A PORTRAYAL OF INTEGER SOLUTIONS TO NON-HOMOGENEOUS TERNARY CUBIC DIOPHANTINE EQUATION

Abstract

This paper is concerned with the problem of determining varieties of non-zero distinct integer solutions to the non-homogeneous ternary cubic diophantine equation;

$$5(x^2+y^2)-6xy+4(x+y+1)=100(4k^2+25)z^3$$

Different sets of integer solutions to the above equation are obtained by reducing it to the equation, which is solvable, through employing suitable transformations.

Keywords: ternary cubic, non-homogeneous cubic, integer solutions.

Authors

J. Shanthi
Assistant Professor
Department of Mathematics
Shrimati Indira Gandhi College
Affiliated to Bharathidasan University
Trichy, Tamil Nadu, India.
shanthivishvaa@gmail.com

M. A. Gopalan
Professor, Department of Mathematics
Shrimati Indira Gandhi College
Affiliated to Bharathidasan University
Trichy, Tamil Nadu, India.
mayilgopalan@gmail.com

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I. INTRODUCTION

One of the interesting areas of Number Theory is the subject of diophantine equations which has fascinated and motivated both Amateurs and Mathematicians alike. It is well-known that diophantine equation is a polynomial equation in two or more unknowns requiring only integer solutions. It is quite obvious that Diophantine equations are rich in variety playing a significant role in the development of Mathematics .The theory of Diophantine equations is popular in recent years providing a fertile ground for both Professionals and Amateurs. In addition to known results , this abounds with unsolved problems. Although many of its results can be stated in simple and elegant terms , their proofs are sometimes long and complicated. There is no well unified body of knowledge concerning general methods .A Diophantine problem is considered as solved if a method is available to decide whether the problem is solvable or not and in case of its solvability , to exhibit all integers satisfying the requirements set forth in the problem. The successful completion of exhibiting all integers satisfying the requirements set forth in the problem add to further progress of Number Theory as they offer good applications in the field of Graph theory ,Modular theory ,Coding and Cryptography ,Engineering ,Music and so on. Integers have repeatedly played a crucial role in the evolution of the Natural Sciences. The theory of integers provide answers to real world problems.

It is well-known that diophantine equations, homogeneous or non-homogeneous, have aroused the interest of many mathematicians.. It is worth to observe that Cubic Diophantine equations fall in to the theory of Elliptic curves which are used in Cryptography. In particular, one may refer [1-10] for cubic equations with three and four unknowns.

The main thrust of this paper is to exhibit different sets of integer solutions to an interesting ternary non-homogeneous cubic equation given by $5(x^2+y^2)-6xy+4(x+y+1)=100(4k^2+25)z^3$ by using elementary algebraic methods. The outstanding results in this study of Diophantine equation will be useful for all readers.

II. METHOD OF ANALYSIS

The non-homogeneous ternary cubic diophantine equation to be solved is given by

$$5(x^2+y^2)-6xy+4(x+y+1)=100(4k^2+25)z^3 \tag{1}$$

Introduction of the linear transformations

$$x=5(u+v)-1, y=5(u-v)-1 \tag{2}$$

in (1) leads to

$$u^2+4v^2=(4k^2+25)z^3 \tag{3}$$

The process of obtaining different sets of integer solutions to (1) is illustrated below:

Illustration 1:

It is observed that (3) is satisfied by

$$u = 5\,\alpha^{3s}\ , v = k\,\alpha^{3s} \tag{4}$$

and

$$z = \alpha^{2s} \tag{5}$$

Using (4) in (2), we get

$$x = 5(5+k)\,\alpha^{3s} - 1\,, y = 5(5-k)\,\alpha^{3s} - 1 \tag{6}$$

Thus, (5) & (6) represent the integer solutions to (1).

Illustration 2:

Taking

$$u = s\,v \tag{7}$$

in (3) leads to

$$(s^2+4)\,v^2 = (4k^2+25)\,z^3$$

which is satisfied by

$$v = (4k^2+25)^2\,(s^2+4)\,t^{3\alpha} \tag{8}$$

and

$$z = (4k^2+25)\,(s^2+4)\,t^{2\alpha} \tag{9}$$

In view of (7) , note that

$$u = (4k^2+25)^2\,s\,(s^2+4)\,t^{3\alpha} \tag{10}$$

Using (8) & (10) in (2) , we get

$$\begin{aligned} x &= 5*(4k^2+25)^2\,(s+1)(s^2+4)\,t^{3\alpha} - 1, \\ y &= 5*(4k^2+25)^2\,(s-1)(s^2+4)\,t^{3\alpha} - 1 \end{aligned} \tag{11}$$

Thus , (9) & (11) represent the integer solutions to (1).

Illustration 3:

Taking

$$v = s\,u \tag{12}$$

in (3) leads to

$$(4s^2+1)u^2=(4k^2+25)\,z^3$$

which is satisfied by

$$u=(4k^2+25)^2(4s^2+1)\,t^{3\alpha} \tag{13}$$

and

$$z=(4k^2+25)\,(4s^2+1)\,t^{2\alpha} \tag{14}$$

In view of (12) , note that

$$v=(4k^2+25)^2 s\,(4s^2+1)\,t^{3\alpha} \tag{15}$$

Using (13) & (15) in (2) , we get

$$\begin{aligned} x&=5*(4k^2+25)^2(s+1)(4s^2+1)t^{3\alpha}-1,\\ y&=5*(4k^2+25)^2(1-s)(4s^2+1)t^{3\alpha}-1 \end{aligned} \tag{16}$$

Thus , (14) & (16) represent the integer solutions to (1).

