DYNAMICS OF A PREY AND TWO PREDATORS MODEL WITH DISTRIBUTED TYPE TIME DELAY

Abstract

This chapter describes stability analysis of a Prey and Two predators Ecological model. Two predators are competing for same prey and they have alternative food resources other than prey. Distributed type delay is incorporated in the interaction of prey and second predator is taken for investigation. The system is described by a system of integro-differential equations and local stability is studied at its interior equilibrium points. The global stability is addressed by constructing a suitable Lyapunov function. The effect of Time delay on the dynamical behaviour of the system is studied using exponential delay kernels. The delay kernels with different strengths are identified in which prey population growth is significant is shown using Numericalsimulation. The weight kernel dynamics is compared with the system when no delay arguments are present.

Keywords: Equilibrium points, Local stability, Global stability, Time delay.

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I. INTRODUCTION

Mathematical methods are well known in the field of ecological classifications. Ecological stability drive intention of many mathematicians in to this field in recent era. The ecological interactions are broadly classified in to prey-predation, competition mutualism, ammensalism etc. Prey-predator models always draw the attention of many researchers The ecological models with mathematical treatment areinitiated by Lokta [1] and volterra [2]. Later on, Kapur[3,4] discussed various models related to ecology. May [5], Freedman [6], Paulcolinvaux [7] contributed a lot to this field. The modelling of ecological system is mainly by differential equations. Braun [8] and Simon's [9] explain the applications of differential equations in this area. Prey-predator interaction has immense importance. A lot research has been done on prey-predator models.Recently stochastic prey-predation interaction and prey-refuge and additional food by A. Das, G.P. Samanta [18,20]. Bapan Ghosh [19] studied the stability switching and hydra effect in a predator–preypopulation.

Three species models are also well versed in ecological systems Vidyanath e.t al [21] studied the dynamics of one -predator and two preys. Shiva Reddy et.al [22,23] the dynamics of the three species model with two predators and one prey as well as prey, predator and super predator models.

Much work is done in two species dynamics. Time delay are very common in ecological phenomenon. A time delay occurs in any ecological interaction. These delays cause a cascade effect in stability of the ecological system. A small delay can cause a big change in the system stability. Naturally the delay can be classified as discrete, continuous and distributed. The nature of the delay depends upon the past history the models can be well explained by using distributed type delays.

The distributed time lags are more appropriate to represent the ecological patterns where time delays are depending on past history. The stability aspects of distributed time lags are widely studied by Cushing, J.M [10], and Sreehari Rao [11], Gopalaswamy. K [12]. Time delay in interactions in three species models with a prey, predator and competitor models are discussed by paparao [13-17]. In spite of this, we proposed a three species ecological model with asingle prey with two predators with logistic growth type. The system dynamics is studied at interior equilibrium point. Numerical simulation is carried out for different delay kernel strengths in support of stability analysis.
The chapter divided in to five sections in which the delay dynamics of the model studied both locally and globally with suitable numerical simulation.

II. MATHEMATICAL MODEL:

The model consisting of a single prey (x) and two predators namely first predator(y), second predator (z). Here two predators are competing for the same food (x). A time delay is induced in the interaction of prey and second predator (Gestation period of the prey). Predators are of generalist type and can sustain in absence of prey population. The ecological system is considered with all three population are non-zero and the interaction coefficients are positive in nature. The model equations are formed using the following system of integro - differential equations.

$$\begin{aligned}\frac{dx}{dt} &= a_1x(1-\frac{x}{L_1}) - \alpha_{12}xy - \alpha_{13}x\int_{-\infty}^{t} k_1(t-u)\, z(u)du \\ \frac{dy}{dt} &= a_2y\left(1-\frac{y}{L_2}\right) + \alpha_{21}xy - \alpha_{23}yz \\ \frac{dz}{dt} &= a_3z(1-\frac{z}{L_3}) + \alpha_{31}z\int_{-\infty}^{t} k_2(t-u)\, x(u)du - \alpha_{32}yz\end{aligned} \tag{2.1}$$

The parameters are described with the following notations

x ,y&z are Density of prey, first and second predator respectively.
a_1, a_2&a_3: Growth rates prey, first and second predator
α_{12},α_{21}, $\alpha_{31,}$ α_{13} : interaction coefficient among three populations.
α_{23}&α_{32} : Interaction among predators.
$k_1(t-u)$&$k_2(t-u)$: Kernel strengths.

