A Distributed Time Delay model with a Single Prey and Two predators

Paparao.A.V [1*]
[1*]Department of Mathematics
JNTU-GV, CEV(A)
Vizianagaram, A.P, India
paparao.alla@gmail.com

G A L Satyavathi[2] ,
[2]Research scholar, A.P India
JNTUK, Kakinada
A.P, India
anuaug15@gmail.com, ,

Sobhan Babu K[3].
[3]Department of Mathematics
JNTUK, UCEN,
A.P, India
ksb.maths@jntukucev.ac.in

ABSTRACT:

In this chapter, we propose a single prey and two predators' mathematical model with distributed time delay. The model consists of a prey (x_1) and two predators (x_2, x_3), surviving on the common prey (x_1). Here two predators are neutral to each other, and all the three species have limited their own natural resources. A Distributed type of delay is included in the interaction prey (x_1) and second predator (x_3). The system is described by system of integro differential equations. The co-existing state is identified and characterizes the local, global stability analysis at this state. The effect of Time delay on the dynamical behaviour of the system is studied using Numerical simulation**:**

Keywords: Prey, Predator, stability analysis, Numerical simulation.
Mathematics Subject Classification:34DXX

I.INTRODUCTION

The relation between prey-predator models is significant in biological relationships. Differential equations play a significant role to establish such relations. The application approach to study the dynamics of the models are by Braun [9] and Simon's [10]. The origin of ecological models using differential equation approach was initiated by Lokta [1] and Volterra [2]. Stability analysis of biological, ecological, epidemical models is briefly discussed by Kapur [3, 4]. May R M [5], Murry [6] and Freed man [7] explained the wide range of ecological models with detailed analysis.

Naturally any biological or ecological phenomenon not only depend on the current values, but also dependent on previous history. The concept of time delay is proposed and introduced to study system dynamics which depend on previous history. The time lags are classified as discrete, continuous, and distributed type. The appropriate time lags for ecological systems are distributed type. Cushing, J.M [11], Norman [12] explains in detail the population dynamics with distributed time lags. The time delays are influence the dynamics of the system and tend to destabilize or stabilizes the system. The systems with delay arguments and the qualitative analysis are widely studied by the authors [13-15]. These lags will change the stable equilibrium to unstable or vice versa. Karuna [16] and Ranjith [17] studied the instability tendencies in HIV and SIR epidemic models. Shiva Reddy [18] discussed the prey-predator dynamics in three species models. The distributed type of delay models in population dynamics are widely studied by Paparao [19-24]. The delay kernels are chosen as exponential type and the dynamics of the models are studied. These delay kernels play switchover behaviour from stable to unstable vice versa. In spite of that a general prey-predator model is taken up for investigation on a three species eco-system consisting of a prey (x_1) and two predators (x_2 & x_3), surviving on the common prey (x_1). Here all the three species have limited their own natural resources.

II. Formation of Mathematical Model

The basic model is with single prey and two predators preying on the same prey species was dealt by Shiva Reddy [18] with exponential growth model. The system dynamics was studied at all possible equilibrium points and shown that the system is both locally and globally asymptotically stable. We proposed the mathematical model with logistic grow type of single prey and two predators. The two predators are generalist type. Paparao et al [25] studied the dynamics of the model and shown that the system is asymptotically stable globally. In spite of that we infuse a distributed time delay in prey-predator model (logistic) in the interaction prey and second predator. The model is characterized by the system of integro differential equations given by

$$\frac{dx_1}{dt} = a_1 x_1 \left[1 - \frac{x_1}{L_1}\right] - \alpha x_1 x_2 - \beta x_1 \int_{-\infty}^{t} w_1 (t-u) x_3(u) du$$
$$\frac{dx_2}{dt} = a_2 x_2 \left[1 - \frac{x_2}{L_2}\right] + \delta x_1(t) x_2(t)$$
$$\frac{dx_3}{dt} = a_3 x_3 \left[1 - \frac{x_3}{L_3}\right] + \varepsilon x_3 \int_{-\infty}^{t} w_2 (t-u) x_1(u) du \qquad (2.1)$$

With the following notations

$x_1(t)$ Prey density, $x_2(t)$ First predator density, $x_3(t)$ second predator density

$a_i (i = 1,2,3)$: Intrinsic growth rates three species

α_{ii} : Inter species competition rate of three species

$\alpha, \beta, \delta, \varepsilon$: Mutual interference strengths of three species

L_i Carrying capacities of three species; $w_1(t-u) \& w_2(t-u)$ are weight kernels.

