Behaviour of Free-surface Profile in Single-layer Fluid Flow Problem

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ABSTRACT

The problem involving single-layer flow of an inviscid and incompressible fluid over an undulated bottom is studied using linear theory. The problem is formulated mathematically in term of a mixed boundary value problem. Perturbation analysis in conjunction with Fourier transform technique is applied to solve the governing boundary value problem, and the free-surface profile which is unknown at the outset is determined. Also, the behaviour of the free-surface profile is studied. In addition, the role of Fourier transform technique is highlighted in an elaborate way. Finally, the effect of undulated bottom profile is also explained.

Keywords—Fluid flow; Linear theory; Mixed BVP; Froude number; Bottom profile

I. INTRODUCTION

Problems involving free-surface fluid flow over an obstacle are studied by many researchers to model various situations arising in oceanography and atmospheric science. The solutions of such flow problems are expected to provide a qualitative insight into the mechanism of wave generation. In the recent past, varieties of challenges have been faced by the researchers to model the free-surface flow over different kinds of obstacles situated at the bottom. Hence, flow over an undulated bottom has been a topic of interest in the mathematical and physical sciences.

Based on the extensive literature survey, it is found that the problems involving free-surface fluid flow over obstacles are studied by many applied mathematicians and physicists. The attention to the free-surface flow over an irregular bottom has been increasing rapidly over the last three decades. Substantial progress has been made in this direction by many researchers. For instance, Forbes and Schwartz [1] considered the flow over a semicircular obstruction and calculated the wave resistance offered by the semicircle using a numerical approach. Vanden-Broeck [2] solved the problem of Forbes and Schwartz [1] numerically, and discussed the existence of supercritical solutions which depend on the Froude number. Later on, Forbes [3] presented a numerical solution for critical free-surface flow over a semicircular obstruction attached to the bottom of a running stream. Yong [4] explained the generation of nonlinear capillary-gravity waves in a fluid system having a concave bottom including the effect of surface tension. Dias and Vanden-Broeck [5] studied the problem involving free-surface flow past a submerged triangular obstacle at the bottom of a channel and solved the problem numerically using series truncation. Shen *et al.* [6] obtained the numerical solution for the steady surface waves on an incompressible and inviscid fluid flow over a semicircular as well as a semielliptical obstacle situated at the bottom. Dias and Vanden-Broeck [7] solved the steady free-surface flow problem numerically, and demonstrated that there exist supercritical flows with waves downstream only. Panda *et al.* [8] made an effort to solve the nonlinear flow over an arbitrary bottom topography using a new approach which is different and simpler than the methods available in literature. Higgins *et al.* [9] presented an analytical series method to obtain the solution of the problem involving flow of a fluid. They have calculated the analytical series solutions for supercritical, transcritical and subcritical flow. The above studies were focused on the solution of the problem involving the steady flow. For the problems involving unsteady fluid flow, Grimshaw and Smyth [10] presented a theoretical study of a stratified fluid which is flowing over bottom topography. They solved the problem by using weak nonlinear theory and pointed out that the flow can be described by a forced Korteweg-de Vries equation. Stokes *et al.* [11] used numerical technique to analyze the unsteady flow with a submerged point sink beneath the free surface. Milewski and Vanden-Broeck [12] considered the time dependent free-surface flow over a submerged moving obstacle, and solved the problem using weak nonlinear theory. It is noticed that, the solutions of the flow problems are determined in most cases for specific bottom obstacle such as semi-circle [1, 2], semi-ellipse [13], a step [14], triangle [15], etc. Therefore, the flow over arbitrary bottom topography remained unsolved because the determination of its solution is somewhat difficult. In such cases, the governing boundary value problems become mixed and coupled, and therefore their explicit solutions are not possible always. The work presented here is concerned with the behaviour of the free-surface profile by solving a mixed boundary value problem with the help of most appropriate mathematical techniques.

In the present study, two-dimensional potential flow of an inviscid and incompressible fluid is considered. The problem involving single-layer flow in a channel having small obstruction is studied using linear theory. The physical problem is formulated mathematically in terms of mixed boundary value problem. Using perturbation analysis along with Fourier transform technique, the boundary value problem is solved to determine the analytical expression of the free-surface profile which is unknown at the outset. In addition, the role of the Fourier transform technique is highlighted. Also, the behaviour of the unknown free-surface profile is analysed.