Put $\frac{a_1}{L_1} = \alpha_{11}$, $\frac{a_2}{L_2} = \alpha_{22}$, $\frac{a_3}{L_3} = \alpha_{33}$ and t-u = w, we get the following system of equations

$$\begin{aligned}\frac{dx}{dt} &= a_1x - \alpha_{11}x^2 - \alpha_{12}xy - \alpha_{13}x\int_0^{\infty} k_1(w)\, z(t-w)dw \\ \frac{dy}{dt} &= a_2y - \alpha_{22}y^2 + \alpha_{21}xy - \alpha_{23}yz \\ \frac{dz}{dt} &= a_3z - \alpha_{33}z^2 + \alpha_{31}z\int_0^{\infty} k_2(w)\, x(t-w)dw - \alpha_{32}yz\end{aligned} \tag{2.2}$$

Choose the kernels k_1 and k_2 such that

$$\int_0^{\infty} k_1(w)dw = 1, \int_0^{\infty} k_2(w)dw = 1\, , \int_0^{\infty} wk_1(w)dw < \infty, \& \int_0^{\infty} wk_2(w)dw < \infty \tag{2.3}$$

Assume the solutions for the above model (2.3) as

$$x_1 = A_1e^{\lambda t}, \quad x_2 = A_2e^{\lambda t}, \quad x_3 = A_3e^{\lambda t}$$

and substituting in (2.3) we get the following system of equations

$$\begin{aligned}\frac{dx}{dt} &= a_1x - \alpha_{11}x^2 - \alpha_{12}xy - \alpha_{13}xz\, w_1(\lambda) \\ \frac{dy}{dt} &= a_2y - \alpha_{22}y^2 + \alpha_{21}xy - \alpha_{23}yz \\ \frac{dz}{dt} &= a_3z - \alpha_{33}z^2 + \alpha_{31}xzw_2(\lambda) - \alpha_{32}yz\end{aligned} \tag{2.4}$$

Where

$$w_1(\lambda) = \int_0^{\infty} k_1(w)e^{-\lambda z}dz = L\{k_1(w)\},\ \text{i.e., Laplace Transform of } k_1(w)$$

$$w_2(\lambda) = \int_0^{\infty} k_2(w)e^{-\lambda z}dz = L\{k_2(w)\}\ i.e., Laplace\ Transform\ of\ k_2(w).$$

III. CO-EXISTING STATE

The co-existing state is obtained by equating system of equations (2.4) and given by

$$\bar{x} = \frac{a_1(\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32})+a_2(\alpha_{13}\alpha_{32}w_1(\lambda)-\alpha_{12}\alpha_{33})+a_3(\alpha_{12}\alpha_{23}-\alpha_{13}\alpha_{22}w_1(\lambda))}{\alpha_{11}(\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32})+\alpha_{12}(\alpha_{21}\alpha_{33}-\alpha_{31}\alpha_{23})+w_1(\lambda)\alpha_{13}(\alpha_{31}w_2(\lambda)\alpha_{22}-\alpha_{21}\alpha_{32})}$$

