Qualitative analysis of a prey predator model with Holling type 1 and Holling type 2 functional response and non-linear prey and predator harvesting

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1 Abstract

In this paper, we have proposed a prey predator model to study systematically dynamical properties of the model with non-linear prey and predator harvesting . Here we showed that the system has positivity and uniform boundedness by applying mathematical tools. We also obtained equilibrium points and analyzed bifurcations at these equilibrium points. Here we analyzed existence and stability of intewrior equilibrium point. Saddle-node , transcritical , and hopf bifurcation are shown in this paper by varying values of parameter.Here we analyzed local and global stability and got different conditions to see the system, that is stable or not at equilibrium points. The main purpose of this work is to obtain a complete mathematical analysis for this model.

2 Introduction

A prey predator model was proposed by Volterra under the assumptions: (1) prey grows logistically in absence of predators, (2) when there is no prey , predators die out exponentially and (3) a predator consumes prey biomass as a linear function of prey density. The model with the above assumption is given by

$$\frac{dx}{dt} = rx(1-\frac{x}{k}) - axy \tag{1}$$

$$\frac{dy}{dt} = maxy - dy \tag{2}$$

Where x and y denote the density of prey and predator respectively at time t. The intrinsic growth rate and environmental carrying capacity for prey population are denoted as r and k. The encounter rate at which predators kill prey is a, m is the conversion rate of prey which is eaten by new predators. d is the natural death rate of predators. There are generally three types of harvesting: (1) Constant harvesting where the number of individuals harvested per unit time is constant, (2) Proportional harvesting

$$H(y) = qEy \tag{3}$$

Which means the number of harvested individuals per unit time is proportional to current population. (3) (Holling Type 2) nonlinear harvesting

$$H(y) = \frac{qEy}{m_1E + m_2y} \tag{4}$$

Where q denote the catchability coefficient and E is the effort, m_1 and m_2 are suitable positive constants.
In this section we proposed a prey predator model

$$\begin{aligned}\frac{dx}{dt} &= rx(1 - \frac{x}{k}) - c_1xy - \frac{c_2xy}{a+x} - d_1x^2 - q_1Ex \\ \frac{dy}{dt} &= mc_1xy + \frac{mc_2xy}{a+x} - d_2y^2 - dy - q_2Ey\end{aligned} \tag{5}$$

with positive initial conditions

$$x(0) > 0, y(0) > 0 \tag{6}$$

Here $x(t)$ and $y(t)$ denote prey and predator density at time t.Where r and k are intrinsic growth rate and environmental carrying capacity for prey population respectively. c_1 is the encounter rate at which predators consumes prey, m is the conversion rate of eaten prey into new predators. c_2 is the maximum value of the per capita reduction rate of prey. a measures the extent to which the environment provides protection to prey and predator. d_1, and d_2 are the intraspecific competition for prey and predator respectively. All the parameters are assumed to be positive.

3 Mathematical analysis

3.1 Positivity

Theorem 1. *All solutions* $(x(t), y(t))$ *of system* (9) *with initial condition* (6) *are positive for all* $t \geq 0$.

Proof. From first equation of (9), after integrating it is obtained that

$$x(t) = x(0)\exp[\int_0^t \{r(1 - \frac{x(s)}{k}) - c_1y(s) - \frac{c_2y(s)}{a+x(s)} - d_1x(s) - q_1E\}ds] > 0$$

Since $x(0) > 0$
Similarly by integrating the second equation of (9) we obtained that

$$y(t) = y(0)\exp[\int_0^t \{mc_1x(s) + \frac{mc_2x(s)}{a+x(s)} - d_2y(s) - d - q_2E\}ds] > 0$$

as $y(0) > 0$
This shows the positivity of all solutions of the system. $\square$

3.2 Boundedness

Theorem 2. *All solutions (x(t),y(t)) of system* (9) *with initial condition* (6) *are uniformly bounded.*

