HOPF BIFURCATION AND HARVESTING ANALYSIS IN PREY AND TWO PREDATORS MODEL

Abstract

In this chapter we studied the dynamics of three species model consisting of a single prey and two predators. The model in volved two predators preying on single prey. Two predators are neutral to each other and of generalist type. A delay is induced in the interaction of prey and second predator species and prey species is harvested. The model equations are described by system of delay differential equations. The co-existing state is identified, and local stability analysis is carried out at this point. The hopf bifurcation analysis is carried outat this point. The critical point is identified using numerical simulation. The harvesting effort of prey species is included in the study and identified the efforts which makes the system stable.

Keywords: prey, predator Timelags, Hopf Bifurcation, harvesting.

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I. INTRODUCTION

Ecological science gained a lot of importance in this era. The population dynamics of species is widely addressed by many authors. The stability analysis was the main aspect in ecological phenomenon. The modelling approach of ecological study was initiated byLokta [1] and Volterra [2]. The modelling of ecological communities is dispensed by using differential equations. the stability aspects using differential equations are dealt by Braun [9] and Simon's [10]. The Wide range of model in ecological and epidemical is studied by Kapur [3, 4]. The qualitative analysis is the focus in ecological stability and dealt by authors [5-8]. The time delaysare quite common in biological and ecological phenomenon. The delays are of discrete, continuous are distributed type. The time lags approach in population dynamics was dealt by Cushing, J.M [11], Norman [12]. The time delays are stabilizing or destabilizes the system. The tendency of time delays is influential in ecological science. The approaches in delay differential equations are widely elaborated by the authors [13-15]. The approaches are not only in ecology but also in epidemiology. The time delays in epidemiology are also plays a significant role. Karuna [16] and Ranjith [17] studied the instability tendencies (Hopf bifurcation) in HIV and SIR epidemic models. The distributed time lags are also very important in ecological communities. These delays are significantly explaining the role of history of species dynamics. The role of kernel dynamics with different delay kernels in prey-predator and competitor populations are Widley studied by paparao e.tal [19-25].

The prey-predator models are always intersecting, three species models with one prey and two predators were discussed by shiva Reddy [18]. In his study he chooses the model with a prey, predator and super-predator. The dynamics was well established. The delays are intersecting in prey-predator models. In this chapter we choose a model consisting of a single prey and two predators preying on a same prey. The two predators are having alternative food source. We infuse a discrete time lag (τ) in prey-predator model (logistic) in the interaction of prey and second predator and also take harvesting of prey species. The stability analysis was carried out at normal study state and hopf bifurcation analysis is also addressed. Finally, the harvesting efforts are also identified in which the system stabilizes for certain harvesting efforts.

II. FORMATION OF MATHEMATICAL MODEL.

The mathematical for three species ecological species consists of a single prey and two predators preying on this prey species. Two predators are neutral to each other and have alternative food sources (Generalist predators). A logistic growth model is proposed for all three species. A discrete time lag is induced in the interaction of prey and second species. The harvesting effort on prey species is also taken investigation. The mathematical model is formulated with a system of delay differential equations given below.

$$\frac{dx_1}{dt} = a_1 x_1\left[1-\frac{x_1}{L_1}\right] - \alpha x_1 x_2 - \beta x_1(t-\tau)x_3(t-\tau) - qEx_1$$

$$\frac{dx_2}{dt} = a_2 x_2\left[1-\frac{x_2}{L_1}\right] + \delta x_1 x_2$$

$$\frac{dx_3}{dt} = a_3 x_3\left[1-\frac{x_3}{L_1}\right] + \varepsilon x_1(t-\tau)x_3(t-\tau) \tag{2.1}$$

With the following notations
$x_1(t)$ Preydensity, $x_2(t)$ First predator density ,$x_3(t)$ second predator density
$a_i (i=1,2,3)$: Intrinsic growth rates three species
α_{ii} : Inter species competition rate of three species
$\alpha, \beta, \delta, \varepsilon$: Mutual interference strengths of three species
L_i : Carrying capacities of three species q: Harsing effort, E: effort of harvesting
all the constants are assumed to be positive

III. EXISTENCE OF EQUILIBRIUM:

The system (2.1) admits a positive equilibrium point for the co-existence state if the following conditions hold good.

