APPLICATION OF RAJ TRANSFORM FOR SOLVING MATHEMATICAL MODELS OCCURRING IN HEALTH SCIENCE AND BIOTECHNOLOGY

Abstract

A lot of mathematical models including differential equations play an important role in healthcare and biotechnology. One of them is the Malthus model. This model was developed by Thomas Malthus in his essay on world population growth and resource supply. Another exciting equation is the Advection diffusion equation and the Predator-prey model. We use an integral transform modified Sumudu transform called as Raj transform to obtain the solutions of these models which are important in biotechnology and health sciences.

Keywords: Mathematical Models in biotechnology; Raj transform; Integral transform; Malthus model; Predator-Prey model; Logistic model; System of differential equations.

I. INTRODUCTION

Mathematical models are of much importance in the optimization of the performance of the biotechnological process. Here we discuss growth law (Malthus equation), logistic model and Predator Prey model.

Population is always modeled by growth law or differential equations, for the population $u = u(\, t)$ of insects in the tube at a time t. For that Malthus model is used, which is

$$\frac{d\,u(t)}{dt} = r\,u(t).$$

Meaning of this law is growth rate is proportional to current population and the proportionality constant r is intrinsic growth rate. By using variable separable method we can obtain its solution as $u(t) = u(0)e^{rt}$. From this equation we can conclude that the graph of population is increasing exponentially. This model is reasonable in early stage.

But as the resources like food, space and other factors are limited; there is competition for these resources the growth of population does not follow the exponential equation. So the Malthus model is replaced by logistic model.

$$\frac{d\,u(t)}{dt} = r\,u(t)\left(1 - \frac{u(t)}{K}\right) \quad (1.1)$$

Where K is carrying capacity, which means that as the population grows and approaches to K then the growth approaches to zero and there is limit to the growth. To obtain solution of these problems we use Raj transform.

This transform is modified version of Sumudu transform.. Raj C. Jesuraj and A Rajkumar modified the Sumudu transform and obtained Raj transform in 2020[1]. Sumudu transform is developed by Watugala in 1993 [5].

Serdal Pamuk and Nagihan Saylu [2] used Laplace transform method for logistic growth in a population and predator models. The interaction between predator and prey commonly occurs among the bacterial species and protozoa. Hence the predator prey model is important model. Khakale and Patil [3] developed and introduced new integral transform called as Soham transform in 2021. Further Kushare [4] developed new integral transform Kushare transform recently. Patil with coauthors [6,7,8] used Kushare transform, general transform and Soham transform for solving models in health science and biotechnology.

This paper is organized as follows: Introduction is in first section. Second section is devoted to the useful results and formulae which we are using to solve models. Logistic growth model which is important model in Health care sciences is solved in third section. Fourth section is for Predator Prey Model. Applications and results are in fifth section and conclusion is in sixth section.

II. USEFUL RESULTS AND FORMULAE

In this section we include some required definitions, some useful formulae and theorems based on Raj transform.

Definition: The Raj transform of the function $f(t)$ is defined as

$$R\{f(t)\} = \int_0^\infty f\left(\frac{t}{s}\right) e^{-t}\, dt \qquad (2.1)$$

where t is in between zero and infinity $0 < k_1 \leq t \leq k_2$ here k_1 and k_2 are either finite or infinite..

Raj transform of some standard functions

Sr. No.	**Function f(t)**	**Raj transform of f(t)R{f(t)}**
1	1	1
2	t	$\frac{1}{s}$
3	t^2	$\frac{2!}{s^2}$
4	t^n	$\frac{n!}{s^n}$
5	e^{at}	$\frac{s}{s-a}$
6	$\sin at$	$\frac{as}{s^2+a^2}$
7	$\cos at$	$\frac{s^2}{s^2+a^2}$
8	$\sinh at$	$\frac{as}{s^2-a^2}$
9	$\cosh at$	$\frac{s^2}{s^2-a^2}$

III. RAJ TRANSFORM FORLOGISTIC GROWTH MODEL

Consider the Logistic growth model equation

$$\frac{du}{dt} = u - f(u)\,, u(0) = u_0 \qquad (3.1)$$

Here f is nonlinear function of u. Suppose that solution$u$ of equation (3.1) is of the infinite power series as follows,

$$u = u(t) = \sum_{n=0}^{\infty} a_n t^n \qquad (3.2)$$

Further (3.2) also satisfies the conditions for the existence of Raj transform. Applying Raj transform on the both sides of the equation (3.1) we get

$$sR(s) - su(0) = R(s) - F(s) \qquad (3.3)$$

Where $R(s) = R[u(t)]$ and $F(s) = R\,[\,f(u(t))]$ are the Raj transform of the functions $u(t)$ and $f(u(t))$ respectively.