Illustration 4

Taking

$$v=5z \tag{17}$$

in (3) , it is written as

$$u^2=z^2\,((4k^2+25)z-100) \tag{18}$$

It is possible to choose the values of z so that the R.H.S. of (18) is a perfect square and hence the corresponding values of u are obtained.

In view of (17) ,the values of v are found .Substituting these values of v,u in (2) , the respective integer solutions to (1) are found. The above process is exhibited below:

Let

$$\alpha^2=(4k^2+25)z-100 \tag{19}$$

which is satisfied by

$$z_0=4,\alpha_0=4k$$

Assume

$$\alpha_1=h-\alpha_0\,,z_1=z_0+uh \tag{20}$$

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to be the second solution to (19). Substituting (20) in (19) and simplifying, we have

$$h = 2\alpha_0 + (4k^2 + 25)u$$

In view of (20) , one has

$$\alpha_1 = \alpha_0 + (4k^2 + 25)u \, , z_1 = z_0 + u(2\alpha_0 + (4k^2 + 25)u)$$

The repetition of the above process leads to the general solution to (19) as

$$\alpha_n = \alpha_0 + (4k^2 + 25)u\,n = 4k + (4k^2 + 25)u\,n,$$

$$z_n = 2nu\alpha_0 + (4k^2 + 25)u^2 n^2 + z_0 = 8ukn + (4k^2 + 25)u^2 n^2 + 4$$
(21)

From (18), it is seen that

$$u_n = ((4k^2 + 25)un + 4k)\ ((4k^2 + 25)n^2 u^2 + 8ukn + 4)$$

Also, from (17) ,note that

$$v_n = 5\ ((4k^2 + 25)n^2 u^2 + 8ukn + 4)$$

In view of (2) ,the integer solutions to (1) are given by

$$x_n = 5((4k^2 + 25)un + 4k + 5)((4k^2 + 25)u^2 n^2 + 8ukn + 4) - 1,$$

$$y_n = 5((4k^2 + 25)un + 4k - 5)((4k^2 + 25)u^2 n^2 + 8ukn + 4) - 1$$

along with (21).

Illustration 5:

Taking

$$u = 5z \tag{22}$$

in (3) , it is written as

$$4v^2 = z^2\ ((4k^2 + 25)z - 25) \tag{23}$$

Let

$$4\alpha^2 = (4k^2 + 25)z - 25 \tag{24}$$

which is satisfied by

$$z_0 = 1, \alpha_0 = k$$

Assume

$$\alpha_1 = h - \alpha_0 \, , z_1 = z_0 + 4uh \tag{25}$$

to be the second solution to (24). Substituting (25) in (24) and simplifying, we have

$$h = 2\alpha_0 + (4k^2 + 25)u$$

In view of (25), one has

$$\alpha_1 = \alpha_0 + (4k^2 + 25)u, z_1 = z_0 + 4u(2\alpha_0 + (4k^2 + 25)u)$$

The repetition of the above process leads to the general solution to (24) as

$$\alpha_n = \alpha_0 + (4k^2 + 25)un = k + (4k^2 + 25)un,$$

$$z_n = 8nu\alpha_0 + 4(4k^2 + 25)u^2 n^2 + z_0 = 8ukn + 4(4k^2 + 25)u^2 n^2 + 1$$

(26)

From (23), it is seen that

$$v_n = ((4k^2 + 25)un + k)\ (4(4k^2 + 25)n^2 u^2 + 8ukn + 1)$$

Also , from (22) ,note that

$$u_n = 5\ (4(4k^2 + 25)n^2 u^2 + 8ukn + 1)$$

In view of (2) ,the integer solutions to (1) are given by

$$x_n = 5((4k^2 + 25)un + k + 5)(4(4k^2 + 25)u^2 n^2 + 8ukn + 1) - 1,$$

$$y_n = -5((4k^2 + 25)un + k - 5)(4(4k^2 + 25)u^2 n^2 + 8ukn + 4) - 1$$

along with (26) .

Illustration 6:

Assume

$$z = a^2 + 4b^2 \tag{27}$$

Express the integer $(4k^2 + 25)$ on the R.H.S. of (3) as the product of complex Conjugates as follows

$$(4k^2 + 25) = (5 + i2k)(5 - i2k) \tag{28}$$

Substituting (27) & (28) in (3) and employing the method of factorization, consider

$$u + i2v = (5 + i2k)(a + i2b)^3 \tag{29}$$

Equating the real and imaginary parts in (29) , the values of u, v are found.

In view of (2) ,the values of x, y are given by

$$x = 5[(5 + k)(a^3 - 12ab^2) + (5 - 4k)(3a^2 b - 4b^3)] - 1,$$
$$y = 5[(5 - k)(a^3 - 12ab^2) - (4k + 5)\ (3a^2 b - 4b^3)] - 1 \tag{30}$$

Thus , (27) & (30) give the integer solutions to (1).

Note 1: Apart from (28), one may consider the integer $(4k^2+25)$ on the R.H.S. of (3) as $(4k^2+25)=(2k+i5)(2k-i5)$ giving a different set of integer solutions to (1).

III. CONCLUSION

In this paper, we have made an attempt to find infinitely many non-zero distinct integer solutions to the non- homogeneous cubic equation with three unknowns given by $5(x^2+y^2)-6xy+4(x+y+1)=100\,(4k^2+25)z^3$. To conclude, one may search for other choices of solutions to the considered cubic equation with three unknowns and higher degree Diophantine equations with multiple variables.

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