompasses the concept of membership functions and addresses uncertainty [23]. In recent years, fuzzy set theory has surged in popularity within mathematics due to its diverse applications across various scientific

domains. Fuzzy differential equations have proven invaluable for modeling numerous dynamic systems, accommodating the inherent uncertainty associated with parameters [24]. A comprehensive exploration of prey-predator system harvesting, incorporating fuzziness, can be found in [25].

The concept of fuzziness holds a pivotal role in dynamical systems, with the Hukuhara derivative emerging as a pioneering concept for addressing fuzzy differential equations [26]. Fuzzy differential equations were first established in 1978 [22], sparking extensive research efforts to develop and apply this concept [22]. Notably, the solutions of crisp models and fuzzy differential equations exhibit significant disparities, motivating numerous researchers to generalize their findings [24, 27, 28–40]. First-order fuzzy differential equations pose particular challenges, as solutions vary depending on the interpretation [41–43]. Several methods, including generalized differentiability concepts and operator methods, have been devised for solving fuzzy differential equations of first and second-order [44–48]. The generalized Euler approximation method is used to tackle fuzzy differential equations numerically [49,50].

In this research article, our primary objective is to model a predator-prey system with harvesting using Hukuhara derivative and analyze it under the more realistic assumption that both populations are uncertain in nature. In Section 2, we provide an overview of the fundamentals of fuzzy differential equations. Next, in Section 3, we introduce the mathematical model for predator-prey interaction with harvesting. Section 4 is dedicated to fuzzy differential equations, considering different scenarios where both populations, namely the prey population ($x(t)$) and predator population ($y(t)$), exhibit varying degrees of differentiability. These scenarios encompass cases where $x(t)$ is i-gH and $y(t)$ is ii-gH differentiable, as well as scenarios where $x(t)$ is ii-gH and $y(t)$ is i-gH differentiable. We also present the model when both populations are in ii-gH differentiable states and provide the corresponding results. Finally, we conclude the paper with a discussion and draw our overall conclusions.

II. BASIC DEFINITIONS

1. Fuzzy Triangular Number [51]: The membership function of a triangular fuzzy number M , denoted as (m_1, m_2, m_3) , is defined as follows:

$$\begin{aligned} n(x) &= \frac{x-m_1}{m_2-m_1}, & m_1 \leq x \leq m_2 \\ &= \frac{m_3-x}{m_3-m_2}, & m_2 \leq x \leq m_3 \\ &= 0, & m_1 \geq x, m_3 \leq x \end{aligned}$$

2. α -Cut of Triangular Fuzzy Number [51]: The α -cut of triangular fuzzy number $M = (m_1, m_2, m_3)$, $\forall \alpha \in [0,1]$ is given by

$$M_\alpha = [m_1 + \alpha(m_2 - m_1), m_3 - \alpha(m_3 - m_2)].$$

- 3. Generalized Hukuhara Difference [52]:** Let us define the generalized difference of A and B as the set $C \in \mathcal{M}(\mathbb{X})$, where A and B belong to the set $\mathcal{M}(\mathbb{X})$, and this is characterized by the following condition:

$$A \theta_g B = C \Leftrightarrow \{(i) A = B + C \\ \text{\{or (ii) } B = A + (-1)C$$

- 4. Generalized Hukuhara derivative for fuzzy valued function [53]:** The function $g:(a,b) \rightarrow R_F$ is said to be strongly gH-differentiable at p_0 where $p_0 \in (a,b)$, if $g'(p_0) \in R_F$, such that

$\forall \epsilon > 0$ ($0 < \epsilon < 1$), $\exists g(p_0 + \epsilon) \theta_g g(p_0), g(p_0) \theta_g g(p_0 - \epsilon)$ and the limits

$$g'(p_0) = \lim_{\epsilon \rightarrow 0} \frac{g(p_0 + \epsilon) \theta_g g(p_0)}{\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{g(p_0) \theta_g g(p_0 - \epsilon)}{\epsilon}$$

or (ii) $\forall \epsilon > 0$ sufficiently small, $\exists g(p_0) \theta_g g(p_0 + \epsilon), g(p_0 - \epsilon) \theta_g g(p_0)$ and the limits

