NUMERICAL ANALYSIS OF FINITE FOURIER DECOMPOSITION WITH TWO SHIFT VALUES

Abstract

In this research work aims to develop the finite Fourier series with two variables using the generalized difference operator with two shift values. The key benefit of this research is to decompose the signals(functions) with two variables. To obtain this aim we define and develop the Finite Fourier Series Decomposition(FFSD), also obtain the orthonormal property for the trigonometry functions. Additionally, we illustrate the results unsing MATLAB to decompose the signals(functions) into finite series.

Keywords: Orthonormal system, Fourier series, Generalized difference operator, Harmonic analysis and signal processing.

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I. INTRODUCTION

The evolution of the Fourier series and the origins of the discipline of harmonic analysis can be found in early 19th-century France. A study addressing a solution to a particular form of the heat equation was published by Jean Baptiste Joseph Fourier in 1804. He used a trigonometric term-rich infinite series expansion to arrive at this solution. Although some trigonometric expansions had been worked out by previous mathematicians, Fourier established their usage as legitimate. Harmonic analysis might be considered to have been initiated by Fourier, as he derived a general solution to the heat equation, which was an open and challenging topic at the time "[1, 12].

A potent tool for tackling a variety of number theory issues is the finite Fourier series. It has something to do with some kinds of trigonometric and exponential sums. Thus, it can be extended to a finite Fourier series of the following form:

$g(\xi^{\mu}) = \sum_{k=0}^{n-1} g(k)\xi^{\mu k}\,(\mu = 0,1,\cdots,n-1)$. The orthogonality relation $\sum_{k=0}^{n-1} \xi^{ak} x i^{-bk} = \begin{cases} n & (a \equiv b(mod\ n)), \\ 0 & (a \not\equiv b(mod\ n)), \end{cases}$ permits the computation of the finite Fourier coefficients. $g(x)$ explicitly using the equation $g(x) = \frac{1}{n}\sum_{\mu=0}^{n-1} f(\xi^{\mu})\xi^{-\mu x}$ [2]. If we are given k distinct complex numbers $z_0, z_1, \cdots, z_{k-1}$, then $P(\lambda) = \lambda_0 + \lambda_1 x + \cdots + \lambda_{k-1}x^{k-1}$ satisfying the equations $P(\omega_v) = z_v(v = 0,1,\cdots,k-1)$ [13].

A finite Fourier series: $\eta(t) = A_0 + \sum_{q=1}^{N/2} A_q \cos(q\sigma_1 t) + \sum_{q=1}^{N/2-1} B_q \sin(q\sigma_1 t)$, where the following are used: $\sigma_1 =$ fundamental radian frequency, $\eta =$ sea surface elevation, $t =$ time (s), $A_0 =$ second mean, $N =$ total number of sample points, A_q and $B_q =$ Fourier coefficients, $q =$ harmonic component index (in the frequency domain) [10]. The sum of N sine waves defined over the time interval, $0 \le t \le T$: $y = \sum_{n=1}^{N} a_n \cos(\omega_n t + \phi_n)$, $0 \le t_n \le T$, $a_n \ge 0$, $0 \le \phi_n < 2\pi$, is also a finite Fourier series[7], where t is time and a_n is amplitude. The authors of [11] present an effective method for formulating the analysis of axi-symmetric solids under non-symmetric loading, which utilizes a discrete Fourier series expansion. The Fourier series method and discrete Fourier series representation issues, including Gibb's phenomena and element nonconformance, have been covered. The use of the generalized difference operator to obtain the finite Fourier series of a single variable was covered in [9].

An expansion of a periodic function $g(x)$ in terms of an infinite sum of sines and cosines is called a Fourier series. The orthonormal correlations between the sine and cosine functions are used in Fourier series. The Fourier series of the function $u(t)$, if such a function forms a full orthogonal system over $[-\pi, \pi]$, is given by

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos nxdx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin nxdx$

The generalized difference equation has two different sorts of solutions: closed form and summation form. Any difference equation can have a summation solution found if a closed form solution cannot be found for any function.

In this study, we use the inverse generalized difference operator $\underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}}$ to define a discrete orthonormal family of functions, and then we create and analyze a new type of Finite Fourier Series Decomposition (FFSD) of two variable functions (signals). The Fourier Series is formed by this FFSD as α_1, α_2 goes to zero. The primary conclusions are confirmed, and MATLAB is used to create the diagrams, which are then provided.