(i) $a_1 qE > \alpha L_2 + \beta L_3$ (ii) $\varepsilon a_2 > \delta a_3$ (iii) $\delta a_3 > \varepsilon a_2$

The co-existing state $E(\overline{x_1}, \overline{x_2}, \overline{x_3})$ given by

$$\overline{x_1} = \frac{L_1 a_1 a_3 (a_1 qE - \alpha L_2 - \beta L_3)}{a_1 a_2 a_3 qE + a_3 \delta \alpha L_1 L_2 + a_2 \beta \varepsilon L_1 L_3}$$
$$\overline{x_2} = \frac{L_2 [a_1 a_3 a_2 qE + a_1 a_3 \alpha L_1 qE + \beta L_1 L_3 (\varepsilon a_2 - \delta a_3)}{a_1 a_2 a_3 qE + a_3 \delta \alpha L_1 L_2 + a_2 \beta \varepsilon L_1 L_3} \quad (3.1)$$
$$\overline{x_3} = \frac{L_3 [a_1 a_3 a_2 qE + a_1 a_2 \varepsilon L_1 qE + \alpha L_1 L_2 (\delta a_3 - \varepsilon a_2)}{a_1 a_2 a_3 qE + a_3 \delta \alpha L_1 L_2 + a_2 \beta \varepsilon L_1 L_3}$$

IV. LOCAL STABILITY ANALYSIS:

Theorem4.1: The co-existing stateislocally asymptotically stable

Proof: The Jacobean matrix for the system of equations (2.1) is given by

$$J = \begin{bmatrix} \beta x_3 (1 - e^{-\lambda\tau}) - \frac{a_1 x_1}{L_1} & -\alpha x_{1-qE} & -\beta x_1 e^{-\lambda\tau} \\ \delta x_2 & \frac{a_2 x_2}{L_2} & 0 \\ \varepsilon x_3 e^{-\lambda\tau} & 0 & -\varepsilon x_1 (1 - e^{-\lambda\tau}) - \frac{a_3 x_3}{L_3} \end{bmatrix} \quad (4.1)$$

With the characterstic equation is given by

$$|J - \lambda I| = \begin{vmatrix} \beta x_3 (1 - e^{-\lambda\tau}) - \frac{a_1 x_1}{L_1} - qE - \lambda & -\alpha x_1 & -\beta x_1 e^{-\lambda\tau} \\ \delta x_2 & -\frac{a_2 x_2}{L_2} - \lambda & 0 \\ \varepsilon x_3 e^{-\lambda\tau} & 0 & -\varepsilon x_1 (1 - e^{-\lambda\tau}) - \frac{a_3 x_3}{L_3} - \lambda \end{vmatrix} = \mathbf{0}$$

$$\psi(\lambda,\tau)=\lambda^3+p_1\lambda^2+p_2\lambda+p_3+e^{-\lambda\tau}(q_1\lambda^2+q_2\lambda+q_3)=0 \tag{4.2}$$

Where

$$p_1=(\frac{a_2x_2}{L_2}+\varepsilon x_1+\frac{a_1x_1}{L_1}-qE-\beta x_3+\frac{a_3x_3}{L_3})$$

$$p_2=(\delta\alpha x_1x_2+\frac{a_3a_2x_3x_2}{L_3L_2}+\frac{\varepsilon a_2x_1x_3}{L_3}+\frac{(a_1-qE)a_2x_1x_2}{L_1L_2}+\frac{(a_1-qE)\varepsilon {x_1}^2}{L_1}+\frac{(a_1-qE)a_2x_1x_2}{L_1L_2}-\frac{\beta a_3{x_3}^2}{L_3}-\frac{\beta a_2x_2x_3}{L_2}-\beta\varepsilon x_1x_3)$$

$$p_3=(\frac{(a_1-qE)a_2a_3x_1x_2x_3}{L_1L_2L_3}+\frac{(a_1-qE)a_2\varepsilon {x_1}^2x_2}{L_1L_2}-\frac{a_3a_2x_2{x_3}^2\beta}{L_3L_2}+\frac{(a_1-qE)a_3x_1x_3}{L_1L_3}+\frac{\alpha\delta x_1x_2x_3a_3}{L_3}-\frac{\beta\varepsilon x_1x_2x_3a_2}{L_2}+\beta\alpha\delta {x_1}^2x_2)$$

$$q_1=(\beta x_3-\varepsilon x_1)$$

$$q_2=(\frac{a_3\beta {x_3}^2}{L_3}-\frac{(a_1-qE){x_1}^2\varepsilon}{L_1}-2\beta\varepsilon x_1x_3-\frac{\varepsilon a_2x_1x_2}{L_2})$$

$$q_3=(\frac{2\beta\varepsilon a_2x_1x_2x_3}{L_2}+\frac{\beta a_3a_2{x_3}^2x_2\alpha}{L_3L_2}-\frac{(a_1-qE)a_2\varepsilon {x_1}^2x_2}{L_1L_2}-\beta\varepsilon\delta {x_1}^2x_3)$$