$$g'(p_0) = \lim_{\epsilon \rightarrow 0} \frac{g(p_0) \theta_g g(p_0 + \epsilon)}{-\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{g(p_0 - \epsilon) \theta_g g(p_0)}{-\epsilon}$$

or (iii) $\forall \epsilon > 0$ sufficiently small, $\exists g(p_0 + \epsilon) \theta_g g(p_0), f(p_0 - \epsilon) \theta_g f(p_0)$ and the limits

$$g'(p_0) = \lim_{\epsilon \rightarrow 0} \frac{g(p_0 + \epsilon) \theta_g g(p_0)}{\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{g(p_0 - \epsilon) \theta_g f(p_0)}{-\epsilon}$$

or (iv) $\forall \epsilon > 0$ sufficiently small, $\exists g(p_0) \theta_g g(p_0 + \epsilon), g(p_0) \theta_g g(p_0 - \epsilon)$ and the limits

$$g'(p_0) = \lim_{\epsilon \rightarrow 0} \frac{g(p_0) \theta_g g(p_0 + \epsilon)}{-\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{g(p_0) \theta_g g(p_0 - \epsilon)}{\epsilon}$$

- 5. Strong and Weak Solutions of Fuzzy Differential Equation:** Consider a fuzzy differential equation $\frac{d g(t)}{dt} = g_1(t, x(t))$ with the initial condition $g(t_0) = x_0$ where x_0 is a fuzzy number. Let, the solution of this fuzzy differential equation is $\tilde{g}(t)$. The α -cut of the solution $\widetilde{g(t)}$ is denoted by $[g_1(t, \alpha), g_2(t, \alpha)]$.

If $g_1(t, \alpha)$ is increasing function, $g_2(t, \alpha)$ is decreasing function and $g_1(t, \alpha) \leq g_2(t, \alpha), \forall \alpha \in [0, 1]$, the solution $\widetilde{g(t)}$ is termed as strong fuzzy solution. Otherwise the solution is called weak fuzzy solution.

Weak solution is also a fuzzy solution, but our aim is to transform the weak solution in to strong solution with the help of the formula

$$\tilde{g}(t, \alpha) = [\min \{g_1(t, \alpha), g_2(t, \alpha)\}, \max \{g_1(t, \alpha), g_2(t, \alpha)\}].$$

III. FORMULATION OF THE MATHEMATICAL MODEL

Here we introduce mathematical model of predator-prey system with harvesting. Where $x(t)$ represent the prey population and $y(t)$ represent the predator population.

$$\begin{aligned} \frac{dx(t)}{dt} &= rx(t) \left(1 - \frac{x(t)}{k_1}\right) - ax(t)y(t) - ex(t) \\ \frac{dy(t)}{dt} &= kax(t)y(t) - d_2y(t) \end{aligned} \quad (1)$$

With $x(0) = x_0$ and $y(0) = y_0$.

Here "r" and "k1" are the intrinsic growth rate and carrying capacity of prey. The predation rate of the prey population by the predator, indicated by "a". The conversion rate of prey population into predator, denoted as "k". The harvesting coefficients for the prey population, given as "e". "d2" is the death rate of the predator population.

IV. PREDATOR-PREY MODEL WITH HARVESTING IN FUZZY ENVIRONMENT

Let's analyze the fuzzy solution of the system of equations (1), which we'll denote as $(\tilde{x}(t), \tilde{y}(t))$. The α -cut of $\tilde{x}(t)$ can be defined as the interval $[x_1(t, \alpha), x_2(t, \alpha)]$, and the α -cut of $\tilde{y}(t)$ can be defined as the interval $[y_1(t, \alpha), y_2(t, \alpha)]$. In this context, the parameter α must satisfy the condition $0 \leq \alpha \leq 1$.