Rearranging the terms in equation 3.3 we get,

$$R(s) = \frac{su_0}{s-1} - \frac{F(s)}{s-1} \qquad (3.4)$$

If we suppose $f(u) = u^2$ then

$$f(u) = \left(\sum_{n=0}^{\infty} a_n t^n\right)^2$$

$$= (a_0 + a_1 t + a_2 t^2 + \dots a_n t^n)^2$$

$$= a_0{}^2 + 2a_0 a_1 t + (2a_0 a_2 + a_1{}^2)t^2 + (2a_0 a_3 + 2a_1 a_2)t^3 + \cdots \qquad (3.5)$$

Taking Raj transform on both sides of equation3.3,

$$F(s) = a_0{}^2 + \frac{2a_0 a_1}{s} + (2a_0 a_2 + a_1{}^2)\frac{2}{s^2} + (2a_0 a_3 + 2a_1 a_2)\frac{6}{s^3} + \dots\dots\dots$$

$$\therefore \frac{F(s)}{s-1} = \frac{a_0{}^2\, s}{s(\,s-1)} + \frac{2a_0 a_1 s}{s^2(s-1)} + (2a_0 a_2 + a_1{}^2)\frac{2s}{s^2(s-1)} + (2a_0 a_3 + 2a_1 a_2)\frac{6s}{s^4(s-1)} + \dots\dots\dots$$

We apply the method of partial fractions to the terms in R.H.S. of the above equation,

$$\therefore \frac{F(s)}{s-1} = \frac{s}{s-1}[a_a{}^2 + 2\,a_0 a_1 + 4\,a_0 a_2 + 2\,a_1{}^2 + 12\,\ a_0 a_3 + 12\,a_1 a_3 \dots]$$
$$- [a_a{}^2 + 2\,a_0 a_1 + 4\,a_0 a_2 + 2\,a_1{}^2 + 12\,\ a_0 a_3 + 12\,a_1 a_3 \dots\dots]$$
$$- \frac{1}{s}[2\,a_0 a_1 + 4\,a_0 a_2 + 2\,a_1{}^2 + 12\,\ a_0 a_3 + 12\,a_1 a_{3\dots.}]$$
$$- \frac{1}{s^2}[\,4\,a_0 a_2 + 2\,a_1{}^2 + 12\,\ a_0 a_3 + 12\,a_1 a_{3\dots\dots\dots}]$$
$$- \frac{1}{s^3}[\,12\,\ a_0 a_3 + 12\,a_1 a_3 \dots..] - \dots.$$

Applying the inverse Raj transform to both sides of the above equation and rearranging the terms we get.

$$u(t) = u_0\left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots\dots\right) - \left[a_0{}^2 t + \left(\frac{a_0{}^2}{2} + a_0 a_1\right)t^2 + \left(\frac{a_0{}^2}{6} + \frac{a_0 a_1}{3} + \frac{2a_0 a_2}{3}\right)t^2\right] + \dots$$

$$\therefore u(t) = u_0 + (u_0 - a_0{}^2)t + \left(\frac{u_0}{2} - \frac{a_0{}^2}{2} - a_0 a_1\right)t^2 + \left(\frac{u_0}{6} - \frac{a_0{}^2}{6} - \frac{a_0 a_1}{3} - 2\frac{a_0 a_2}{3}\right)t^3$$

If we take $u_0 = 2$ and compare with 3.2 we obtain
$a_0 = 2,\ a_1 = -2\ , a_2 = 3,\ a_3 = -\frac{13}{3}\ \dots\dots$