We need to find the condition for existence of negative real roots

Case (i): For $\tau=0$ equation (4.2) becomes

$$\psi(\lambda,0)=\lambda^3+\lambda^2(\frac{(a_1-qE)x_1}{L_1}+\frac{a_2x_2}{L_2}+\frac{a_3x_3}{L_3})+\lambda(\frac{(a_1-qE)a_2x_1x_2}{L_1L_2}+\alpha\delta x_1x_2+\frac{a_3a_2x_3x_2}{L_3L_2}+\frac{(a_1-qE)a_3x_1x_3}{L_1L_3}+\beta\varepsilon x_1x_3)+(\frac{(a_1-qE)a_2a_3x_1x_2x_3}{L_1L_2L_3}+\frac{\alpha\delta a_3x_1x_2x_3}{L_3}+\frac{\beta\varepsilon x_1x_2x_3a_2}{L_2})=0$$

In simplest form the above equation can be represented as $b_0\lambda^3+b_1\lambda^2+b_2\lambda+b_3=0$
Where

$$b_0=1$$

$$b_1=\frac{(a_1-qE)x_1}{L_1}+\frac{a_2x_2}{L_2}+\frac{a_3x_2}{L_3}$$

$$b_2=\frac{(a_1-qE)a_2x_1x_2}{L_1L_2}+\frac{a_3(a_1-qE)x_1x_3}{L_1L_3}+\frac{a_2a_3x_3x_2}{L_2L_3}+\alpha\delta x_1x_2+\beta\varepsilon x_1x_1$$

$$b_3=x_1x_2x_3\left(\frac{(a_1-qE)a_2a_{3_1}}{L_1L_2L_3}+\frac{a_3\delta\alpha}{L_3}+\frac{a_2\beta\varepsilon}{L_2}\right)$$

By using Routh-Hurwitzcriteria, we calculate the following determinants

$$D_1=b_1, D_2=\begin{vmatrix} b_1 & b_3\\ b_0 & b_2\end{vmatrix}, D_3=\begin{vmatrix} b_1 & b_3 & b_5\\ b_0 & b_2 & b_4\\ 0 & b_1 & b_3\end{vmatrix}$$

$$D_1=b_1=\left(\frac{(a_1-qE)x_1}{L_1}+\frac{a_2x_2}{L_2}+\frac{a_3x_3}{L_3}\right)>0$$

$D_1 > 0 at\ E(\bar{x}_1, \bar{x}_2, \bar{x}_3)$.

$D_2 = b_1 b_2 - b_3 b_0$

$$D_2 = \left(\frac{a_1 x_1}{L_1} + \frac{a_2 x_2}{L_2} + \frac{a_3 x_3}{L_3}\right)\left(\frac{a_1 a_2 x_1 x_2}{L_1 L_2} + \frac{a_1 a_3 x_1 x_3}{L_1 L_3} + \frac{a_3 a_2 x_3 x_2}{L_3 L_2} + \alpha\delta x_1 x_2 + \beta\varepsilon x_1 x_3\right)$$

$$-x_1 x_2 x_3\left(\frac{a_1 a_2 a_3}{L_1 L_2 L_3} + \frac{a_3 \delta\alpha}{L_3} + \frac{a_2 \beta\varepsilon}{L_2}\right)$$

$$= \left(\frac{a_1^2 a_2 x_1 x_2}{L_1^2 L_2} + \frac{a_1^2 a_3 x_1 x_3}{L_1^2 L_3} + \frac{a_1 a_3 a_2 x_3 x_2}{L_1 L_3 L_2} + \frac{\alpha\delta x_1^2 x_2 a_1}{L_1} + \frac{a_1 \beta\varepsilon x_1^2 x_3}{L_1} + \frac{a_1 a_2^2 x_1 x_2^2}{L_1 L_2^2}\right.$$

$$\left.+\frac{a_3 a_1 a_2 x_3 x_2 x_1}{L_3 L_1 L_2} + \frac{a_3 a_2^2 x_3 x_2^2}{L_3 L_2} + \frac{a_2 \alpha\delta x_1 x_2^2}{L_2} + \frac{a_1 a_2 a_3 x_1 x_2 x_3}{L_1 L_2 L_3} + \frac{a_1 a_3^2 x_1 x_3^2}{L_1 L_2 L_3} + \frac{a_3^2 a_2 x_3 x_2}{L_3^2 L_2} + \frac{a_3 \beta\varepsilon x_1 x_3^2}{L_3}\right) > 0$$