1. Case 1: when $x(t)$ and $y(t)$ both are i-gH Differentiable: With the help of the fuzzy technique, we can write the system of equation (1) as

$$\begin{aligned} \frac{dx_1(t, \alpha)}{dt} &= rx_2(t, \alpha) \left(1 - \frac{x_2(t, \alpha)}{k_1}\right) - ax_2(t, \alpha)y_1(t, \alpha) - ex_2(t, \alpha) \\ \frac{dx_2(t, \alpha)}{dt} &= rx_1(t, \alpha) \left(1 - \frac{x_1(t, \alpha)}{k_1}\right) - ax_1(t, \alpha)y_2(t, \alpha) - ex_1(t, \alpha) \\ \frac{dy_1(t, \alpha)}{dt} &= kax_1(t, \alpha)y_2(t, \alpha) - d_2y_2(t, \alpha) \\ \frac{dy_2(t, \alpha)}{dt} &= kax_2(t, \alpha)y_1(t, \alpha) - d_2y_1(t, \alpha) \end{aligned} \quad (2)$$

With $x_1(0, \alpha) = x_{01}(\alpha), x_2(0, \alpha) = x_{02}(\alpha), y_1(0, \alpha) = y_{01}(\alpha)$ and $y_2(0, \alpha) = y_{02}(\alpha)$

Stability Analysis of System: Interior equilibrium point of the system (2) is given by $E_1^*(x_1^{1*}, x_2^{1*}, y_1^{1*}, y_2^{1*})$

Where,

$$\begin{aligned} x_1^{1*} &= \frac{d_2}{ka}, & x_2^{1*} &= \frac{d_2}{ka}, \\ y_1^{1*} &= \frac{kk_1a(r-e)-d_2r}{kk_1a^2}, & y_2^{1*} &= \frac{kk_1a(r-e)-d_2r}{kk_1a^2} \end{aligned}$$

The existence condition of the equilibrium point E_1^* is $kk_1a(r - e) > d_2r$. This condition is also feasible.

The jacobian matrix corresponding to E_1^* is denoted as V_1^* ,

Where,
$$V_1^* = \begin{pmatrix} 0 & a_{12} & a_{13} & 0 \\ a_{21} & 0 & 0 & a_{24} \\ a_{31} & 0 & 0 & 0 \\ 0 & a_{42} & 0 & 0 \end{pmatrix}$$

And,

$$\begin{aligned} a_{12} &= r - \frac{2rx_2^{1*}}{k_1} - ay_1^{1*} - e, & a_{13} &= -ax_2^{1*}, \\ a_{21} &= r - \frac{2rx_1^{1*}}{k_1} - ay_2^{1*} - e, & a_{24} &= -ax_1^{1*}, \\ a_{31} &= kay_2^{1*}, \\ a_{42} &= kay_1^{1*} \end{aligned}$$

The characteristic equation is given by

$$\xi_1^4 + A_1\xi_1^3 + A_2\xi_1^2 + A_3\xi_1 + A_4 = 0$$

Where,

$$A_1 = 0.$$

$$\begin{aligned} A_2 &= a_{12}a_{21} + a_{24}a_{42} + a_{13}a_{31} \quad \text{where,} \quad a_{12} = r - \frac{2rx_2^{1*}}{k_1} - ay_1^{1*} - e, \quad a_{13} = \\ & -ax_2^{1*}, a_{21} = r - \frac{2rx_1^{1*}}{k_1} - ay_2^{1*} - e, a_{24} = -ax_1^{1*}, a_{42} = -ax_1^{1*}, a_{31} = kay_2^{1*}, a_{42} = \\ & kay_1^{1*}. \\ A_3 &= 0. \\ A_4 &= a_{13}a_{24}a_{31}a_{42}. \\ \text{Here } A_1 &= 0, \quad A_3 = 0, \end{aligned}$$

Therefore, according to the Routh-Hurwitz criteria, the system is unstable if both $x(t)$ and $y(t)$ are i-gH differentiable.

2. Case 2: when $x(t)$ and $y(t)$ are i-gH differentiable and ii-gH differentiable respectively

The system (1) can be written as

$$\begin{aligned} \frac{dx_1(t, \alpha)}{dt} &= rx_2(t, \alpha) \left(1 - \frac{x_2(t, \alpha)}{k_1} \right) - ax_2(t, \alpha)y_1(t, \alpha) - ex_2(t, \alpha) \\ \frac{dx_2(t, \alpha)}{dt} &= rx_1(t, \alpha) \left(1 - \frac{x_1(t, \alpha)}{k_1} \right) - ax_1(t, \alpha)y_2(t, \alpha) - ex_1(t, \alpha) \\ \frac{dy_1(t, \alpha)}{dt} &= kax_2(t, \alpha)y_1(t, \alpha) - d_2y_1(t, \alpha) \\ \frac{dy_2(t, \alpha)}{dt} &= kax_1(t, \alpha)y_2(t, \alpha) - d_2y_2(t, \alpha) \end{aligned} \tag{3}$$