II. PRELIMINARIES

The ξ^{th} roots of unity is $(\chi^{\xi} = 1\ \ but\ \ \chi^{i} \neq 1; 0 < i < \xi)$

$$\chi_p = e^{i(2\pi/\xi)p},\ p = 1,2,3,\ldots,\xi - 1, \tag{1}$$

where the geometric series written as follows when p and ξ are co-prime.

$$\sum_{j=0}^{\xi-1} \chi_p^j = \Delta^{-1}\chi_p^j\big|_{j=0}^{\xi} = \frac{\chi_p^{\xi}-1}{\chi_p-1} = \begin{cases} 1 & if \quad \xi = 1 \\ 0 & if \quad \xi > 1. \end{cases} \tag{2}$$

The complex discrete-time sequence $f_i(t)$ is defined from (1) and (2) as

$$f_p(t) = (\chi_p)^j = f^{i(2\pi/\xi)pj};\ \ p,\ j = 0,1,2,\ldots,\xi - 1. \tag{3}$$

For the positive integers $p,\ i$ and ξ, the $f_p(t)$ defined in (3) satisfies the i dentity

$$\sum_{j=0}^{\xi-1} f_p(t) = \Delta^{-1}f_p(t)\big|_{j=0}^{\xi} = \Delta^{-1}f^{i(2\pi p/\xi)j}\big|_{j=0}^{\xi} = \begin{cases} \xi & if \quad p = i\xi \\ 0 & if \quad p \neq i\xi. \end{cases} \tag{4}$$

Using the factorization into two orthogonal exponential functions, $\{e_n(k)\}$ satisfying this mathematical characteristic

$$\Delta^{-1}f_p(t)f_q^{*}(t)\big|_{j=0}^{\xi} = \Delta^{-1}f^{i(\frac{2\pi(p-q)j}{\xi})}\big|_{j=0}^{\xi} = \begin{cases} \xi & if \quad p - q = i\xi \\ 0 & if \quad p - q \neq i\xi, \end{cases} \tag{5}$$

where $(*)$ denotes the complex conjugate and $p, q, andi$ are integers. By substituting $\underset{\alpha_1,\alpha_2}{\Delta}$ for Δ and $f_p(\xi_1, \xi_2)$ for $f_p(t)$, we may create a generalized discrete orthonormal system of two variables and a finite Fourier series using the equation (5).

III. BASIC RESULTS

In order to determine the Fourier coefficients using the generalized difference equation, we present certain fundamental definitions and results in this section. Real valued functions of two variables are denoted by $f(\xi_1,\xi_2)$ and $g(\xi_1,\xi_2)$ in this case.

Definition 3.1 *Let $f(\xi_1,\xi_2)$ be the two-variable function, and let the shift values be $(\alpha_1,\alpha_2) \in R^2$. Then, the partial difference operator described in two dimensions is*

$$\underset{\alpha_1,\alpha_2}{\Delta} f(\xi_1,\xi_2) = \frac{f(\xi_1+\alpha_1,\xi_2+\alpha_2)-f(\xi_1,\xi_2)}{\alpha_1\alpha_2}, \tag{6}$$

Lemma 3.2 *If $\underset{\alpha_1,\alpha_2}{\Delta} g(\xi_1,\xi_2) = f(\xi_1,\xi_2)$ and $\alpha_1,\alpha_2 > 0$ where p is any positive integer, then we have*

$$g(\xi_1,\xi_2) - g(\xi_1 - p\alpha_1, \xi_2 - p\alpha_2) = \alpha_1\alpha_2 \sum_{j=1}^{p} f(\xi_1 - j\alpha_1, \xi_2 - j\alpha_2) \tag{7}$$

Proof. Since $\underset{\alpha_1,\alpha_2}{\Delta} g(\xi_1,\xi_2) = f(\xi_1,\xi_2)$, from the Definition 3.1, we have

$$\frac{g(\xi_1+\alpha_1,\xi_2+\alpha_2)-g(\xi_1,\xi_2)}{\alpha_1\alpha_2} = f(\xi_1,\xi_2) \tag{8}$$

Replacing ξ_1 by $\xi_1-\alpha_1$ and ξ_2 by $\xi_2-\alpha_2$, we get

$$g(\xi_1,xi_2) = \alpha_1\alpha_2 f(\xi_1-\alpha_1,\xi_2-\alpha_2) + g(\xi_1-\alpha_1,\xi_2-\alpha_2) \tag{9}$$