Therefore,
$$u(t) = 2 - 2t + 3t^2 - \frac{13}{3}t^3 \dots$$

It is required solution and the graph of this solution is

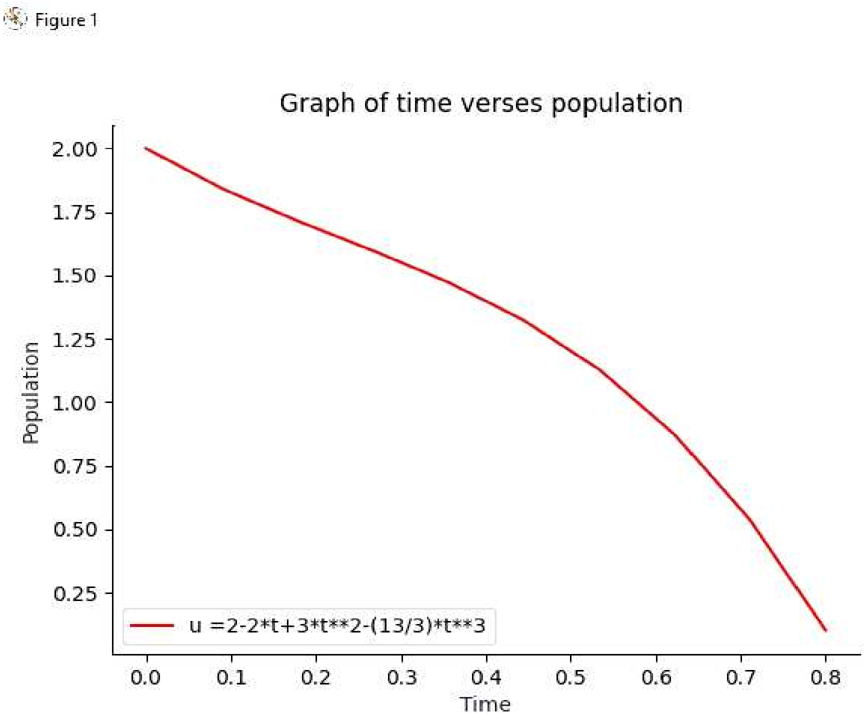

Figure 1

This figure 1 is graph showing that how population of a species changes when a hazard function is acting in life of the species.

From graph we can conclude that, if there is more competition in life and more hazards then the population decreases. Here f(u) that is hazard function is taken as square of the population that means more hazard, so population of the insect reaches to zero in less than one unit interval of time.

IV. RAJ TRANSFORM FOR PREDATOR PREY MODEL

The interaction between two species and their effect on each other is called as predator prey relationship. In this one species is feeding on the other species. An organism that eats or hunts other organism as food is called as predator and an organism that is killed by other organism for food is called as prey. Fox and rabbit, lion and zebra are examples of predator and prey. This concept of predator prey is not only applicable for animals but it is applicable for plants also. Grasshopper and leaf is an example of this.

Consider the system

$$\frac{du}{dt} = u - f(u,v) \quad (4.1)$$

$$\frac{dv}{dt} = \beta\,[g(u,v) - v\,] \quad (4.2)$$

with initial conditions $u(0) = u_0$ and $v(0) = v_0$, f and g are nonlinear functions of u and v. β is a positive constant. Let u and v. be the solutions of this system, which are infinite series of the form $u = u(t) = \sum_{n=0}^{\infty} a_n t^n$ $v = v(t) = \sum_{n=0}^{\infty} b_n t^n$ and they both also satisfy the required conditions for existence of Raj transform.

Applying Raj transform to both sides of the equations 4.1 and 4.2.

$$R\left(\frac{du}{dt}\right) = R(u) - R[f(u,v)]$$

$$R\left(\frac{dv}{dt}\right) = \beta\,[R\,[g(u,v)] - R(\,v)\,]$$

Using the Raj transform of derivative theorem

$$s\big(\,U(s)\big) - s\,u(0) = U(s) - F(s)$$

$$s\,v(s) - s\,v(0) = \beta\,G(s) - \beta\,v(s)$$

where,

$R[u(t)] = U(s)$,

$R[v(t)] = V(s)$,

$R\big[f\big(\,u(t), v(t)\big)\big] = F(s)$ and

$R\big[g\big(\,u(t), v(t)\big)\big] = G(s)$

Rearranging the terms and simplifying we get

$$U(s) = u_0 \frac{s}{s-1} - \frac{F(s)}{s-1}$$

and

$$V(s) = v_0 \frac{s}{s+\beta} + \beta\,\frac{G(s)}{s+\beta}$$

Applying inverse Raj transform

$$u(t) = u_0 e^{t} - R^{-1}\left(\frac{F(s)}{s-1}\right) \quad (4.3)$$

$$v(t) = v_0 e^{-\beta t} + \beta\,R^{-1}\left(\frac{G(s)}{s+\beta}\right) \quad (4.4)$$

These equations 4.3 and 4.4 represent the solution of the system of equations 4.1 and 42.