With $x_1(0, \alpha) = x_{01}(\alpha), x_2(0, \alpha) = x_{02}(\alpha), y_1(0, \alpha) = y_{01}(\alpha)$ and $y_2(0, \alpha) = y_{02}(\alpha)$

Stability Analysis of System: Interior point equilibrium of the system (3) is given by $E_2^*(x_1^{2*}, x_2^{2*}, y_1^{2*}, y_2^{2*})$

Where,

$$x_1^{2*} = \frac{d_2}{ka}, \quad x_2^{2*} = \frac{d_2}{ka},$$

$$y_1^{2*} = \frac{kk_1a(r-e)-d_2r}{kk_1a^2}, \quad y_2^{2*} = \frac{kk_1a(r-e)-d_2r}{kk_1a^2}$$

The existence condition of the equilibrium point E_2^* is $kk_1a(r - e) > d_2r$. This condition is also feasible.

The corresponding jacobian matrix corresponding to E_2^* is denoted by V_2^* ,
Where,

$$V_2^* = \begin{pmatrix} 0 & a_{12} & a_{13} & 0 \\ a_{21} & 0 & 0 & a_{24} \\ 0 & a_{32} & 0 & 0 \\ a_{41} & 0 & 0 & 0 \end{pmatrix}$$

and,

$$a_{12} = r - \frac{2rx_2^{2*}}{k_1} - ay_1^{2*} - e, \quad a_{13} = -ax_2^{2*},$$

$$a_{21} = r - \frac{2rx_1^{2*}}{k_1} - ay_2^{2*} - e, \quad a_{24} = -ax_1^{2*},$$

$$a_{32} = kay_1^{2*},$$

$$a_{41} = kay_2^{2*}.$$

The corresponding characteristic equation is given as follows

$$\xi_2^4 + B_1\xi_2^3 + B_2\xi_2^2 + B_3\xi_2 + B_4 = 0$$

Where,

$$B_1 = 0,$$

$$B_2 = a_{12}a_{21}. \text{ where, } a_{12} = r - \frac{2rx_2^{2*}}{k_1} - ay_1^{2*} - e, a_{21} = r - \frac{2rx_1^{2*}}{k_1} - ay_2^{2*} - e,$$

$$B_3 = a_{12}a_{24}a_{41} + a_{13}a_{21}a_{32},$$

$$B_4 = a_{13}a_{24}a_{32}a_{41}.$$

Here $B_1 = 0$,

Therefore as per Routh-Hurwitz criteria, system (3) is unstable.

Therefore, we can conclude that, the system is unstable, if $x(t)$ and $y(t)$ are i-gH differentiable and ii-gH differentiable.

3. Case 3: When $x(t)$ and $y(t)$ are ii-gH differentiable and i-gH differentiable respectively: The system of equation (1) can be written as

$$\frac{dx_1(t, \alpha)}{dt} = rx_1(t, \alpha) \left(1 - \frac{x_1(t, \alpha)}{k_1} \right) - ax_1(t, \alpha)y_2(t, \alpha) - ex_1(t, \alpha)$$

$$\begin{aligned} \frac{dx_2(t, \alpha)}{dt} &= rx_2(t, \alpha) \left(1 - \frac{x_2(t, \alpha)}{k_1}\right) - ax_2(t, \alpha)y_1(t, \alpha) - ex_2(t, \alpha) \\ \frac{dy_1(t, \alpha)}{dt} &= kax_1(t, \alpha)y_2(t, \alpha) - d_2y_2(t, \alpha) \\ \frac{dy_2(t, \alpha)}{dt} &= kax_2(t, \alpha)y_1(t, \alpha) - d_2y_1(t, \alpha) \end{aligned} \quad (4)$$

With $x_1(0, \alpha) = x_{01}(\alpha), x_2(0, \alpha) = x_{02}(\alpha), y_1(0, \alpha) = y_{01}(\alpha)$ and $y_2(0, \alpha) = y_{02}(\alpha)$