$$\bar{y} = \frac{a_1\left(\alpha_{21}\alpha_{33}-\alpha_{31}\alpha_{23}w_2(\lambda)\right)+a_2\left(\alpha_{11}\alpha_{33}+\alpha_{13}\alpha_{31}w_1(\lambda)w_2(\lambda)\right)-a_3(\alpha_{11}\alpha_{23}+\alpha_{13}\alpha_{21}w_1(\lambda))}{\alpha_{11}(\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32})+\alpha_{12}(\alpha_{21}\alpha_{33}-\alpha_{31}\alpha_{23})+w_1(\lambda)\alpha_{13}(\alpha_{31}w_2(\lambda)\alpha_{22}-\alpha_{21}\alpha_{32})}$$

$$\bar{z} = \frac{a_1(\alpha_{22}\alpha_{31}w_2(\lambda)-\alpha_{21}\alpha_{32})-a_2\left(\alpha_{11}\alpha_{32}+\alpha_{12}\alpha_{31}w_2(\lambda)\right)+a_3(\alpha_{11}\alpha_{22}+\alpha_{12}\alpha_{21})}{\alpha_{11}(\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32})+\alpha_{12}(\alpha_{21}\alpha_{33}-\alpha_{31}\alpha_{23})+w_1(\lambda)\alpha_{13}(\alpha_{31}w_2(\lambda)\alpha_{22}-\alpha_{21}\alpha_{32})}$$

. (3.1)

This equilibrium state exist only when, $\bar{x} > 0$, $\bar{y} > 0$, $\bar{z} > 0$ (3.2)

IV. LOCAL STABILITY ANALYSIS

Theorem4.1: The system (2.4) locally asymptotically stable at co-existing state $E(\bar{x},\bar{y},\bar{z})$.

Proof: Let the variational matrix is given by

$$J = \begin{bmatrix} -\alpha_{11}\bar{x} & -\alpha_{12}\bar{x} & -\alpha_{13}\bar{x}w_1(\lambda) \\ \alpha_{21}\bar{y} & -\alpha_{22}\bar{y} & -\alpha_{23}\bar{y} \\ \alpha_{31}\bar{z}w_2(\lambda) & -\alpha_{32}\bar{z} & -\alpha_{33}\bar{z} \end{bmatrix} \quad (4.1)$$

With The characteristic equation $\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$ (4.2)

Where $b_1 = (\alpha_{11}\bar{x} + \alpha_{22}\bar{y} + \alpha_{33}\bar{z})$

$$(b_2 = (\alpha_{11}\alpha_{22}+\alpha_{12}\alpha_{21})\bar{x}\bar{y} + (\alpha_{11}\alpha_{33}+\alpha_{13}\alpha_{31}k_1(\lambda)w_2(\lambda)\bar{x}\bar{z} + (\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32})\bar{y}\bar{z}$$

$$b_3 = \bar{x}\bar{y}\bar{z}(\alpha_{11}\alpha_{22}\alpha_{33}+\alpha_{12}\alpha_{21}\alpha_{33}+\alpha_{13}\alpha_{22}\alpha_{31}w_1(\lambda)w_2(\lambda)-\alpha_{11}\alpha_{23}\alpha_{32}- \alpha12\alpha23\alpha31w2(\lambda)-\alpha13\alpha21\alpha32w1(\lambda)$$

(4.3)

Calculate the Routh-Hurwitz determinates b_1, $(b_1b_2-b_3)$ and $b_3(b_1b_2-b_3)$
If all the determinates are positive, the system becomes stable otherwise system becomes unstable.