V. APPLICATIONS AND RESULTS

In this section we use results in above section to solve some systems of differential equations arising in biotechnology and health of sciences.

Example: 1. Consider the system of differential equations,

$$\frac{du}{dt} = u - uv \qquad (5.1)$$

$$\frac{dv}{dt} = uv - v \quad (5.2)$$

With initial conditions

$$u(0) = 1.3\,, \qquad v(0) = 0.6$$

Suppose,

$u = \sum_{n=0}^{\infty} a_n t^n$, $v = \sum_{n=0}^{\infty} b_n t^n$ be solution of the system of equations (5.1) and (5.2).

$$\therefore uv = a_0b_0 + (a_0b_1 + a_1b_0)t + (a_0b_2 + a_1b_1 + a_2b_0)t^2 + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)t^3 + \cdots$$

Applying Raj transform,

$$R(uv) = a_0b_0R(1) + (a_0b_1 + a_1b_0)R(t) + (a_0b_2 + a_1b_1 + a_2b_0)R(t^2) + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)R(t^3) + \cdots ..$$

$$\therefore R(uv) = a_0b_0 + (a_0b_1 + a_1b_0)\frac{1}{s} + (a_0b_2 + a_1b_1 + a_2b_0)\frac{2}{s^2} + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)\frac{6}{s^3} + \cdots ..$$

Suppose $R(uv) = F(s) = G(s)$

By previous section we have

$$U(s) = u_0\frac{s}{s-1} - \frac{F(s)}{s-1}$$

$$\therefore U(s) = 1.3\frac{s}{s-1} - \left[\frac{a_0b_0}{s-1} + \frac{a_0b_2 + a_1b_0}{s(s-1)} + 2\frac{a_0b_2 + a_1b_1 + a_2b_0}{s^2(s-1)} + 6\,\frac{a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0}{s^3(s-1)}\right] + \ \ldots$$

Rearranging the terms

$$\therefore U(s) = 1.3\frac{s}{s-1} - \left[\frac{a_0b_0s}{s(s-1)} + \frac{(a_0b_2 + a_1b_0)s}{s^2(s-1)} + 2\frac{(a_0b_2 + a_1b_1 + a_2b_0)s}{s^3(s-1)} + 6\,\frac{(a_{\ 0}b_3 + a_1b_2 + a_2b_1 + a_3b_0)s}{s^4(s-1)}\right] + \ \ldots$$

Applying partial fraction and rearranging terms

$$U(s) = 1.3\frac{s}{s-1} - \left[a_0b_0\left(\frac{s}{s-1}-1\right) + (a_0b_2 + a_1b_0)\left(-1-\frac{1}{s}+\frac{s}{s-1}\right) + 2(a_0b_2 + a_1b_1 + a_2b_0)\left(-1-\frac{1}{s}-\frac{1}{s^2}+\frac{s}{s-1}\right) + 6\,(a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)\left(-1-\frac{1}{s}-\frac{1}{s^2}-\frac{1}{s^3}+\frac{s}{s-1}\right)\right] + \ldots$$

Applying inverse Raj transform

$$u(t) = 1.3 + (1.3. - a_0b_0)t + (1.3. - a_0b_0 - a_0b_1 - a_1b_0)\frac{t^2}{2} + (1.3. - a_0b_0 - a_0b_1 - a_1b_0 - a_0b_2 - a_1b_1 - a_2b_0)\frac{t^3}{6} + \ldots \quad (5.3)$$

Similarly we can obtain

$$v(t) = 0.6 + (a_0b_0 - 0.6)t + (0.6. - a_0b_0 + a_0b_1 + a_1b_0)\frac{t^2}{2} + (-0.6. + a_0b_0 - a_0b_1 - a_1b_0 + a_0b_2 + a_1b_1 + a_2b_0)\frac{t^3}{6} + \ldots \quad (5.4)$$

From equation (5.3)

$$\sum_{n=0}^{\infty} a_n t^n = 1.3 + (1.3. - a_0b_0)t + (1.3. - a_0b_0 - a_0b_1 - a_1b_0)\frac{t^2}{2} + (1.3. - a_0b_0 - a_0b_1 - a_1b_0 - a_0b_2 - a_1b_1 - a_2b_0)\frac{t^3}{6} + \ldots$$

Hence $a_0 = 1.3 a_1 = 1.3. - a_0b_0$

$a_2 = 1.3. - a_0b_0 - a_0b_1 - a_1b_0$

$a_3 = 1.3. - a_0b_0 - a_0b_1 - a_1b_0 - a_0b_2 - a_1b_1 - a_2b_0$,……..