Again replacing ξ_1 by $\xi_1-\alpha_1$ and ξ_2 by $\xi_2-\alpha_2$ in (9), we get

$g(\xi_1-\alpha_1,\xi_2-\alpha_2) = \alpha_1\alpha_2 f(\xi_1-2\alpha_1,\xi_2-2\alpha_2) + g(\xi_1-2\alpha_1,\xi_2-2\alpha_2)$ and (9) Becomes

$$g(\xi_1,\xi_2) = \alpha_1\alpha_2[f(\xi_1-\alpha_1,\xi_2-\alpha_2) + f(\xi_1-2\alpha_1,\xi_2-2\alpha_2)] + g(\xi_1-2\alpha_1,\xi_2-2\alpha_2)$$

Continuing in this manner for p steps, we obtain (7).

Lemma 3.3 *[8] Let $f(\xi_1,\xi_2)$ and $g(\xi_1,\xi_2)$ are the two functions, then we have*

$$\overset{-1}{\underset{\alpha_1,\alpha_2}{\Delta}}(f(\xi_1,\xi_2)g(\xi_1,\xi_2)) = f(\xi_1,\xi_2)\overset{-1}{\underset{\alpha_1,\alpha_2}{\Delta}} g(\xi_1,\xi_2) - \overset{-1}{\underset{\alpha_1,\alpha_2}{\Delta}}(\overset{-1}{\underset{\alpha_1,\alpha_2}{\Delta}} g(\xi_1+\alpha_1,\xi_2+\alpha_2)\underset{\alpha_1,\alpha_2}{\Delta} f(\xi_1,\xi_2)). \tag{10}$$

Lemma 3.4 *Let s_r^m and S_r^m are the Stirling numbers of first and second kinds, $(\xi_1+\xi_2)^{(0,0)}_{\alpha_1,\alpha_2} = 1$, $(\xi_1+\xi_2)^{(1,1)}_{\alpha_1,\alpha_2} = \xi_1+\xi_2$ and the polynomial factorial as*

$(\xi_1+\xi_2)^{(m,m)}_{\alpha_1,\alpha_2} = (\xi_1+\xi_2)(\xi_1+\xi_2-(\alpha_1+\alpha_2))\cdots(\xi_1+\xi_2-(m-1)(\alpha_1+\alpha_2))$. Then

$$(\xi_1+\xi_2)^{(m,m)}_{\alpha_1,\alpha_2} = \sum_{r=1}^{m} s_r^m (\alpha_1+\alpha_2)^{m-r}(\xi_1+\xi_2)^r,\ (\xi_1+\xi_2)^m = \sum_{r=1}^{m} S_r^m (\alpha_1+\alpha_2)^{m-r}(\xi_1+\xi_2)^{(r,r)}_{\alpha_1,\alpha_2}, \quad (11)$$

$$\underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}}(\xi_1+\xi_2)^{(m,m)}_{\alpha_1,\alpha_2} = \frac{(\xi_1+\xi_2)^{(m+1,m+1)}_{\alpha_1,\alpha_2}}{(\alpha_1+\alpha_2)(m+1)},\ \underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}}(\xi_1+\xi_2)^m = \sum_{r=1}^{m} \frac{S_r^m(\alpha_1+\alpha_2)^{m-r}(\xi_1+\xi_2)^{(r,r)}_{\alpha_1,\alpha_2}}{(r+1)(\alpha_1+\alpha_2)}. \quad (12)$$

Lemma 3.5 *Let p be real, $\alpha_1,\alpha_2>0$, $\xi_1\in(\alpha_1,\infty),\xi_2\in(\alpha_2,\infty)$ and $p\alpha_1,p\alpha_2\neq m2\pi$.Then, we have*

$$\underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}}\cos p(\xi_1+\xi_2) = \alpha_1\alpha_2\frac{\cos p(\xi_1-\alpha_1+\xi_2-\alpha_2)-\cos p(\xi_1+\xi_2)}{2(1-\cos p(\alpha_1+\alpha_2))} \quad (13)$$

$$\underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}}\sin p(\xi_1+\xi_2) = \alpha_1\alpha_2\frac{\sin p(\xi_1-\alpha_1+\xi_2-\alpha_2)-\sin p(\xi_1+\xi_2)}{2(1-\sin p(\alpha_1+\alpha_2))} \quad (14)$$