Clearly $b_1 = (\alpha_{11}\bar{x}+\alpha_{22}\bar{y}+\alpha_{33}\bar{z}) > 0$

By algebraic calculations

$$\begin{aligned}(b_1b_2-b_3) = {} & (\alpha_{11}{}^2\alpha_{22}+\alpha_{11}\alpha_{12}\alpha_{21})\bar{x}^2\bar{y} + (\alpha_{11}{}^2\alpha_{33}+\alpha_{11}\alpha_{13}\alpha_{31}w_1(\lambda)w_2(\lambda))\bar{x}^2\bar{z} \\ & +(\alpha_{22}{}^2\alpha_{33}-\alpha_{22}\alpha_{23}\alpha_{32})\bar{y}^2\bar{z} + (\alpha_{22}{}^2\alpha_{11}+\alpha_{22}\alpha_{12}\alpha_{21})\bar{y}^2\bar{x} \\ & +(\alpha_{11}\alpha_{33}{}^2+\alpha_{33}\alpha_{13}\alpha_{31}w_1(\lambda)w_2(\lambda))\bar{z}^2\bar{x} + (\alpha_{22}\alpha_{33}{}^2-\alpha_{33}\alpha_{23}\alpha_{32})\bar{z}^2\bar{y} \\ & +\bar{x}\bar{y}\bar{z}(2\alpha_{11}\alpha_{22}\alpha_{33}+\alpha_{12}\alpha_{23}\alpha_{31}w_2(\lambda)+\alpha_{13}\alpha_{21}\alpha_{32}w_1(\lambda))\end{aligned}$$

$(b_1b_2-b_3) > 0$ (Majority of the terms are positive) (4.4)

Also $b_3\left(b_1b_2-b_3\right)>0$
Hence the co-existing state $E(\bar{x}\,,\bar{y},\bar{z})$ is locally asymptotically stable

V. GLOBAL STABILITY

Theorem 5.1: The co-existing state $E(\bar{x}\,,\bar{y},\bar{z})$is globally asymptotically stable

Proof: Consider the Lyapunov function be

$$V(\bar{x}\,,\bar{y}\,,\bar{z}) = (x-\bar{x}) - x\log\frac{x}{\bar{x}} + (y-\bar{y}) - y\log\frac{y}{\bar{y}} + (z-\bar{z}) - z\log\frac{z}{\bar{z}} \tag{5.1}$$

The time derivate of ‘V’ along the solutions of equations (2.4) is

$$\begin{aligned} V^1(t) = [x-\bar{x}]\left(a_1-\alpha_{11}x-\alpha_{12}y-\alpha_{13}\int_0^\infty k_1(w)z(t-w)dw\right) \\ +[y-\bar{y}](a_2-\alpha_{22}y+\alpha_{21}x-\alpha_{23}z) \\ +[z-\bar{z}]\left(a_3-\alpha_{32}z+\alpha_{31}\int_0^\infty k_2(w)x(t-w)dw-\alpha_{32}y\right) \end{aligned} \tag{5.2}$$

by proper choice of $a_1, a_2 \,\&\, a_3$

$$a_1 = \alpha_{11}\bar{x}+\alpha_{12}\bar{y}+\alpha_{13}\int_0^\infty k_2(w)z(t-w)dw$$
$$a_2 = -\alpha_{21}\bar{x}+\alpha_{22}\bar{y}+\alpha_{23}\bar{z}$$
$$\& a_3 = -\alpha_{31}\int_0^\infty k_2(w)x(t-w)dw+\alpha_{33}\bar{z}+\alpha_{32}\bar{y}$$

Substitute the above in equation (5.2) we get

$$-\alpha_{11}(x-\bar{x})^2-\alpha_{22}(y-\bar{y})^2-\alpha_{33}(z-\bar{z})^2-(\alpha_{32}+\alpha_{23})(y-\bar{y})(z-\bar{z})+(\alpha_{21}-\alpha_{12})(y-\bar{y})(x-\bar{x}) \tag{5.3}$$

Using the inequality $ab \le \frac{a^2+b^2}{2}$

$$-\alpha_{11}(x-\bar{x})^2-\alpha_{22}(y-\bar{y})^2-\alpha_{33}(z-\bar{z})^2-\frac{(\alpha_{32}+\alpha_{23})}{2}[(y-\bar{y})^2+(z-\bar{z})^2]$$
$$+\frac{(\alpha_{21}-\alpha_{12})}{2}[(y-\bar{y})^2+(x-\bar{x})^2]$$
$$V'(t) \le -\mu[(x-\bar{x})^2+(y-\bar{y})^2+(z-\bar{z})^2] < 0$$