Stability analysis of system: The interior point equilibrium corresponding to system of equation (4) is given by $E_3^*(x_1^{3*}, x_2^{3*}, y_1^{3*}, y_2^{3*})$

Where,

$$\begin{aligned} x_1^{3*} &= \frac{d_2}{ka}, & x_2^{3*} &= \frac{d_2}{ka}, \\ y_1^{3*} &= \frac{kk_1a(r-e)-d_2r}{kk_1a^2}, & y_2^{3*} &= \frac{kk_1a(r-e)-d_2r}{kk_1a^2} \end{aligned}$$

The existence condition of E_3^* is $kk_1a(r - e) > d_2r$. This condition is also feasible.

The jacobian matrix corresponding to E_3^* is denoted by V_3^* ,

Where,

$$V_3^* = \begin{bmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & 0 \\ a_{31} & 0 & 0 & 0 \\ 0 & a_{42} & 0 & 0 \end{bmatrix}$$

Where,

$$\begin{aligned} a_{11} &= r - \frac{2rx_1^{3*}}{k_1} - ay_2^{3*} - e, & a_{14} &= -ax_1^{3*}, \\ a_{22} &= r - \frac{2rx_2^{3*}}{k_1} - ay_1^{3*} - e, & a_{23} &= -ax_2^{3*}, \\ a_{31} &= kay_2^{3*}, \\ a_{42} &= kay_1^{3*} \end{aligned}$$

The characteristic equation is given by

$$\xi_3^4 + C_1\xi_3^3 + C_2\xi_3^2 + C_3\xi_3 + C_4 = 0$$

Where,

$$C_1 = a_{11} + a_{22} . \text{ where, } a_{11} = r - \frac{2rx_1^{3*}}{k_1} - ay_2^{3*} - e, a_{22} = r - \frac{2rx_2^{3*}}{k_1} - ay_1^{3*} - e.$$

$$C_2 = a_{11}a_{22} . \text{ where, } a_{11} = r - \frac{2rx_1^{3*}}{k_1} - ay_2^{3*} - e, a_{22} = r - \frac{2rx_2^{3*}}{k_1} - ay_1^{3*} - e.$$

$$C_3 = 0.$$

$$C_4 = a_{14}a_{23}a_{31}a_{42}. \text{ Where } a_{14} = -ax_1^{3*}, a_{23} = -ax_2^{3*}, a_{31} = kay_2^{3*}, a_{42} = kay_1^{3*}.$$

$$\text{Here } C_3 = 0,$$

Therefore the above system is unstable by Routh-Hurwitz criteria, So the system (4) is unstable when $x(t)$ and $y(t)$ are ii-gH and i-gH differentiable respectively.

4. Case 4:when $x(t)$ and $y(t)$ both are ii-gH differentiable

We can write the system (1) as

$$\begin{aligned} \frac{dx_1(t, \alpha)}{dt} &= rx_1(t, \alpha) \left(1 - \frac{x_1(t, \alpha)}{k_1}\right) - ax_1(t, \alpha)y_2(t, \alpha) - ex_1(t, \alpha) \\ \frac{dx_2(t, \alpha)}{dt} &= rx_2(t, \alpha) \left(1 - \frac{x_2(t, \alpha)}{k_1}\right) - ax_2(t, \alpha)y_1(t, \alpha) - ex_2(t, \alpha) \\ \frac{dy_1(t, \alpha)}{dt} &= kax_2(t, \alpha)y_1(t, \alpha) - d_2y_1(t, \alpha) \\ \frac{dy_2(t, \alpha)}{dt} &= kax_1(t, \alpha)y_2(t, \alpha) - d_2y_2(t, \alpha) \end{aligned} \tag{5}$$

With $x_1(0, \alpha) = x_{01}(\alpha), x_2(0, \alpha) = x_{02}(\alpha), y_1(0, \alpha) = y_{01}(\alpha)$ and $y_2(0, \alpha) = y_{02}(\alpha)$

Stability Analysis of System: Interior point equilibrium of system (5) is given by $E_4^*(x_1^{4*}, x_2^{4*}, y_1^{4*}, y_2^{4*})$

Where,

$$\begin{aligned} x_1^{4*} &= \frac{d_2}{ka}, & x_2^{4*} &= \frac{d_2}{ka}, \\ y_1^{4*} &= \frac{kk_1a(r-e)-d_2r}{kk_1a^2}, & y_2^{4*} &= \frac{kk_1a(r-e)-d_2r}{kk_1a^2} \end{aligned}$$

The existence condition of the equilibrium point E_4^* is $kk_1a(r - e) > d_2r$. This condition is feasible condition also.