Proof. From Definition 3.1,

$$\underset{\alpha_1,\alpha_2}{\Delta}\cos p(\xi_1+\xi_2) = \frac{\cos p(\xi_1+\alpha_1+\xi_2+\alpha_2)-\cos p(\xi_1+\xi_2)}{\alpha_1\alpha_2}$$

$$\mathrm{R.P}\,(\underset{\alpha_1,\alpha_2}{\Delta}e^{ip(\xi_1+\xi_2)}) = \mathrm{R.P}(e^{ip(\xi_1+\xi_2)})Re(e^{ip(\alpha_1+\alpha_2)}-1)$$

Applying $\underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}}$ both sides, we obtain

$$\mathrm{R.P}\left(\underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}}e^{ip(\xi_1+\xi_2)}\right) = \mathrm{R.P}\left(\frac{e^{ip(\xi_1+\xi_2)}}{e^{ip(\alpha_1+\alpha_2)-1}}\right)$$

After equating the real components of the complex conjugate, we obtain (13).
Similarly, by equating the imaginary part, we obtain the evidence of (14).

IV. COMPUTATION OF FINITE FOURIER SERIES DECOMPOSITION

In this section, we use the orthonormal condition of trigonometric functions and the generalized difference equation to compute the Fourier series and extract the Fourier coefficients.

Theorem 4.1 *Let $f(\xi_1,\xi_2)$ be bounded function on $[a,a+2\pi]$ and $\alpha_1+\alpha_2=\frac{2\pi}{N}$. Then we have FFSD as*

$$f(\xi_1,\xi_2) = \frac{a_{0,0}}{2}+\sum_{n=1}^{P-1}(a_{p,p}cos\ p(\xi_1+\xi_2)+b_{p,p}sin\ p(\xi_1+\xi_2))+\frac{a_{P,P}}{2}cos\ P(\xi_1+\xi_2), \quad (15)$$

where the coefficients are obtained by

$$a_{0,0} = \frac{\alpha_1+\alpha_2}{2\pi}\underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}}f(\xi_1,\xi_2)|_a^{a+2\pi}$$

$$a_{p,p} = \frac{\alpha_1+\alpha_2}{2\pi} \underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}} f(\xi_1,\xi_2)\cos\ p(\xi_1+\xi_2)|_a^{a+2\pi}$$

$$b_{p,p} = \frac{\alpha_1+\alpha_2}{2\pi} \underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}} f(\xi_1,\xi_2)\sin\ p(\xi_1+\xi_2)|_a^{a+2\pi}$$

Proof. To prove orthogonality condition, we can take

$$(\alpha_1+\alpha_2) \underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}} \frac{\cos p(\xi_1+\xi_2)}{\sqrt{2\pi}} \frac{\cos q(\xi_1+\xi_2)}{\sqrt{2\pi}} |_0^{2\pi}$$

$$= \frac{(\alpha_1+\alpha_2)}{2\pi} \underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}} (\cos(p\xi_1+q\xi_2)+\cos(q\xi_1-q\xi_2)|_0^{2\pi}) = 0.$$

and $\underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}} \cos^2 p(\xi_1+\xi_2)|_0^{2\pi}|_0^{2\pi} = \underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}} \left(\frac{1+\cos 2p(\xi_1+\xi_2)}{2}\right)|_0^{2\pi}$

$$= \underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}} \left(\frac{1}{2}\right)|_0^{2\pi}|_0^{2\pi} + \frac{1}{2} \underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}} \cos^2 p(\xi_1+\xi_2)|_0^{2\pi}|_0^{2\pi}$$

$$= \frac{2\pi}{\alpha_1+\alpha_2},$$

which is the required FFSD coefficients as $a_{p,p}$ and $b_{p,p}$.

V. MAIN RESULTS AND DECOMPOSITION OF FUNCTIONS

This section provides the FFSD for polynomial, polynomial factorial, and trigonometric functions. By utilizing the extended difference operator, we can also break down real valued functions of two variables into the sum of sine and cosine.