Proof. Let us consider the function

$$W(t) = x(t) + \frac{1}{m}y(t)$$

Then after simplification, we get

$$\frac{dW}{dt} + H_1W \leq rx + H_1x + \frac{H_1}{m}y$$

Now considering

$$rx + H_1x + \frac{H_1}{m}y = H_2$$

We get

$$\frac{dW}{dt} + H_1W \leq H_2$$

Therefore

$$0 \leq \lim_{t\to\infty} W(t) \leq \frac{H_2}{H_1}$$

as $t \to \infty$
Hence all solutions of (9), which are initiating from $\mathbb{R}^2_+$ are confined in the region
$R = \{(x,y) \in \mathbb{R}^2_+ : 0 < x(t) + \frac{1}{m}y(t) < H_2 + \phi, for any \phi > 0\}$ Hence uniform boundedness of solutions of the system is proved. $\square$

4 Equilibria

To find the equilibria of the system (9), we have to consider

$$\frac{dx}{dt} = 0 \tag{7}$$

$$\frac{dy}{dt} = 0 \tag{8}$$

By simple calculation we get the axial equilibria of the system (9) as follows:
(1) Trivial or prey-predator free equilibrium point $E_0 = (0, 0)$
(2) Predator free equilibrium point $E_1 = (x_1, 0)$ where

$$x_1 = \frac{r - q_1 E}{\frac{r}{k} + d_1}$$

which exist if $r > q_1 E$
(3) interior equilibrium point $E_*(x_*, y_*)$
where x_* and y_* satisfies the following system of equation:

$$\frac{dx}{dt} = rx(1 - \frac{x}{k}) - c_1 xy - \frac{c_2 xy}{a + x} - d_1 x^2 - q_1 Ex = 0$$
$$\frac{dy}{dt} = mc_1 xy + \frac{mc_2 xy}{a + x} - d_2 y^2 - dy - q_2 Ey = 0$$

4.1 Local stability analysis

The Jacobian matrix for system is
$J = \begin{pmatrix} r - \frac{2rx}{k} - c_1 y - \frac{ac_2 y}{(a+x)^2} - 2d_1 x - q_1 E & -c_1 x - \frac{c_2 x}{a+x} \\ mc_1 y + \frac{amc_2 y}{(a+x)^2} & mc_1 x + \frac{mc_2 x}{a+x} - 2d_2 y - d - q_2 E \end{pmatrix}$
So here
$tr(J) = r - \frac{2rx}{k} - c_1 y - \frac{ac_2 y}{(a+x)^2} - 2d_1 x - q_1 E + mc_1 x + \frac{mc_2 x}{a+x} - 2d_2 y - d - q_2 E$
and
$det(J) = (r - \frac{2rx}{k} - c_1 y - \frac{ac_2 y}{(a+x)^2} - 2d_1 x - q_1 E)(mc_1 x + \frac{mc_2 x}{a+x} - 2d_2 y - d - q_2 E) - (-c_1 x - \frac{c_2 x}{a+x})(mc_1 y + \frac{amc_2 y}{(a+x)^2})$
So if
$|det(J)| < 1$ then the system is dissipative dynamical system and if $|det(J)| = 1$ then the system is conservative dynamical system, and is an undissipated system ottherwise

4.1.1 Stability and dynamic behaviour of $E0$

At $E0(0, 0)$
$J_{E_0} = \begin{pmatrix} r - q_1 E & 0 \\ 0 & -d - q_2 E \end{pmatrix}$ Therefore
$E0(0, 0)$ is
(a) sink if $|r - q_1 E| < 1$ and$| - d - q_2 E| < 1$
(b)source if $|r - q_1 E| > 1$ and$| - d - q_2 E| > 1$
(c) saddle if $|r - q_1 E| > 1$ and $|-d - q_2 E| < 1$, or $|r - q_1 E| < 1$ and $|-d - q_2 E| > 1$
(d) Non-hyperbolic if $|r - q_1 E| = 1$ or $| - d - q_2 E| = 1$