Clearly $D_2 > 0\ at\ E(\overline{x}_1, \overline{x}_2, \overline{x}_3)$

$D_3 = (b_1 b_2 - b_3 b_0) b_3$

$$D_3 = \left(\frac{a_1^2 a_2 x_1 x_2}{L_1^2 L_2} + \frac{a_1^2 a_3 x_1 x_3}{L_1^2 L_3} + \frac{a_1 a_3 a_2 x_3 x_2}{L_1 L_3 L_2} + \frac{\alpha\delta x_1^2 x_2 a_1}{L_1} + \frac{a_1 \beta\varepsilon x_1^2 x_3}{L_1} + \frac{a_1 a_2^2 x_1 x_2^2}{L_1 L_2^2} + \frac{a_3 a_1 a_2 x_3 x_2 x_1}{L_3 L_1 L_2} + \frac{a_3 a_2^2 x_3 x_2^2}{L_3 L_2}\right.$$

$$\left.+\frac{a_2 \alpha\delta x_1 x_2^2}{L_2} + \frac{a_1 a_2 a_3 x_1 x_2 x_3}{L_1 L_2 L_3} + \frac{a_1 a_3^2 x_1 x_3^2}{L_1 L_2 L_3} + \frac{a_3^2 a_2 x_3 x_2}{L_3^2 L_2} + \frac{a_3 \beta\varepsilon x_1 x_3^2}{L_3}\right)\left(x_1 x_2 x_3\left(\frac{a_1 a_2 a_3}{L_1 L_2 L_3} + \frac{a_3 \delta\alpha}{L_3} + \frac{a_2 \beta\varepsilon}{L_2}\right)\right) > 0$$

$D_3 > 0\ \ E(\overline{x}_1, \overline{x}_2, \overline{x}_3)$.

Hence the co-existing state $E(\overline{x_1}, \overline{x_2}, \overline{x_3})$ is locally asymptotically stable.

Case (ii) for τ > 0:

Suppose there is positive τ_0 such that the equation (3.1) has pair of purely imaginary root, can be taken as $\pm i\omega, \omega > 0$ then ω satisfies the equation (4.1)

Putting$\lambda = i\omega, \omega > 0$we have

$$\psi(i\omega, \tau) = (i\omega)^3 + p_1(i\omega)^2 + p_2(i\omega) + p_3 + e^{-i\omega\tau}(q_1(i\omega)^2 + q_2(i\omega) + q_3) = 0$$

$$\psi(i\omega, \tau) = (i\omega)^3 + p_1(i\omega)^2 + p_2(i\omega) + p_3 + [cos(\omega\tau) - i\, sin(\omega\tau)][q_1(i\omega)^2 + q_2(i\omega) + q_3] = 0$$

$$-\omega^2 p_1 + p_3 - q_1\omega^2 \cos\omega\tau + q_3 \cos\omega\tau + q_2\omega \sin\omega\tau + i[-w^3 + wp_2 + q_2\omega \cos\omega\tau + q_1\omega^2 \sin\omega\tau - q_3 \sin\omega\tau] = 0$$

$$(q_3 - q_1\omega^2)\cos\omega\tau + q_2\omega \sin\omega\tau = \omega^2 p_1 - p_3 \quad (4.3)$$

$$q_2\omega \cos\omega\tau - (q_3 - q_1\omega^2)\sin\omega\tau = \omega^3 - \omega p_2 \quad (4.4)$$

Onsimplication we get

$(3.3)^2 + (3.4)^2 \Rightarrow$

$(q_3 - q_1\omega)^2 + (q_2\omega)^2 = (\omega^2 p_1 - p_3)^2 + (\omega^3 - \omega p_2)^2$

$\omega^6 + \omega^4(p_1^2 - 2p_2 - q_1^2) + \omega^2(p_2^2 - 2p_1 p_3 - q_2^2 + 2q_1 q_3) + q_3^2 + p_3^2 = 0$

$let\psi(p) = p^3 + p^2 N_1 + pN_2 + N_3 = 0$

Where

$N_1 = p_1^2 - 2p_1 - {q_1}^2$
$N_2 = p_2^2 - 2p_1p_2 - q_2^2 + 2q_1q_3$
$N_3 = q_3^3 + p_3^3$
$p = \omega^2$
$\therefore \psi(p) = 0$