Where $\mu = \min\left(\alpha_{11}+\alpha_{22}+\alpha_{33}+\frac{1}{2}\alpha_{23}+\frac{1}{2}\alpha_{32}+\frac{1}{2}\alpha_{21}-\frac{1}{2}\alpha_{12}-\frac{1}{2}(\alpha_{31}+\alpha_{13})\right)$

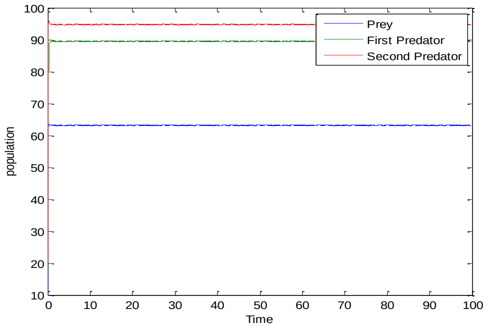
$\frac{dV}{dt}<0$ Therefore, the system is globally stable at interior equilibrium $E(\bar{x}\,,\bar{y},\bar{z})$

VI. NUMERICAL SIMULATION

Example 1: Let a_1=2.5; a_2=1.5; a_3=2.5; α_{12}=0.05; α_{13}=0.05; α_{21}=0.05; α_{23}=0.05; α_{31}=0.05; α_{32}=0.05; x=15, y=15, z=15; L1=100;L2=100;L3=100.

The system of equations (2.3) with above parametric values are simulated using

MATLAB without infuse delay arguments. The system is stable and converging to the fixed equilibrium pointE (63, 89, 95). The graphs are given below with 6.1 (A) time series plot & 6.1 (B) phase portrait.



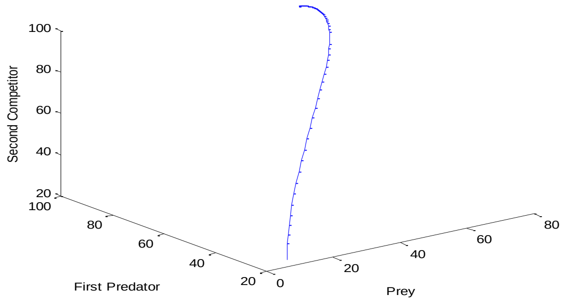


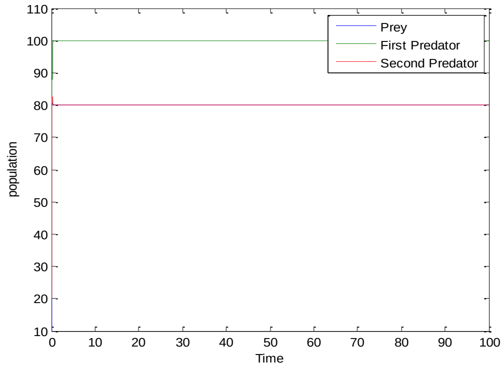
Figure 1: (A) **Figure 1:** (B)

Choose the exponential kernel given by $k_1(w) = k_2(w) = ae^{-aw}$ for $a > 0$
Then the Laplace transform of $k_1(w) = k_2(w)$are defined as

$$k_1(\lambda) = k_2(\lambda) = \int_0^{\infty} e^{-\lambda t} a e^{-at} dt = \frac{a}{a+\lambda}$$

Using the above intervention simulate the results with different values of a and 'λ' along with the parametric values in example 6.1.

Case (i) :λ=0. 001, a=0.5.