4.1.2 Stability and dynamic behaviour of $E1$

At $E_1(x_1, 0)$

$$J_{E_1} = \begin{pmatrix} r - \frac{2rx_1}{k} - 2d_1x_1 - q_1E & -c_1x_1 - \frac{c_2x_1}{a+x_1} \\ 0 & mc_1x_1 + \frac{mc_2x_1}{a+x_1} - d - q_2E \end{pmatrix}$$

So here we get the two eigen values which are
$r - \frac{2rx_1}{k} - 2d_1x_1 - q_1E$ and $mc_1x_1 + \frac{mc_2x_1}{a+x_1} - d - q_2E$
So $E_1(x_1, 0)$ is a
(a) sink if $|r - \frac{2rx_1}{k} - 2d_1x_1 - q_1E| < 1$ and $|mc_1x_1 + \frac{mc_2x_1}{a+x_1} - d - q_2E| < 1$
(b) Source if $|r - \frac{2rx_1}{k} - 2d_1x_1 - q_1E| > 1$ and $|mc_1x_1 + \frac{mc_2x_1}{a+x_1} - d - q_2E| > 1$
(c) Saddle if $|r - \frac{2rx_1}{k} - 2d_1x_1 - q_1E| > 1$ and $|mc_1x_1 + \frac{mc_2x_1}{a+x_1} - d - q_2E| < 1$
or $|r - \frac{2rx_1}{k} - 2d_1x_1 - q_1E| < 1$ and $|mc_1x_1 + \frac{mc_2x_1}{a+x_1} - d - q_2E| > 1$
(d) Non-hyperbolic if $|r-\frac{2rx_1}{k}-2d_1x_1-q_1E| = 1$ or $|mc_1x_1+\frac{mc_2x_1}{a+x_1}-d-q_2E| = 1$

4.1.3 Dynamical behaviour of the interior equilibrium point $E^*(x^*, y^*)$

At the interior equilibrium point $E^*(x^*, y^*)$
$J_{E_*} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$
Where

$$M_{11} = r - \frac{2rx^*}{k} - c_1y^* - \frac{ac_2y^*}{(a+x^*)^2} - 2d_1x^* - q_1E$$
$$M_{12} = -c_1x^* - \frac{c_2x^*}{a+x^*}$$
$$M_{21} = mc_1y^* + \frac{amc_2y^*}{(a+x^*)^2}$$
$$M_{22} = mc_1x^* + \frac{mc_2x^*}{a+x^*} - 2d_2y^* - d - q_2E$$

Here $T = tr(J_{E^*}) = M_{11} + M_{22}$ and $D = det(J_{E^*}) = M_{11}M_{22} - M_{12}M_{21}$
If $1 - tr + det > 0$, then interior equilibrium point is a
(a)Sink if $1 + tr + det > 0$ and $det < 1$
(b) Source if $1 + tr + det > 0$ and $det > 1$
(c) Saddle if $1 + tr + det < 0$
(d) Non-hyperbolic if $1 + tr + det = 0$ and $tr \neq 0$ or $tr^2 - 4det < 0$ and $det = 1$

4.2 Global Stability Analysis

Theorem 3. *The positive interior equilibrium point is globally asymptotically stable if*

$$\frac{c_2}{(a+x_*)^2} < \frac{r}{y_*k} + \frac{d_1}{y_*} + \frac{d_2}{x_*} \tag{9}$$

Proof. For proving the global stability of positive interior equilibrium point $E_*(x_*, y_*)$ we construct a Lyapunov function $L(x, y) = \frac{1}{xy}$

Clearly $L > 0$ if $x > 0$ and $y > 0$.
Let,

$$h_1(x,y) = rx(1-\frac{x}{k}) - c_1xy - \frac{c_2xy}{a+x} - d_1x^2 - q_1Ex$$
$$h_2(x,y) = mc_1xy + \frac{mc_2xy}{a+x} - d_2y^2 - dy - q_2Ey$$

So,

$$\frac{\partial(h_1L)}{\partial x} + \frac{\partial(h_2L)}{\partial y} = \frac{-r}{yk} + \frac{c_2}{(a+x)^2} - \frac{d_1}{y} - \frac{d_2}{x} \quad (10)$$