If we assume that $N_1 > 0, N_2 > 0, N_3 > 0$ then have no positive real roots

$N_1 = p_1^2 - 2p_1 - {q_1}^2 N_2 = p_2^2 - 2p_1p_2 - q_2^2 + 2q_1q_3 N_3 = q_3^3 + p_3^3$
Thus if $N_1 > 0, N_2 > 0, N_3 > 0$ then there is no ω such that $i\omega$ is an Eigen value of the characteristic equation of $\psi(\lambda, \tau) = 0$

If λ will never be a purely imaginary root of equation$\psi(\lambda, \tau) = 0$, thus the real parts of all Eigen values of
$\psi(\lambda, \tau) = 0$ are negative for all $\tau \geq 0$ summarizing, the above analysis, we have the following theorem

Theorem 4.2: Co-existing state$E(\overline{x_1}, \overline{x_2}, \overline{x_3})$ is locally asymptotically stable for all τ ,if following condition hold.

1. $(p_1 + q_1) > 0, (p_2 + q_2) > 0, (p_3 + q_3) > 0$
2. $N_1 > 0, N_2 > 0, N_3 > 0$

Now if any one of $N_1, N_2, N_3$is negative then (2.1) has positive root ω_0
Eliminating $\cos \omega\lambda$ from equation (4.3)&(4.4) we get

$$cos\,\omega\,\tau = \frac{\begin{vmatrix} \omega^2 p_1 - p_3 & q_2\omega \\ \omega^3 - \omega p_2 & -(q_3 - q_1\omega^2) \end{vmatrix}}{\begin{vmatrix} q_3 - q_1\omega^2 & q_2\omega \\ q_2\omega & -(q_3 - q_1\omega^2) \end{vmatrix}}$$

$$cos\,\omega\,\tau = \frac{\omega^2 p_1 q_3 - p_3 q_3 - \omega^4 p_1 q_1 + \omega^2 q_1 p_3 + q_3 w^3 - p_2 q_3 \omega^2}{q_3^2 + q_1^2\omega^4 - 2q_1q_3w^2 + q_2^2\omega^2}$$

$$\tau_k = \frac{1}{\omega_0} cos^{-1}\left[\frac{\omega_0^4(q_3 - p_1q_1) + \omega_0^2(p_1q_3 + q_1p_3 - p_2q_3) - p_3q_3}{q_1^2\omega_0^4 + \omega_0^2(q_2^2 - 2q_1q_3) + q_3^2}\right] + \frac{2k\pi}{\omega_0} \quad (4.5)$$

$where\ k = 0,1,2,3\ldots\ldots\ldots\ldots..$

V. HOPFBIFURCATIONS

Theorem5.1: The sufficient condition for the system (2.1) admits bifurcation along the normal steady state E if
$\tau > \tau_0$and locally asymptotically stable if $0 < \tau < \tau_0$

Proof: Hopf bifurcation occurs when the real part of $\lambda(t)$ become positive when $\tau > \tau_0$ and the steady state become unstable moreover, when τ passes through the critical value τ_0 .

To check this result, differentiating (4.2) W .r t.τ, we get

$$\psi(\lambda,\tau) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 + e^{-\lambda\tau}(q_1\lambda^2 + q_2\lambda + q_3) = 0$$

$$=3\lambda^2\frac{d\lambda}{d\tau} + 2p_1\lambda\frac{d\lambda}{d\tau} + p_2\frac{d\lambda}{d\tau} + e^{-\lambda\tau}(2q_1\lambda\frac{d\lambda}{d\tau} + q_2\frac{d\lambda}{d\tau}) + (q_1\lambda^2 + q_2\lambda + q_3)(-\lambda - \lambda\frac{d\lambda}{d\tau})e^{-\lambda\tau} = 0$$

$$\frac{d\lambda}{d\tau}\left[3\lambda^2 + 2p_1\lambda + p_2 + e^{-\lambda\tau}(2q_1\lambda + q_2) - (q_1\lambda^2 + q_2\lambda + q_3)\lambda e^{-\lambda\tau}\right] = (q_1\lambda^2 + q_2\lambda + q_3)\lambda e^{-\lambda\tau}$$

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{\left[3\lambda^2 + 2p_1\lambda + p_2 + e^{-\lambda\tau}(2q_1\lambda + q_2) - (q_1\lambda^2 + q_2\lambda + q_3)\pi e^{-\lambda\tau}\right]}{(q_1\lambda^2 + q_2\lambda + q_3)\lambda e^{-\lambda\tau}}$$