II. DESCRIPTION AND FORMULATION OF THE PROBLEM

Two-dimensional potential flow of an inviscid and incompressible fluid is considered here. The fluid is flowing from the left to the right over an irregular bottom y = B(x) having a small undulation. A sketch of the flow domain is shown in Figure 1. The x-axis is chosen along the undisturbed bottom and the y-axis is measured vertically upward. The effect of the surface tension is ignored. Far upstream, the flow is uniform with a constant velocity c. The upstream depth of the fluid is denoted by H and the density of fluid is denoted by ρ. Let $\phi(x,y)$ be the velocity potential so that, the velocity of the fluid $\bar{q}$ can be written as $\bar{q} = (\phi_x, \phi_y)$, where ϕ_x and ϕ_y are, respectively, the partial derivatives of ϕ with respect to x and y. The free-surface, which is unknown at the outset, is given by $y = \eta(x)$. It is further assumed that the flow is stationary with respect to the bottom profile, so that the partial derivatives with respect to time can be taken to be equal to zero. The problem is made dimensionless using H as the length scale and c as the velocity scale. Therefore, the work proceeds purely with dimensionless variables.

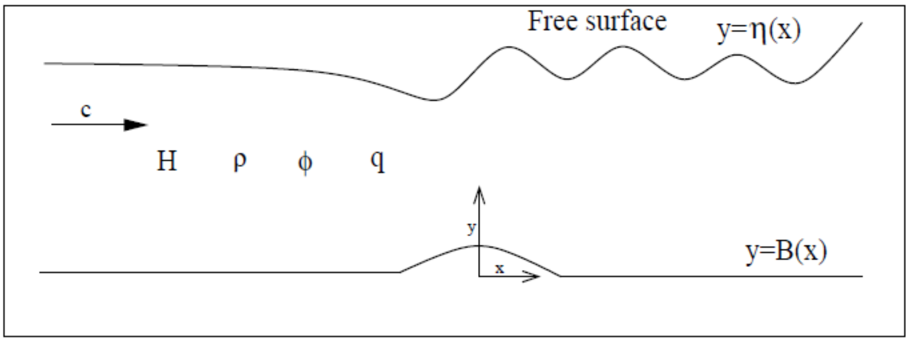

Figure 1: Defination sketch of the flow domain.

In the fluid region, the equation of continuity yields the Laplace equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0. \tag{1}$$

As no fluid particle leaves the surface, the kinematic condition on the free surface can be written as

$$\frac{\partial\phi}{\partial n} = 0, \quad \text{on} \quad y = \eta(x), \tag{2}$$

where $\partial / \partial n$ is the normal derivative at a point (x,y) on the surface.

Using Bernoulli's equation, the other condition on the free surface is derived as

$$\frac{1}{2}F^2\left(q^2 - 1\right) + \eta(x) = 1, \quad \text{on} \quad y = \eta(x), \tag{3}$$

where $F = c/\sqrt{gH}$ is the Froude number and g is acceleration due to gravity. In this work, the flow of the type subcritical is only considered. Therefore, the value of the Froude number is $F<1$.

The condition of no penetration at the bottom gives rise

$$\frac{\partial \phi}{\partial n} = 0, \text{ on } y = B(x). \tag{4}$$

In addition, the conditions at the upstream are

$$\vec{q} \to \vec{i}, \ \eta(x) \to 1 \text{ as } x \to -\infty. \tag{5}$$

The aim of the present work is to determine the unknown functions $\phi(x,y)$ and $\eta(x)$. These unknowns can be determined once the boundary value problem (1)-(5) is solved. In the following section, the above BVP is solved using perturbation analysis along with Fourier transform technique.

III. SOLUTION OF THE PROBLEM

It is assumed that the bottom profile is given by $B(x) = \varepsilon f(x)$ where ε is the maximum height of the bottom and is a dimensionless quantity. When the height ε is small, then an approximate solution of the boundary value problem (1)-(5) may be derived using perturbation expansion in powers of ε, retaining only the first-order terms. Therefore, the asymptotic expansions of the velocity potential and the free-surface profile can be expressed respectively as

$$\phi(x, y) = x + \varepsilon\phi_1(x, y) + O\left(\varepsilon^2\right), \tag{6}$$

$$\eta(x) = 1 + \varepsilon\eta_1(x) + O\left(\varepsilon^2\right), \tag{7}$$

where $\phi_1(x, y)$ and $\eta_1(x)$ are the first-order corrections of the velocity potential and the free-surface profile, respectively. The unknown parameters such as velocity potential $\phi(x, y)$ and free-surface profile $\eta(x)$ can be determined once $\phi_1(x, y)$ and $\eta_1(x)$ are evaluated. Hence, in the following part $\phi_1(x, y)$ and $\eta_1(x)$ will be determined. Using relations (6) and (7) in (1)-(4); and then comparing the first order terms of ε on both the sides of all equations, the following mixed boundary value problem is obtained:

$$\nabla^2\phi_1 = 0 \quad \text{in the fluid region,} \tag{8}$$

$$\phi_{1,y} = \eta_1'(x) \quad \text{on } y = 1, \tag{9}$$

$$F^2\phi_{1,x} + \eta_1(x) = 0 \quad \text{on } y = 1, \tag{10}$$

$$\phi_{1,y} = f'(x) \quad \text{on } y = 0, \tag{11}$$

where $f'(x)$ and $\eta_1'(x)$ are, respectively, the first order derivatives of $f(x)$ and $\eta_1(x)$ with respect to x.