So if at $E_*(x_*, y_*)$
$\frac{\partial(h_1L)}{\partial x} + \frac{\partial(h_2L)}{\partial y} < 0$
that is if
$\frac{-r}{y_*k} + \frac{c_2}{(a+x_*)^2} - \frac{d_1}{y_*} - \frac{d_2}{x_*} < 0$
Then $E_*(x_*, y_*)$ is globally asymptotically stable □

4.3 Permanence

Theorem 4. *The system* (9) *is permanent if*
(a) $p_1(r - q_1E) + p_2(-d - q_2E) > 0$
(b)$p_1[r - \frac{rx_1}{k} - d_1x_1 - q_1E] + p_2[mc_1x_1 + \frac{mc_2x_1}{a+x_1} - d - q_2E] > 0$
(c)$p_1[r(1-\frac{x_2}{k}) - c_1y_2 - \frac{c_2y_2}{a+x_2} - d_1x_2 - q_1E] + p_2[mc_1x_2 + \frac{mc_2x_2}{a+x_2} - d_2y_2 - d - q_2E] > 0$

Proof. Let the average Lyapunov function for system (9) be

$$\sigma(x,y) = x_1^p y_2^p \quad (11)$$

Clearly , $\sigma(x,y)$ is a non-negative C^1 function defined in $\mathbb{R}_+^2$ and each p_i is assumed to be positive.Then

$$\psi(x,y) = \frac{\dot{\sigma}(x,y)}{\sigma(x,y)} \quad (12)$$
$$= p_1\frac{\dot{x}}{x} + p_2\frac{\dot{y}}{y} \quad (13)$$
$$= p_1[r(1-\frac{x}{k} - c_1y - \frac{c_2y}{a+x} - d_1x - q_1E] + p_2[mc_1x + \frac{mc_2x}{a+x} - d_2y - d - q_2E] \quad (14)$$

At $E_0(0,0)$

$$\psi(x,y) = p_1(r - q_1E) + p_2(-d - q_2E) \quad (15)$$

At $E_1(x_1, 0)$

$$\Psi(x,y) = p_1[r - \frac{rx_1}{k} - d_1x_1 - q_1E] + p_2[mc_1x_1 + \frac{mc_2x_1}{a+x_1} - d - q_2E] \qquad (16)$$

At $E_2(x_2, y_2)$

$$\psi(x,y) = p_1[r(1 - \frac{x_2}{k}) - c_1y_2 - \frac{c_2y_2}{a+x_2} - d_1x_2 - q_1E] + p_2[mc_1x_2 + \frac{mc_2x_2}{a+x_2} - d_2y_2 - d - q_2E] \quad (17)$$

Therefore if at $E_0(0,0)$, $E_1(x_1, 0)$ and $E_2(x_2, y_2)$
$\psi(x,y) > 0$, that is if

$$p_1(r - q_1E) + p_2(-d - q_2E) > 0 \quad (18)$$

$$p_1[r - \frac{rx_1}{k} - d_1x_1 - q_1E] + p_2[mc_1x_1 + \frac{mc_2x_1}{a+x_1} - d - q_2E] > 0 \quad (19)$$

$$p_1[r(1 - \frac{x_2}{k}) - c_1y_2 - \frac{c_2y_2}{a+x_2} - d_1x_2 - q_1E] + p_2[mc_1x_2 + \frac{mc_2x_2}{a+x_2} - d_2y_2 - d - q_2E] > 0 \quad (20)$$

Then the system is permanent. □

5 Bifurcation

5.1 Transcritical and Saddle node bifurcation

In this subsection, we are interested in transcritical bifurcation of system (9) by using Sotomayor's theorem.

Theorem 5. *(1) System* (9) *undergoes a transcritical bifurcation around* $E_0(0,0)$ *if* $r - q_1E = 0$
(2) System (9) *undergoes transcritical bifurcation around* $E_1(x_1, 0)$ *if* $x_1 = k and$ $x_1 \neq \frac{k}{2}$, *and saddle node bifurcation if* $x_1 \neq k$.