The corresponding jacobian matrix is given by V_4^* ,

$$V_4^* = \begin{bmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & 0 \\ a_{41} & 0 & 0 & a_{44} \end{bmatrix}$$

Where,

$$\begin{aligned} a_{11} &= r - \frac{2rx_1^{4*}}{k_1} - ay_2^{4*} - e, & a_{14} &= -ax_1^{4*}, \\ a_{22} &= r - \frac{2rx_2^{4*}}{k_1} - ay_1^{4*} - e, & a_{23} &= -ax_2^{4*}, \\ a_{32} &= kay_1^{4*}, & a_{33} &= 0, \\ a_{41} &= kay_2^{4*}, & a_{44} &= 0. \end{aligned}$$

The characteristic equation is given by

$$\xi^{*4} + P_1\xi^{*3} + P_2\xi^{*2} + P_3\xi^* + P_4 = 0$$

Where,

$$P_1 = a_{11} + a_{22} \cdot \text{where, } a_{11} = r - \frac{2rx_1^{*4}}{k_1} - ay_2^{*4} - e, a_{22} = r - \frac{2rx_2^{*4}}{k_1} - ay_1^{*4} - e,$$

$$P_2 = a_{23}a_{32} - a_{11}a_{22} + a_{14}a_{41} \quad \cdot \quad \text{where, } a_{14} = -ax_1^{*4}, a_{23} = -ax_2^{*4}, a_{32} = kay_1^{*4}, a_{41} = kay_2^{*4}.$$

$$P_3 = a_{11}a_{23}a_{32} + a_{14}a_{22}a_{41},$$

$$P_4 = a_{14}a_{23}a_{32}a_{41}.$$

Now,

$$P_1 > 0$$

$$P_2 > 0$$

$$P_3 > 0$$

$$P_4 > 0$$

The system (5) is stable if P_1, P_2, P_3, P_4 all are positive and $P_1P_2 - P_3 > 0$ and $P_1P_2P_3 - P_3^2 - P_1^2P_4 > 0$, according to the Routh-Hurwitz criteria. So the system (5) is stable if $x(t)$ and $y(t)$ both are ii-gH differentiable.

V. DISCUSSION AND CONCLUSION

In this research article, we investigate the dynamic behavior of predator-prey interactions with harvesting within a fuzzy environment. In mathematical representations of ecological problems, not all parameters need to be precise; some may exhibit imprecision to capture various behavioral aspects. The presence of impreciseness among these parameters leads to changes in the system's dynamical behavior. Fuzzy concepts play a fundamental role in understanding and modeling such dynamical systems.

In this research article, we begin by examining a two-dimensional Predator-Prey system with harvesting. Subsequently, we establish four sets of fuzzy differential equation models for the Predator-Prey system with harvesting, along with fuzzy initial values. In the initial fuzzy differential equation, we assume that both populations, i.e., $x(t, \alpha)$, $y(t, \alpha)$, are i-gH differentiable, and we analyze the model's dynamical behavior analytically. Next, we investigate a scenario where $x(t, \alpha)$ is i-gH differentiable, while $y(t, \alpha)$ is ii-gH differentiable, and we observe the model's dynamical behavior. Then, we consider the opposite scenario, where $x(t, \alpha)$ is ii-gH differentiable and $y(t, \alpha)$ is i-gH differentiable, and we analyze the model's dynamics analytically. Finally, we examine a case where both populations, $x(t, \alpha)$, $y(t, \alpha)$ are ii-gH differentiable, and we observe the model's dynamical behavior analytically. In our observations, we find that the equilibrium points corresponding to the first three fuzzy systems of equations are unstable, while the equilibrium point corresponding to the last fuzzy system of equations is stable. This means that when both populations are ii-gH differentiable, the solution remains stable.

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