Theorem 5.1 *Let* $\xi_1,\xi_2 \in (-\infty,\infty)$ *and* $\alpha_1,\alpha_2 > 0.$ *If* $p(\alpha_1+\alpha_2) \neq 2m\pi$, *then*

$$\underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}} (\xi_1+\xi_2)^{(n,n)}_{\alpha_1,\alpha_2} \cos p(\xi_1+\xi_2)$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{j+1} \binom{j+1}{k} \frac{(n)_1^{(j)}(\xi_1+\xi_2)^{(n-j,n-j)}_{\alpha_1,\alpha_2} \cos p((\xi_1+\xi_2)-(k-1)(\alpha_1+\alpha_2))}{(-1)^{(k-1)}(\alpha_1+\alpha_2)^{-j}(2(\cos p(\alpha_1+\alpha_2)-1))^{(j+1)}} \tag{16}$$

$$\underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}} (\xi_1+\xi_2)^{(n,n)}_{\alpha_1,\alpha_2} \sin p(\xi_1+\xi_2)$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{j+1} \binom{j+1}{k} \frac{(n)_1^{(j)}(\xi_1+\xi_2)^{(n-j,n-j)}_{\alpha_1,\alpha_2} \sin p((\xi_1+\xi_2)-(k-1)(\alpha_1+\alpha_2))}{(-1)^{(k-1)}(\alpha_1+\alpha_2)^{-j}(2(\sin p(\alpha_1+\alpha_2)-1))^{(j+1)}} \tag{17}$$

Proof. Taking $f(\xi_1,\xi_2) = (\xi_1+\xi_2)^{(1,1)}_{\alpha_1,\alpha_2}$ and $g(\xi_1,\xi_2) = \cos p(\xi_1+\xi_2)$ in (10) and using (13) we get

$$\underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}}(\xi_1+\xi_2)^{(1,1)}_{\alpha_1,\alpha_2}\cos p(\xi_1+\xi_2) = (\xi_1+\xi_2)^{(1,1)}_{\alpha_1,\alpha_2}\frac{\cos p(\xi_1+\xi_2)-\cos p(\xi_1+\alpha_1+\xi_2+\alpha_2)}{2\big(1-\cos p(\alpha_1+\alpha_2)\big)} -$$

$$\underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}}\left(\underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}}\cos p(\xi_1+2\alpha_1+\xi_2+2\alpha_2)\underset{\alpha_1,\alpha_2}{\Delta}(\xi_1+\xi_2)^{(1,1)}_{\alpha_1,\alpha_2}\right)$$

Applying (13) in the above equation, we get

$$\underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}}(\xi_1+\xi_2)^{(1,1)}_{\alpha_1,\alpha_2}\cos p(\xi_1+\xi_2) = (\xi_1+\xi_2)^{(1,1)}_{\alpha_1,\alpha_2}\frac{\cos p(\xi_1+\xi_2)-\cos p(\xi_1+\alpha_1+\xi_2+\alpha_2)}{2(1-\cos p(\alpha_1+\alpha_2))}$$

$$-\frac{(\alpha_1+\alpha_2)(\cos p(\xi_1+\xi_2)-2\cos p(\xi_1+\xi_2+(\alpha_1+\alpha_2)+\cos p(\xi_1+\xi_2+2(\alpha_1+\alpha_2))}{(2(1-\cos p(\alpha_1+\alpha_2)))^2} \tag{18}$$

Taking $f(\xi_1,\xi_2)=(\xi_1+\xi_2)^{(2,2)}_{\alpha_1,\alpha_2}$ and $g(\xi_1,\xi_2)=\cos p(\xi_1+\xi_2)$ in (10), using (13) and (18), we get

$$\underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}}(\xi_1+\xi_2)^{(2,2)}_{\alpha_1,\alpha_2}\cos p(\xi_1+\xi_2) = (\xi_1+\xi_2)^{(2,2)}_{\alpha_1,\alpha_2}\frac{\cos p(\xi_1+\xi_2-(\alpha_1+\alpha_2))-\cos p(\xi_1+\xi_2)}{2(1-\cos p(\alpha_1+\alpha_2))}$$

$$-\frac{2(\alpha_1+\alpha_2)(\xi_1+\xi_2)^{(1,1)}_{\alpha_1,\alpha_2}}{(2(1-\cos p(\alpha_1+\alpha_2)))^2}(\cos p(\xi_1+\xi_2-(\alpha_1+\alpha_2))-2\cos p(\xi_1+\xi_2)+\cos p(\xi_1+\xi_2+$$

$$(\alpha_1+\alpha_2))+\frac{2(\alpha_1+\alpha_2)^2}{(2(1-\cos p(\alpha_1+\alpha_2)))^3}(\cos p(\xi_1+\xi_2-(\alpha_1+\alpha_2))$$

