BANACH FIXED CONTRACTION MAPPING THEOREM IN VECTOR S-METRIC SPACES

Abstract

We demonstrate the Banach contraction mapping theorem on vector S-metric space. We also give an example to explain our results.

Keywords: Vector metric space, Vector lattice, Vector S-metric space.

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I. INTRODUCTION

Banach Contraction Principle(BCP) was demonsted firstly by S. Banach [2] in 1922. It has a vital role in fixed point(FP) theory and became very famous due to iterations. Many researchers are establishing new results in various generalizations of metric spaces. S-metric space is one of the generalizations in metric spaces. In 2012, S-metric space was defined by Sedghi et al.[7]. We start with some definitions and results for vector S-metric spaces(VSMS).

Definition 1: [4] On a set $\complement$, a relation $\preceq$ is a partial order if it follows the conditions stated below:

1. $\Delta_1 \preceq \Delta_1$ (reflexive)
2. $\Delta_1 \preceq \Delta_2$ and $\Delta_2 \preceq \Delta_1$ implies $\Delta_1 = \Delta_2$ $(anti-symmetry)$
3. $\Delta_1 \preceq \Delta_2$ and $\Delta_2 \preceq \Delta_3$ implies $\Delta_1 \preceq \Delta_3$ $(transitivity)$
   $\forall\ \Delta_1, \Delta_2, \Delta_3 \in \complement$.

The set $\complement$ with partial order $\preceq$ is known as partially ordered set (poset).

A partially ordered set $(\complement, \preceq)$ is called linearly ordered if for $\Delta_1, \Delta_2 \in \complement$, we have either $\Delta_1 \preceq \Delta_2$ or $\Delta_2 \preceq \Delta_1$.

Definition 2: [4] Let $\complement$ be linear space which is real and $(\complement, \preceq)$ be a poset . Then the poset $(\complement, \preceq)$ is said to be an ordered linear space if it follows the properties mentioned below:

1. $\wp_1 \preceq \wp_2 \Longrightarrow \wp_1 + \wp_3 \preceq \wp_2 + \wp_3$
2. $\wp_1 \preceq \wp_2 \Longrightarrow \omega\wp_1 \preceq \omega\wp_2$

   $\forall \wp_1, \wp_2, \wp_3 \in \complement$ and $\omega > 0$.

Definition 3: [4] A poset is called lattice if each set with two elements has an infimum and a supremum.

Definition 4: [4] An ordered linear space where the ordering is lattice is called vector lattice(VL).

Definition 5: [4] A VL K is called Archimedean if $inf\{\frac{1}{m}\Omega\} = 0$ for every $\Omega \in K^+$ where

$$K^+ = \{\Omega \in K : \Omega \succeq 0\}.$$

Definition 6: [3] Let K be VL and $\mathfrak{R}$ be a nonvoid set. A function $d : \mathfrak{R} \times \mathfrak{R} \to K$ is called vector metric on $\mathfrak{R}$ if it follows the conditions stated below:

1. $d(\Omega_1, \Omega_2) = 0$ iff $\Omega_1 = \Omega_2$
2. $d(\Omega_1, \Omega_2) \preceq d(\Omega_1, \Omega_3) + d(\Omega_3, \Omega_2)$ $\forall \Omega_1, \Omega_2, \Omega_3 \in \mathfrak{R}$
   The triple $(\mathfrak{R}, d, K)$ is called vector metric space.

Now, vector valued S-metric space is defined as follows:

Definition 8: [10} Let K be VL and $\mathfrak{R}$ be a nonvoid set. A function $S: \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R} \rightarrow K$ is called vector S-metric on $\mathfrak{R}$ that satisfies the conditions mentioned below:

1. $S(\wp_1, \wp_2, \wp_3) \geq 0$,
2. $S(\wp_1, \wp_2, \wp_3) = 0$ iff $\wp_1 = \wp_2 = \wp_3$,
3. $S(\wp 1, \wp 2, \wp 3) \leq S(\wp 1, \wp 1, \alpha) + S(\wp 2, \wp 2, \alpha) + S(\wp 3, \wp 3, \alpha)$

   for all $\wp_1, \wp_2, \wp_3, \alpha \in \mathfrak{R}$

The triplet $(\mathfrak{R}, S, K)$ is called vector S-metric space(VSMS).

Example 1: *Let $\mathfrak{R}$ be a nonvoid set and K be a VL. A function $S: \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R} \rightarrow K$ is defined by*

$$S(\wp_1, \wp_2, \wp_3) = |(\wp_1, \wp_3)| + |(\wp_2, \wp_3)| \quad \forall \wp_1, \wp_2, \wp_3 \in \mathfrak{R}$$

then the triplet $(\mathfrak{R}, S, K)$ is VSMS.

Definition 9: A sequence $\langle \hbar_n \rangle$ in VSMS $(\mathfrak{R}, S, K)$ is called K-convergent to some $\hbar \in K$ if there is a sequence $\langle \mu_n \rangle$ in K satisfying $\mu_n \downarrow 0$ and $S(\hbar_n, \hbar_n, \hbar) \leq \mu_n$ and denote it by $\hbar\text{n} \xrightarrow{S,K} \hbar$.

Definition 10: A sequence $\langle \hbar_n \rangle$ in VSMS $(\mathfrak{R}, S, K)$ is known as K-Cauchy sequence if there is a sequence $\langle \mu_n \rangle$ in K satisfying $\mu_n \downarrow 0$ and $S(\hbar_n, \hbar_n, \hbar_{n+q}) \leq \mu_n$ holds for all q and n.

Definition 11: If each K-Cauchy sequence in $\mathfrak{R}$ is K-converges to a limit in $\mathfrak{R}$ then VSMS $(\mathfrak{R}, S, K)$ is called K-complete .