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{3\lambda^2 + 2p_1\lambda + p_2}{(q_1\lambda^2 + q_2\lambda + q_3)\lambda e^{-\lambda\tau}} + \frac{(2q_1\lambda + q_2)}{(q_1\lambda^2 + q_2\lambda + q_3)\lambda} - \frac{\tau}{\lambda}$$

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{3\lambda^2 + 2p_1\lambda + p_2}{-\lambda(\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3)} + \frac{(2q_1\lambda + q_2)}{(q_1\lambda^2 + q_2\lambda + q_3)\lambda} - \frac{\tau}{\lambda}$$

Put $\lambda = i\omega$ in the above equation

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{1}{\omega_0}\left[\frac{-3{\omega_0}^2 + 2ip_1\omega_0 + p_2}{(-{\omega_0}^3 + p_2\omega_0 + i(p_1\omega_0^2 - p_3)^2} + \frac{(2iq_1\omega_0 + q_2)}{-q_2\omega_0 + i(q_3 - q_1\omega_0^2)} + \tau i\right]$$

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{1}{\omega_0}\left[\frac{(-3{\omega_0}^2 + 2ip_1\omega_0 + p_2)((-{\omega_0}^3 + p_2\omega_0 - i(p_1\omega_0^2 - p_3)}{(-{\omega_0}^3 + p_2\omega_0)^2 + (p_1\omega_0^2 - p_3)^2} + \frac{(2iq_1\omega_0 + q_2)(-q_2\omega_0 - i(q_3 - q_1\omega_0^2)}{(q_2\omega_0)^2 + (q_3 - q_1\omega_0^2)^2} + \tau i\right]$$

Real part of $\left[\frac{d\lambda}{d\tau}\right]^{-1}$

$$=\frac{1}{\omega_0}\left[\frac{(-3{\omega_0}^2 + p_2)(-{\omega_0}^3 + p_2\omega_0) + 2p_1\omega_0(p_1\omega_0^2 - p_3)}{(-{\omega_0}^3 + p_2\omega_0)^2 + (p_1\omega_0^2 - p_3)^2} + \frac{-{q_2}^2\omega_0 + 2q_1\omega_0(q_3 - q_1\omega_0^2)}{(q_2\omega_0)^2 + (q_3 - q_1\omega_0^2)^2}\right]$$

On substituting $(-{\omega_0}^3 + p_2\omega_0)^2 + (p_1\omega_0^2 - p_3)^2 = (q_2\omega_0)^2 + (q_3 - q_1\omega_0^2)^2$

$$=\frac{1}{\omega_0}\left[\frac{3\omega_0^5 + \omega_0^3(2p_1^2 - p_2 - 3p_2 - 2q_1^2) + (p_2^2 - 2p_1p_3 + 2q_1q_2 - q_2^2)\omega_0}{(q_2\omega_0)^2 + (q_3 - q_1\omega_0^2)^2}\right]$$

$$Re\left[\frac{d\lambda}{d\tau}\right]^{-1} = \left[\frac{3\omega_0^4 + \omega_0^2(2p_1^2 - 4p_2 - 2q_1^2) + p_2^2 - 2p_1p_3 + 2q_1q_2 - q_2^2}{(q_2\omega_0)^2 + (q_3 - q_1\omega_0^2)^2}\right]$$

$$\left[\frac{d}{d\tau}Re(\lambda)\right] = \left[Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda=i\omega_0}$$

$$= \left[\frac{3\omega_0^4 + \omega_0^2(2p_1^2 - 4p_2 - 2q_1^2) + p_2^2 - 2p_1p_3 + 2q_1q_3 - q_2^2}{(q_2\omega_0)^2 + (q_3 - q_1\omega_0^2)^2}\right]$$

$$= \left[\frac{3\omega_0^4 + 2\omega_0^2 N_1 + N_2}{(q_2\omega_0)^2 + (q_3 - q_1\omega_0^2)^2}\right]$$

By using this condition$N_1>0$, $N_2>0$, $N_3>0$ we have $\left[\frac{d}{d\tau}(Re(\lambda))\right]_{\lambda=i\omega_0} > 0$

Therefore, the Hopf bifurcation occurs at $\tau > \tau_0$

VI. NUMERICAL EXAMPLE:

Numerical simulation is carried out in support of bifurcation analysis along co-existing state for the system of equations (2.1). Four set of parametric values are chosen for investigation and identified the bifurcation parameter (τ_0) shownbelow.
In each graph, figure (a) represents Time series responsesand (b) represents Phase portraits

Example 6.1: Let us parametric values
$a_1 = 1, a_2 = 2, a_3 = 3, \alpha = 0.1, \beta = 0.1, \delta = 0.2, \varepsilon = 0.1, L_1 = 100, L_2 = 100, L_3 = 100$
$x_1 = 20, x_2 = 20, x_3 = 20.$