$$-3\cos p(\xi_1+\xi_2)+3\cos p(\xi_1+\xi_2+(\alpha_1+\alpha_2))-\cos p(\xi_1+\xi_2+2(\alpha_1+\alpha_2)), \tag{19}$$

and hence RHS of (19) can be expressed as

$$\sum_{j=0}^{2}\sum_{k=0}^{j+1}\binom{j+1}{k}\frac{(2)^{(j)}_1(\alpha_1+\alpha_2)^j(\xi_1+\xi_2)^{(2-j,2-j)}_{\alpha_1,\alpha_2}\cos p((\xi_1+\xi_2)-(k-1)(\alpha_1+\alpha_2))}{(-1)^{(k-1)}(2(\cos p(\alpha_1+\alpha_2)-1))^{(j+1)}}$$

Performing the aforementioned procedure up to n steps yields (16).

Now,(17) follows by replacing $\cos p(\xi_1+\xi_2)$ by $\sin p(\xi_1+\xi_2)$ in (16).

Corollary 5.2 *When* $I=[0,2\pi]$, $\alpha_1+\alpha_2=\frac{2\pi}{P}$, $\xi_1,\xi_2\in\{k(\alpha_1+\alpha_2)\}_0^{2P-1}$, *the finite Fourier coefficients* $a_{p,p}$ *and* $b_{p,p}$ *for the polynomial factorial* $(\xi_1+\xi_2)^{(n,n)}_{\alpha_1,\alpha_2}$ *are given by,*

$$a_{0,0} = \frac{\alpha_1+\alpha_2}{2\pi} \underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}} (\xi_1+\xi_2)^{(n,n)}_{\alpha_1,\alpha_2}\Big|_0^{2\pi} = \frac{(4\pi)^{(n+1,n+1)}_{\alpha_1,\alpha_2}(\alpha_1+\alpha_2)}{2\pi 3(\alpha_1+\alpha_2)} \tag{20}$$

$$a_{p,p} = \frac{\alpha_1+\alpha_2}{2\pi} \underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}} (\xi_1+\xi_2)^{(n,n)}_{\alpha_1,\alpha_2} \cos p(\xi_1+\xi_2)\Big|_0^{2\pi}$$

$$= \sum_{j=0}^{n-1} \sum_{k=0}^{j+1} \binom{j+1}{k} \frac{(n)^j_1 (\alpha_1+\alpha_2)^j (4\pi)^{(n-j,n-j)}_{\alpha_1,\alpha_2} \cos p(k-1)(\alpha_1+\alpha_2)}{P(-1)^{(k-1)}(2(\cos p(\alpha_1+\alpha_2)-1))^{j+1}} \tag{21}$$

$$b_{p,p} = \frac{\alpha_1+\alpha_2}{2\pi} \underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}} (\xi_1+\xi_2)^{(n,n)}_{\alpha_1,\alpha_2} \sin p(\xi_1+\xi_2)\Big|_0^{2\pi}$$

$$= \sum_{j=0}^{n-1} \sum_{k=0}^{j+1} \binom{j+1}{k} \frac{(n)^j_1 (\alpha_1+\alpha_2)^j (4\pi)^{(n-j,n-j)}_{\alpha_1,\alpha_2} \sin p(k-1)(\alpha_1+\alpha_2)}{P(-1)^{(k-1)}(2(\sin p(\alpha_1+\alpha_2)-1))^{j+1}} \tag{22}$$

Proof. To prove the statement, multiply $\frac{\alpha_1+\alpha_2}{2\pi}$ by the limit 0 to 2π in $(16)$and (17)

Theorem 5.3 *Let $\xi_1, \xi_2 \in (-\infty, \infty), \alpha_1+\alpha_2 > 0$. If $p(\alpha_1+\alpha_2) \neq n2\pi$, then*

$$\underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}} (\xi_1+\xi_2)^q \cos p(\xi_1+\xi_2)$$

$$= \sum_{n=1}^{q} \sum_{j=0}^{n} \sum_{k=0}^{j+1} \binom{j+1}{k} \frac{S^p_n (n)^{(j)}_1 (\xi_1+\xi_2)^{(n-j),(n-j)}_{\alpha_1,\alpha_2} \cos p((\xi_1+\xi_2)-(k-1)(\alpha_1+\alpha_2))}{(-1)^{k-1}(\alpha_1+\alpha_2)^{n-j-q}(2(\cos p(\alpha_1+\alpha_2)-1))^{j+1}} \tag{23}$$