Lemma[8] For VSMS $(\mathfrak{R}, S, K)$,

$$S(\hbar, \hbar, \mu) = S(\mu, \mu, \hbar) \qquad \forall \mu, \hbar \in \mathfrak{R}.$$

II. MAIN RESULTS

Theorem 1: *Let $(\mathfrak{R}, S, K)$ be a VSMS which is K-complete and K be Archimedean. Suppose the transformation $f: \mathfrak{R} \rightarrow \mathfrak{R}$ satisfies*

$$S(f\,\Omega, f\,\Omega, f\,\hbar) \leq qS(\Omega, \Omega, \hbar)\ \forall \Omega, \hbar \in \mathfrak{R}$$

where $q \in [0,1)$. Then f has FP in $\mathfrak{R}$ which is unique and for any $\wp_0 \in \mathfrak{R}$, iterative sequence $\langle \wp_m \rangle$ defined by $\wp_m = f\wp_{m-1}$, for all $m \in \mathbb{N}$, K-converges to FP of f.

Proof: Let $\wp_0 \in \mathfrak{R}$ and $\langle \wp_m \rangle$ defined by $\wp_m = f\wp_{m-1}$ for $m \in \mathbb{N}$.Then we have

$$S(\wp_m, \wp_m, \wp_{m+1}) = S(f\wp_{m-1}, f\wp_{m-1}, f\wp_m) \leq qS(\wp_{m-1}, \wp_{m-1}, \wp_m) \leq \ldots \leq q^m S(\wp_0, \wp_0, \wp_1)$$

Thus for $m, p \in \mathbb{N}$

$$
\begin{aligned}
S(\wp_m,\wp_m,\wp_{m+p}) \preceq\ & 2S(\wp_m,\wp_m,\wp_{m+1}) + \\
& 2S(\wp_{m+1},\wp_{m+1},\wp_{m+2}) + \\
& \dots + S(\wp_{m+p-1},\wp_{m+p-1},\wp_{m+p}) \\
& \preceq 2S(\wp_m,\wp_m,\wp_{m+1}) \\
& 2S(\wp_{m+1},\wp_{m+1},\wp_{m+2}) + \\
& \dots + 2S(\wp_{m+p-1},\wp_{m+p-1},\wp_{m+p}) \\
& \preceq 2(q^m + q^{m+1} + \dots + q^{m+p-1})\ S(\wp_0,\wp_0,\wp_1) \\
& \preceq 2q^{m+p-1}(1 + q + q^2 + \cdots)\ S(\wp_0,\wp_0,\wp_1) \\
& \preceq 2\frac{q^{m+p-1}}{1-q} S(\wp_0,\wp_0,\wp_1).
\end{aligned}
$$

Thus $\langle\wp_m\rangle$ is a K-Cauchy sequence because K be Archimedean. Then by K-completeness of $\Re$, there exist $\wp \in \Re$ such that $\wp_m \xrightarrow{S,K} \wp$. So there exist $\langle b_m\rangle$ in K such that $b_m \downarrow 0$ and $S(\wp_m,\wp_m,\wp) \preceq b_m$. Since

$$
\begin{aligned}
S(f\wp,f\wp,\wp) \preceq\ & 2S(f\wp_m,f\wp_m,f\wp) + S(f\wp_m,f\wp_m,\wp \\
& \preceq 2qS(\wp_m,\wp_m,\wp) + S(\wp_{m+1},\wp_{m+1},\wp) \\
& \preceq 2qb_m + b_{m+1} \\
& \preceq 2(q+1)b_m,
\end{aligned}
$$

then $S(f\wp,f\wp,\wp) = 0$, i.e. $f\wp = \wp$.
We can also verify the following theorem as above.

Theorem 2 *Let $(\Re,S,K)$ be a VSMS which is complete and K be Archimedean. Suppose the transformation $f:\Re \to \Re$ satisfies*

$$
\begin{aligned}
S(f\Omega,f\Omega,f\wp) \preceq \{&a_1S(\Omega,\Omega,f\Omega) + a_2S(\wp,\wp,f\wp) + a_3S(\Omega,\Omega,f\wp) + \\
&a_4S(\wp,\wp,f\Omega) + a_5S(\Omega,\Omega,\wp)\}
\end{aligned}
$$

for all $\Omega,\wp \in \Re$, where a_1,a_2,a_3,a_4 and a_5 are positive and $a_1+a_2+a_3+a_4+a_5 < 1$. Then f has FP in $\Re$ and for any $\wp_0 \in \Re$, iterative sequence $\langle\wp_m\rangle$ defined by $y_m = f\wp_{m-1}$, $m \in \mathbb{N}$, K-converges to FP of f.

Example 2 *Let $K = \mathbb{R}^2_+$ with coordinatewise ordering and let*

$$\Re = \{(0,\wp) \in \mathbb{R}^2 : 0 \le \wp \le 1\} \cup \ (\wp,0) \in \mathbb{R}^2 : 0 \le \wp \le 1\}.$$

The mapping $S: \Re \times \Re \times \Re \to K$ is defined by

$$S((\Omega,0),(\Omega,0),(\wp,0)) = (\tfrac{4}{3}|\Omega-\wp|, |\Omega-\wp|)$$
$$S((0,\Omega),(0,\Omega),(0,\wp)) = (|\Omega-\wp|, \tfrac{2}{3}|\Omega-\wp|)$$
$$S((\Omega,0),(\Omega,0),(0,\wp)) = (\tfrac{4}{3}\Omega+\wp, \Omega+\tfrac{2}{3}\wp)$$

Then $\Re$ is VSMS which is complete.

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