In order to solve the mixed boundary value problem (8)-(11), the first-order potential $\phi_1(x, y)$ and the bottom profile $f(x)$ are assumed such that the Fourier transforms of $\phi_1(x, y)$ and $f(x)$ exist, which are defined as

$$\hat{\phi}_1(k, y) = \int_0^\infty \phi_1(x, y)\sin(kx)dx, \tag{12}$$

with inverse

$$\phi_1(x, y) = \frac{2}{\pi}\int_0^\infty \hat{\phi}_1(k, y)\sin(kx)dk, \tag{13}$$

and

$$f(x) = \int_0^\infty M(k)\cos(kx)dk, \tag{14}$$

where $M(k)$ determines the bottom profile. For the free-surface profile, let us define $\eta_1(x)$ as

$$\eta_1(x) = \int_0^\infty a(k)\cos(kx)dk. \tag{15}$$

Applying Fourier transform along with its inverse; and using the relations (14) and (15), the solution of the BVP (8)-(11) is obtained as

$$\phi_1(x,y)=\int_0^{\infty}\left[\frac{M(k)-a(k)\cosh k}{\sinh k}\cosh k(1-y)+a(k)\sinh k(1-y)\right]\sin(kx)dk, \tag{16}$$

where

$$a(k)=\frac{F^2kM(k)}{E_1(k)} \tag{17}$$

with

$$E_1(k)=F^2k\cosh k-\sinh k. \tag{18}$$

It should be noted that the relation

$$E_1(k)=0 \tag{19}$$

is called as *dispersion relation*. It can be shown (demonstrated later in Section IV) that the relation (19) has one positive real root and one negative real root, the magnitude of roots being same. It is worth-mentioning here that the root of the dispersion relation signifies the wave number of the downstream waves. It is observed from relations (15) and (17) that $\eta_1(x)$ depends on the bottom profile. Hence, the free-surface profile $\eta(x)$ can be determined once the bottom profile is known. In the present study, the following bottom profile is considered to demonstrate the results:

$$f(x)=\begin{cases}\frac{1}{2}\left[1+\cos\left(\frac{\pi x}{L}\right)\right], & -L\le x\le L,\\ \\ 0, & \text{otherwise,}\end{cases} \tag{20}$$

where L is the half length of the obstacle.

Using relations (14), (17) and (20), $a(k)$ is obtained as

$$a(k)=\frac{F^2\pi\sin(kL)}{\left(\pi^2-k^2L^2\right)E_1(k)}. \tag{21}$$

Now, using the value of $a(k)$ into the relation (15), the first-order free-surface profile is determined as

$$\eta_1(x)=\frac{\pi F^2}{4L^2}\left[\int_{-\infty}^{\infty}\frac{\sin k(x+L)}{\left(\frac{\pi^2}{L^2}-k^2\right)E_1(k)}dk-\int_{-\infty}^{\infty}\frac{\sin k(x-L)}{\left(\frac{\pi^2}{L^2}-k^2\right)E_1(k)}dk\right]. \tag{22}$$

It can be noticed that the integrals in relation (22) contain simple pole on the real axis at the zero of $E_1(k)$. Hence, these integrals have to be understood as a Cauchy principal value with an indentation below the singularity. The free-surface profile which is given below is thus obtained by using the residue theorem:

$$\eta_1(x)=\begin{cases}\frac{-\pi^2F^2}{L^2}\frac{\sin(k_0x)\sin(k_0L)}{\left(\frac{\pi^2}{L^2}-k_0^2\right)E_1^{'}(k_0)}, & \text{for } x>L,\\ \\ 0 & \text{for } x<-L,\end{cases} \tag{23}$$

where k_0 is the real and positive root of the relation (19).

The relation (23) illustrates that the profile $\eta_1(x)$ is oscillatory in nature, representing a wave with constant amplitude. In addition, it is also noticed that the linearized free-surface possesses a wave train downstream preceded by a wave-free region at the upstream of the obstacle.

IV. COMPUTATIONAL RESULTS AND DISCUSSION

In this section, some of the numerical results which are important for the present study are discussed. For instance, a detail discussion on the real roots (*i.e.*, the wave number) of the dispersion relation

(19) is provided in a tabular form. Also, the effects of several system parameters on the free-surface profile $\eta(x)$ are presented.