$$\underset{\alpha_1,\alpha_2}{\overset{-1}{\Delta}} (\xi_1+\xi_2)^q \sin p(\xi_1+\xi_2)$$

$$= \sum_{n=1}^{q} \sum_{j=0}^{n} \sum_{k=0}^{j+1} \binom{j+1}{k} \frac{S^q_n (n)^{(j)}_1 (\xi_1+\xi_2)^{(n-j),(n-j)}_{\alpha_1,\alpha_2} \sin p((\xi_1+\xi_2)-(k-1)(\alpha_1+\alpha_2))}{(-1)^{k-1}(\alpha_1+\alpha_2)^{n-j-q}(2(\sin p(\alpha_1+\alpha_2)-1))^{j+1}}. \tag{24}$$

Proof. The proof follows by second term of (11) and applying (16).

Corollary 5.4 *When $I = [0,\ 2\pi]$, $\alpha_1+\alpha_2 = \frac{\pi}{P}$, the finite Fourier coefficients $a_{p,p}$ and $b_{p,p}$ for $p = 0,1,2,\cdots,P$ for polynomial $(\xi_1+\xi_2)^q$ are given by*

$$a_{p,p} = \frac{\alpha_1+\alpha_2}{2\pi} \Delta^{-1}_{\alpha_1,\alpha_2} (\xi_1+\xi_2)^q \cos p(\xi_1+\xi_2)\Big|_0^{2\pi}$$

$$= \sum_{n=1}^{q-1} \sum_{j=0}^{n} \sum_{k=0}^{j+1} \binom{j+1}{k} \frac{S^q_n (n)^{(j)}_1 (4\pi)^{(n-j),(n-j)}_{\alpha_1,\alpha_2} \cos p(k-1)(\alpha_1+\alpha_2}{(-1)^{k-1}P(\alpha_1+\alpha_2)^{n-j-q}(2(\cos p(\alpha_1+\alpha_2)-1))^{j+1}} \tag{25}$$

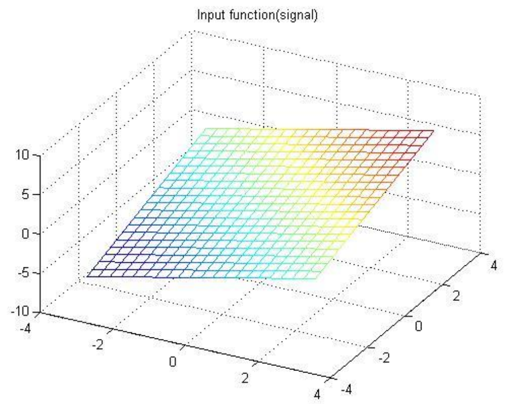
$$b_{p,p} = \frac{\alpha_1+\alpha_2}{2\pi} \Delta^{-1}_{\alpha_1,\alpha_2} (\xi_1+\xi_2)^q \sin p(\xi_1+\xi_2)\Big|_0^{2\pi}$$

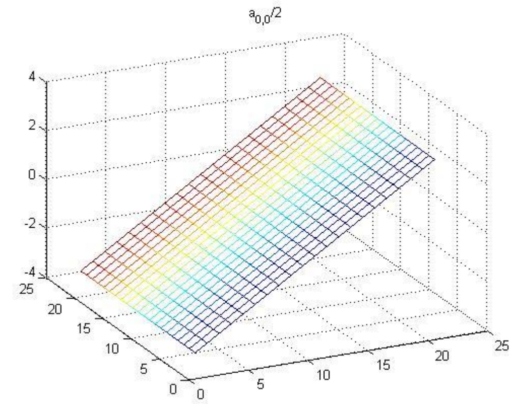
$$= \sum_{n=1}^{q-1} \sum_{j=0}^{n} \sum_{k=0}^{j+1} \binom{j+1}{k} \frac{S^q_n (n)^{(j)}_1 (4\pi)^{(n-j)}_{\ell} \sin p(k-1)(\alpha_1+\alpha_2)}{(-1)^{k-1}P(\alpha_1+\alpha_2)^{n-j-q}(2(\cos p(\alpha_1+\alpha_2)-1))^{j+1}}. \tag{26}$$