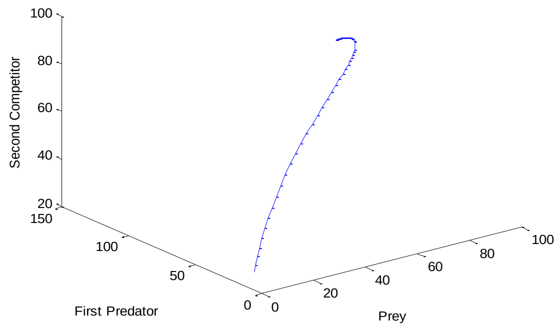
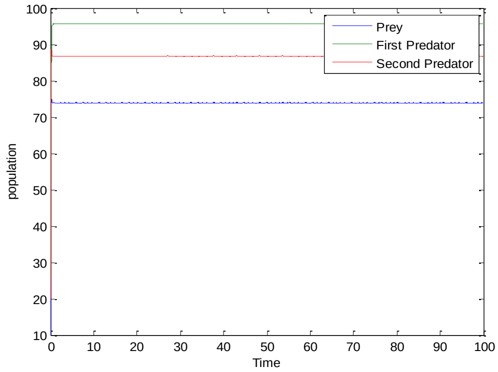


Figure 2: (A) **Figure 2:** (B)

The system is stable and converge to the fixed equilibrium point E (80,99,80). The prey and first predator population are slightly increase and second predator population is slightly decreases when no delay arguments are present in the system.

Case (ii) :λ=1.0, a= 1.5.E (74,96,87)



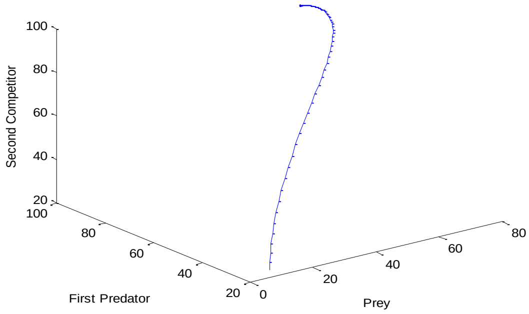
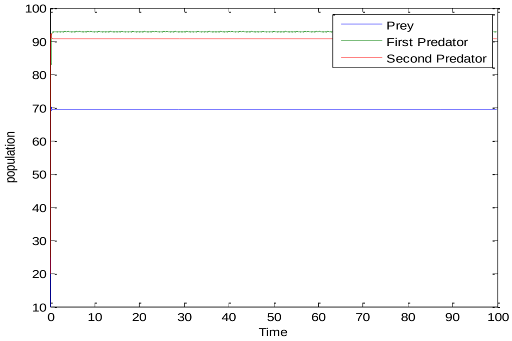


Figure 3: (A) **Figure 3:** (B)

The dynamics the system is stable and converging to fixed equilibrium point E (74, 96, 87). In this case also prey and first predator population is slightly increases and second predator population is decrease when compared with system has no delay arguments.

Case (iii) λ=1.0, a= 0.5



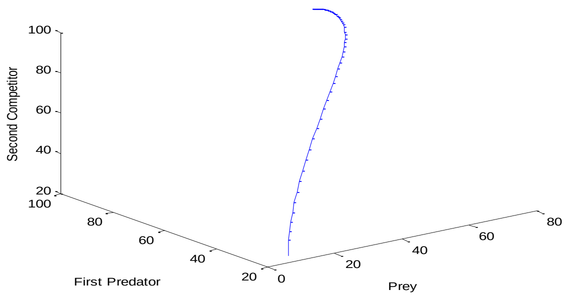


Figure 4: (A) **Figure 4:** (B)

For the above set of delay kernel strengths, the prey and first predator populations show significant growth and second predator population decreases when compared system with no delay arguments. The system remains stable to fixed equilibrium point E (69, 93, 91).

VII. CONCLUSION

The proposed model with distributed delay is stable both locally and globally. The weight kernels are identified and solved the system numerically observed that the delay strengths significant. The system without delay arguments converges to the equilibrium point E (63, 89, 95). For the kernel strengths (i) λ=0. 001, a=0.5, (ii) λ=1.0, a= 1.5 (iii) λ=1.0, a= 0.5, the significant growth is identified in prey & first predator population, decay in the second predator population. The model does not possess any instability characteristics

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