The roots of the relation (19) are calculated using Newton's method for different values of the Froude number (F) and are shown in Table 1. From Table 1, it is clear that the dispersion relation (19) has two non-zero real roots. Out of these two real roots, one is positive and another one is negative having same magnitude. This affirms the theoretical observation reported in Section III. In addition, it can also be observed from Table 1 that the magnitude of the wave number decreases, i.e., the wave length increases as the Froude number F increases.

Table 1: Roots of the dispersion relation (19) for D=0.7 and γ =1

Parameter value	F = 0.2	F = 0.3	F = 0.4	F = 0.5	F = 0.6
Real roots	24.99999, -24.99999	11.11111, -11.11111	6.24995, -6.24995	3.99730, -3.99730	2.75541, -2.75541

In Figure 2, the free-surface profile $\eta(x)$ is depicted for two different values of Froude number, F=0.5 and 0.6 with ε = 0.1 and L=1. In this figure, it is noticed that the profile is oscillatory in nature, representing the wave with a constant amplitude. The oscillatory nature is mainly due to the interaction of the fluid with undulated bottom. In addition, the amplitude of the downstream wave increases as the Froude number increases (*refer* to Figure 2). It is well known that the wave number decreases (i.e., wavelength increases) as the speed of the fluid increases. From the relation $F = c/\sqrt{gH}$, we know that the fluid speed increases as the Froude number increase. Hence, the wavelength of the downstream wave increases as the Froude number increases. This phenomenon is also observed in Figure 2.

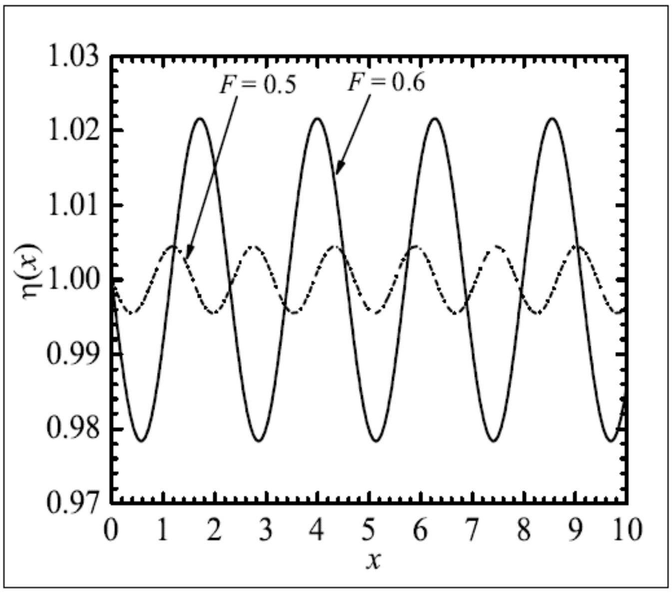

Figure 2: Free-surface profile $\eta(x)$ for ε = 0.1, L=1.

Figure 3 illustrates the effect of the height of the undulated on the free-surface profile. In this figure, the free-surface profile $\eta(x)$ is shown for three different values of the bottom height ε = 0.01, 0.05 and 0.1 with F = 0.6 and L=1. From the physical phenomenon, it is obvious that the amplitude of the downstream wave increases as the height of the bottom increases. This phenomenon is also observed in Figure 3. In this figure, we have kept the Froude number same (i.e., F = 0.6) for each free-surface profile (or downstream wave). And we have noticed that the wavelengths of the downstream waves are same (*refer* Figure 3). This is completely consistent with the phenomenon that the wavelength depends on the Froude number.

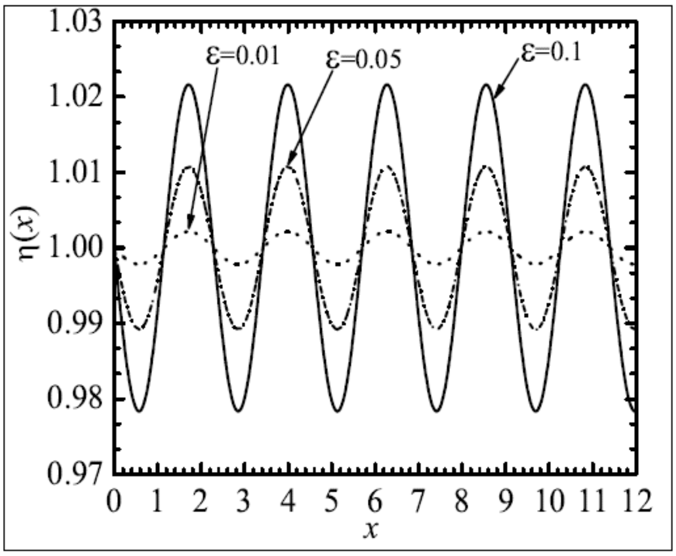

Figure 3: Free-surface profile $\eta(x)$ for *F*=0.6, *L*=1.

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