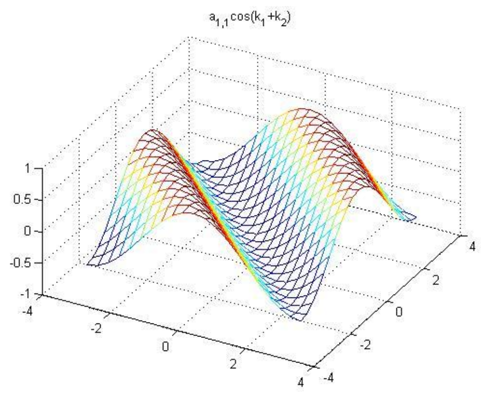
Proof. Applying the limits 0 to 2π in (23) and (24) and multiplying the result by $(\alpha_1 + \alpha_2)/2\pi$ completes the proof.

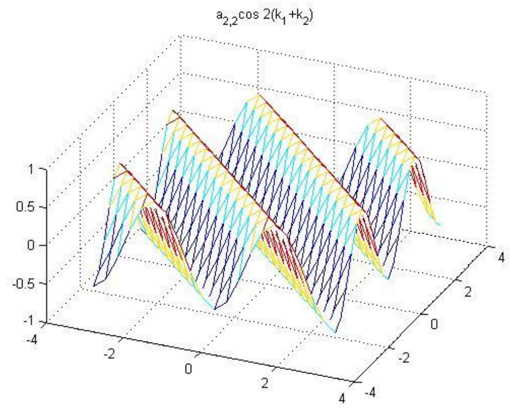
Assuming the two-dimensional harmonic signal $f(\xi_1, \xi_2) = \cos\ p(\xi_1 + \xi_2)$ to be a periodic signal in two-dimensional space, similar to a picture, in the specific situation of FFSD, we obtain $a_{0,0} = 0, a_{p,p} = 1, b_{p,p} = 0$, for all p by (15), which is the Fourier series equivalent of the cosine function.

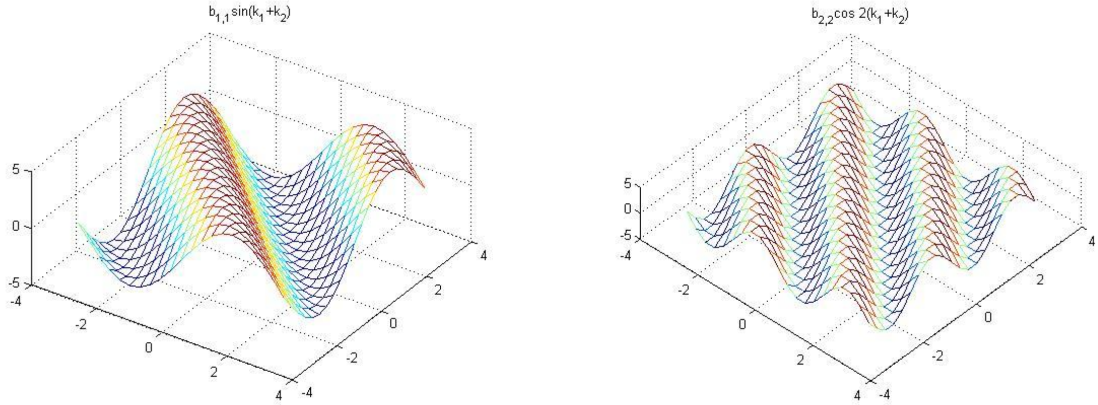
Assuming that $f(\xi_1, \xi_2) = \xi_1 + \xi_2$ is a signal (polynomial) at this point, we can derive $a_{0,0} = \frac{38\pi}{5}, a_{p,p} = \frac{-2\pi}{5}$, and $b_{p,p} = \frac{-2\pi \sin(2\pi/10)}{5(1-\cos p(2\pi/10))}$ from (15).‘In this case, $\alpha_1 + \alpha_2 = \frac{2\pi}{P}$. Hence, using MATLAB, the result of the input signal’s decomposition can be produced as follows:











The aforementioned diagrams are produced for the specific value of $N = 3$. Depending on the number of N, the function can be broken down into as many identical components as desired. We can also break down different functions, such as polynomials, polynomial factorials, exponentials, and so forth.

VI. CONCLUSION

Here, we have provided the FFSD expression (decomposition) for the functions using the inverse of the generalized difference operator's summation solution form and orthonormal constraints. The nature of Fourier series is demonstrated and illustrated with a numerical example. The Fourier series decomposition of the input functions—which are treated as signals—is produced using MATLAB.

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