Proof. (1) For proving that the model (9) undergoes a transcritical bifurcation around $E_0(0,0)$, We use Sotomayor's theorem by considering r as the bifurcation parameter.
At $E_0(0,0)$

$$J_{E_0} = \begin{pmatrix} r - q_1E & 0 \\ 0 & -d - q_2E \end{pmatrix}. \qquad (21)$$

According to Sotomayor's theorem at $E_(0,0)$ transcritical bifurcation occurs if one of the eigen values of the jacobian at $E_0(0,0)$ is zero and the other eigen value has negative real part i.e if $r = q_1 E$. Here

$$F_r E_0 = \begin{pmatrix} x - \frac{x^2}{k} \\ 0 \end{pmatrix}$$

So at $E_0(0,0)$

$$F_r E_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So
$W^T F_r E_0 = 0$
Let V and W are eigen vectors coresponding to zero eigen value of $J_(E_0)$ and $J_(E_0)^T$ respectively. After simple calculation we get

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{22}$$

Similarly, we get

$$W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{23}$$

Also after easy calculation, we get

$W^T D F_r E_0 V = 1 \neq 0$ and

$W^T D^2 F_r E_0(V,V) = \frac{-2r}{k} - 2d_1 \neq 0$
So a transcritical bifurcation occurs around $E_(0,0)$.
(2) At $E_1(x_1, 0)$

let V and W are eigen vector coresponding to zero eigen value of $J_(E_1)$ and $J_(E_1)^T$ respectively , where

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{24}$$

Similarly we get

$$W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ -\frac{Q_1}{Q_2} \end{pmatrix} \tag{25}$$

where

$$\begin{aligned} Q_1 &= -c_1 x_1 - \frac{c_2 x_1}{a + x_1} \\ Q_2 &= m c_1 x_1 + \frac{m c_2 x_1}{a + x_1} - d - q_2 E \end{aligned} \tag{26}$$

$$F_rE_1 = \begin{pmatrix} x_1 - \frac{(x_1)^2}{k} \\ 0 \end{pmatrix}$$

and we get

$$\begin{aligned} W^T F_r E_1 &= x_1 - \frac{(x_1)^2}{k} \\ W^T D F_r E_1 V &= 1 - 2\frac{x_1}{k} \\ W^T D^2 F_{E_1}(V,V) &= -2\frac{r}{k} - 2d_1 \end{aligned} \tag{27}$$

Therefore we get if $x_1 = k$ and $x_1 \neq \frac{k}{2}$, then transcritical bifurcation occurs at predator free equilibrium $E_1(x_1, 0)$, and if $x_1 \neq k$ then saddle node bifurcation occurs at $E_1(x_1, 0)$. □

5.2 Existence and stability of Hopf bifurcation

At $E^*(x^*, y^*)$ if
(a) $tr(J_{E^*}) = 0$
(b) $det(J_{E^*}) > 0$
(c) $\frac{d}{dr}tr(J_{E^*}) \neq 0$
then the system undergoes a hopf bifurcation at interior equilibrium point.
So if
(a) $M_{11} + M_{22} = 0$
(b) $M_{11}M_{22} - M_{12}M_{21} > 0$
(c) $x^* \neq \frac{k}{2}$
then the system undergoes a hopf bifurcation. Now if $A > 0$ then the periodic orbit is unstable i.e the bifurcation is subcritical and if $A < 0$ then the periodic orbit is stable i.e the bifurcation is supercritical.
where

$$A = \frac{1}{16}(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) + \frac{1}{16w}[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + g_{yy}f_{yy}]$$

Here

$$A = \frac{1}{16w}[d_1(c_1 + \frac{ac_2}{(a+x)^2}) + d_2(mc_1 + \frac{mac_2}{(a+x)^2})]$$

which implies that $A > 0$.So the hopf bifurcation is subcritical.

6 References