From equation 5.4

$$\sum_{n=0}^{\infty} b_n t^n == 0.6 + (a_0b_0 - 0.6)t + (0.6. - a_0b_0 + a_0b_1 + a_1b_0)\frac{t^2}{2} + (-0.6. + a_0b_0 - a_0b_1 - a_1b_0 + a_0b_2 + a_1b_1 + a_2b_0)\frac{t^3}{6} + \ldots$$

Hence $b_0 = 0.6 b_1 = a_0b_0 - 0.6$

$b_2 = 0.6. - a_0b_0 + a_0b_1 + a_1b_0$

$b_3 = -0.6. + a_0b_0 - a_0b_1 - a_1b_0 + a_0b_2 + a_1b_1 + a_2b_0$ ……….

Obtaining the values of a_0 , a_1 ,a_2 , a_3 …...b_0 , b_1 , b_2 , b_3……..

We get required solution of the system of equations

$$u(t) = 1.3 + 0.52t - 0.237\,9\,t^2 - 0.03153\,t^3 + \ldots\ldots$$

and

$$v(t) = 0.6 + 0.18t + 0.183\,t^2 - 0.00835\,t^3 + \ldots..$$

Graph of solution of the system 5.1 and 5.2 with given initial conditions is,

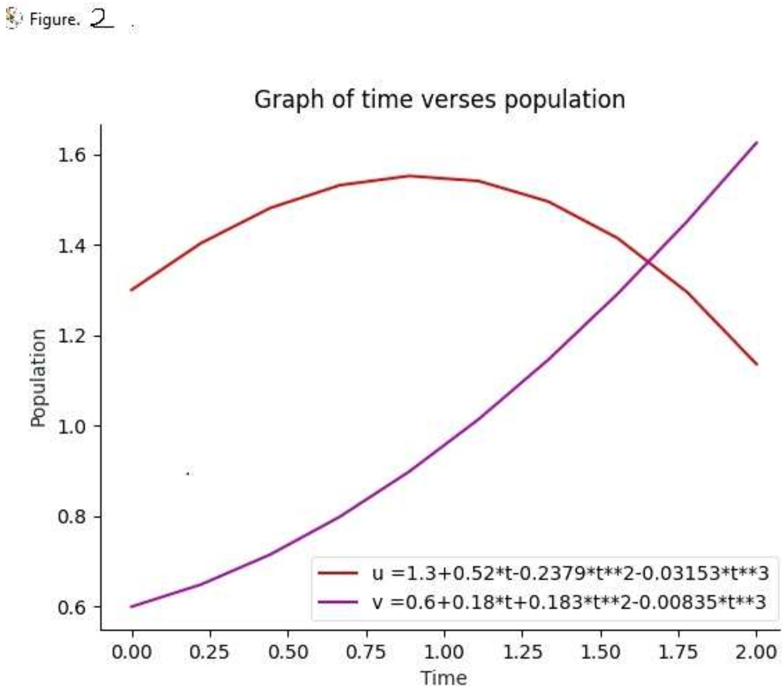

Figure 2

This figure 2 is graph showing effect of predators on preys. From this graph we can conclude that the number of predators and prey is maintained (conserved).in some limit. That means if number of preys increases then the number of predators will also increase due to increase in food supply. Increase in the predators consumes more food. It results reduction in food supply means number of preys reduces. A time comes when the number of predators and prey becomes equal. Then increase in predator results decrease in prey. Hence there is shortage of food for predators. Thus the chain is continued and number of predators and prey always remains in some specific limit.

VI. CONCLUSION

By using Raj transform we can easily solve the mathematical models in biochemistry, health sciences and environmental sciences, containing ordinary differential equations.

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