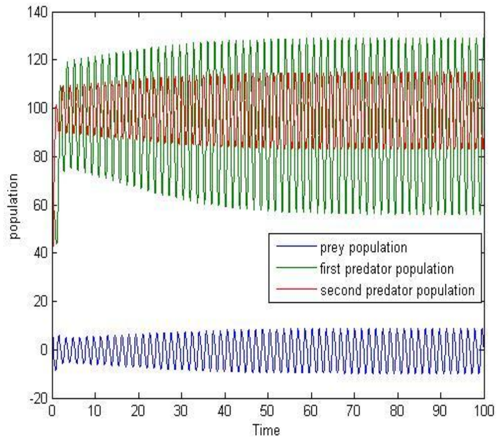


Figure 6.1.1(A)

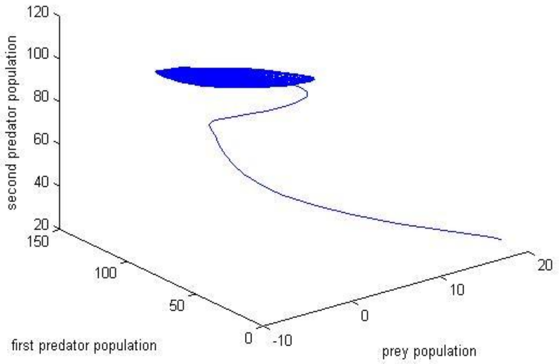
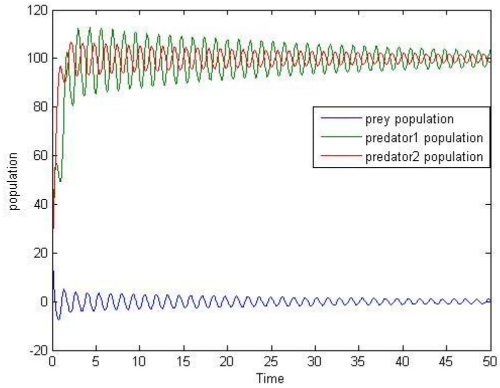


Figure 6.1.1(B)

The system possesses unbounded periodic solutions hence the system (2.1) becomes unstable for the critical parameter τ = 0.7.



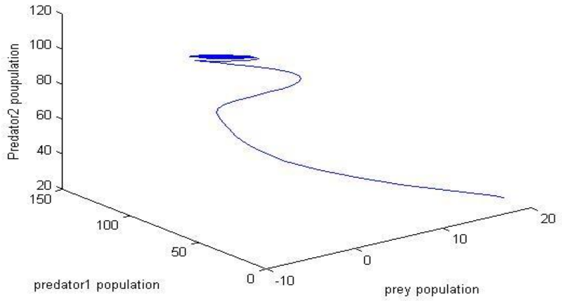
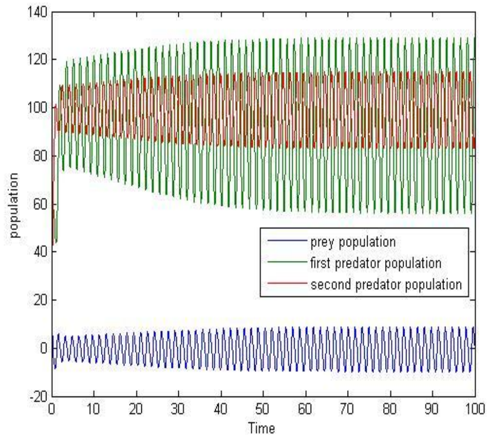


Figure 6.1.2(A) **Figure 6.1.2(B)**

The system becomes stable for τ = 0.65 for the system of equations (2.1).
The critical parameter ‘τ ‘is identified using numerical simulation where the stable system becomes unstable when the critical parameter ‘τ ‘value greater than 0.65.

Now the harvesting parameter dynamics is also discussed with various catch ability coefficient (q) values and effort (E) is shown below.

Case (i):When harvesting effort was induced on prey species the dynamics of q = 1 and E= 5



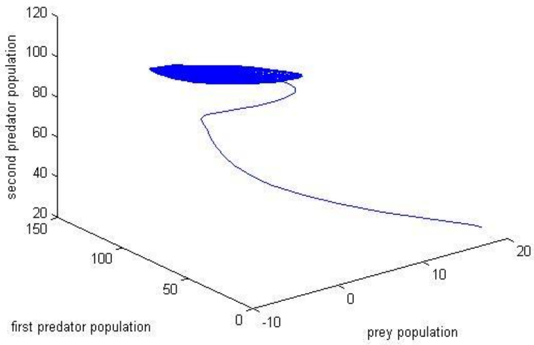
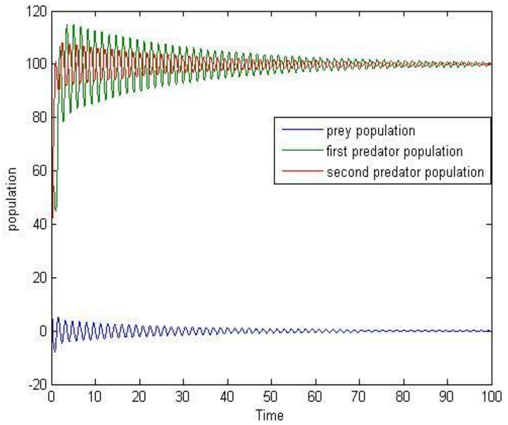


Figure 6.1.3(A) **Figure 6.1.3(B)**

The system admits unbounded periodic solutions with τ = 0.7 and q =1, E=5 makes the system (2.1) unstable.

Case (ii) :When harvesting effort was induced on prey species the dynamics of q = 0.1 and E= 5



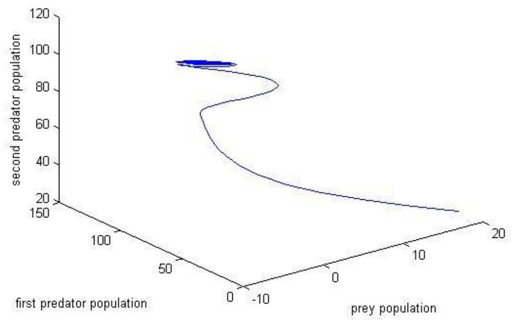
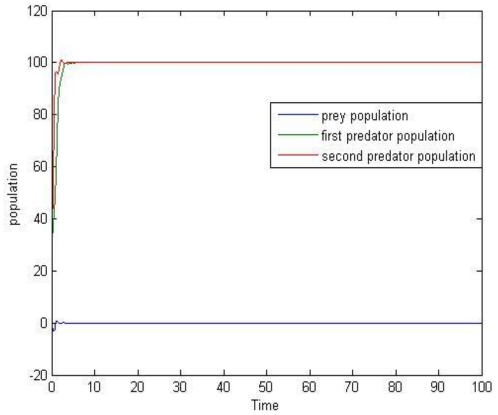


Figure 6.1.4(A) **Figure 6.1.4(B)**

The system of equations (2.1) possesses bounded variations and makes the system asymptotically stable for $\tau = 0.7$ and q =0.1, E=5.

Case (iii) : When harvesting effort was induced on prey species the dynamics of q = 1 and E= 10



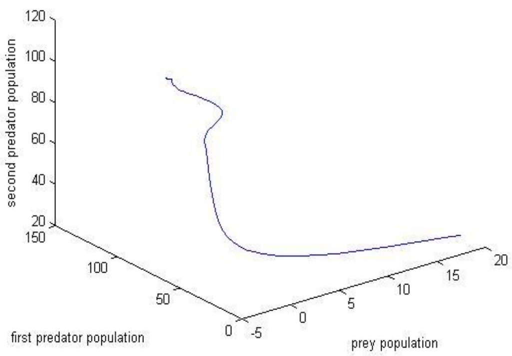


Figure 6.1.5(A) **Figure 6.1.5(B)**

The system is stable for $\tau = 0.7$ and q=1, E= 10.

VII. DISCUSSION AND CONCLUSION

The proposed model is asymptotically stable and exhibit Hopf bifurcation nature for certain critical value$\tau = 0.65$.The sufficient condition for Hopf bifurcation is derived and the critical value (τ_0) is identified forone different set of parameters. The critical values for system (2.1) with four set examples are shown in the above graphs [6.1.1 -6.1.2]. The harvesting efforts are also identified in which the system becomes stable is identified using numerical simulation.

The system still admits unstable nature for $\tau = 0.7$ and q=1,E= 5 and becomes asymptotically stable for $\tau = 0.7$ and becomes stable for $\tau = 0.7$ and q=1,E= 10.

The system undergoes hopf bifurcation along the co-existing state and harvesting efforts are significant in stability analysis. The harvesting efforts are identified in which the unstable system becomes stable. Hence harvesting efforts are also significant in stability analysis.

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