Let $t - u = z$ and substitute in equation (1.1) becomes

$$\frac{dx_1}{dt} = a_1 x_1 \left[1 - \frac{x_1}{L_1}\right] - \alpha x_1 x_2 - \beta x_1 \int_{0}^{\infty} w_1(z) x_3(t-z) dz$$
$$\frac{dx_2}{dt} = a_2 x_2 \left[1 - \frac{x_2}{L_2}\right] + \delta x_1(t) x_2(t) \qquad (2.2)$$
$$\frac{dx_3}{dt} = a_3 x_3 \left[1 - \frac{x_3}{L_3}\right] + \varepsilon x_3 \int_{0}^{\infty} w_2(z) x_1(t-z) dz$$

Assume the solutions for the above model (2.2) as

$x_1 = A_1 e^{\lambda t}, \quad x_2 = A_2 e^{\lambda t}, \quad x_3 = A_3 e^{\lambda t}$ we get

$$\frac{dx_1}{dt} = a_1 x_1 \left[1 - \frac{x_1}{L_1}\right] - \alpha x_1 x_2 - \beta x_1 x_3 w_1(\lambda)$$
$$\frac{dx_2}{dt} = a_2 x_2 \left[1 - \frac{x_2}{L_2}\right] + \delta x_2 x_1 \qquad (2.3)$$
$$\frac{dx_3}{dt} = a_3 x_3 \left[1 - \frac{x_3}{L_3}\right] + \varepsilon x_1 x_3 w_2(\lambda)$$

Where $w_1(\lambda) = \int_0^\infty w_1(z) e^{-\lambda z} dz$ is the Laplace Transform of $w_1(z)$

and $w_2(\lambda) = \int_0^\infty w_2(z) e^{-\lambda z} dz$ is the Laplace Transform of $w_2(z)$

All the constants are assumed to be positive

From equation (2.3) we can write

$$\frac{1}{x_1}\frac{dx_1}{dt} = a_1 \left[1 - \frac{x_1}{L_1}\right] - \alpha x_2 - \beta x_3 w_1(\lambda)$$
$$\frac{1}{x_2}\frac{dx_2}{dt} = a_2 \left[1 - \frac{x_2}{L_2}\right] + \delta x_1 \qquad (2.4)$$
$$\frac{1}{x_3}\frac{dx_3}{dt} = a_3 \left[1 - \frac{x_3}{L_3}\right] + \varepsilon x_1 w_2(\lambda)$$

From the equation (2.4) **it** is evident that

$$x_1 = x_{10} e^{\int_0^t \left(a_1\left[1-\frac{x_1}{L_1}\right] - \alpha x_2 - \beta x_3 w_1(\lambda)\right) dt} > 0$$
$$x_2 = x_{20} e^{\int_0^t \left(a_2\left[1-\frac{x_2}{L_2}\right] + \delta x_1\right) dt} > 0 \qquad (2.5)$$
$$x_3 = x_{30} e^{\int_0^t \left(a_3\left[1-\frac{x_3}{L_3}\right] + \varepsilon x_1 w_2(\lambda)\right) dt} > 0$$

From the above equation (2.5) the system (2.1) admits positive solutions in R_3^+ .

III. Existence of Equilibrium:

The system (2.1) admits a positive equilibrium point for the co-existence state if the following conditions hold good.

(i) $a_1 > \alpha L_2 + \beta w_1(\lambda) L_3$ (ii) $\varepsilon w_2(\lambda) a_2 = \delta a_3$

For the normal steady state $E(\overline{x_1}, \overline{x_2}, \overline{x_3})$ given by

$$\overline{x_1} = \frac{L_1 a_1 a_3 (a_1 - \alpha L_2 - \beta w_1(\lambda) L_3)}{a_1 a_2 a_3 + a_3 \delta \alpha L_1 L_2 + a_2 \beta \varepsilon w_1(\lambda) w_2(\lambda) L_1 L_3}$$

$$\overline{x_2} = \frac{L_2[a_1 a_3 a_2 + a_1 a_3 \alpha L_1 + \beta w_1(\lambda) L_1 L_3 (\varepsilon w_2(\lambda) a_2 - \delta a_3)}{a_1 a_2 a_3 + a_3 \delta \alpha L_1 L_2 + a_2 \beta \varepsilon K_1(\lambda) K_2(\lambda) L_1 L_3} \quad (3.1)$$

$$\overline{x_3} = \frac{L_3[a_1 a_3 a_2 + a_1 a_2 \varepsilon w_2(\lambda) L_1 + \alpha L_1 L_2 (\delta a_3 - \varepsilon w_2(\lambda) a_2)}{a_1 a_2 a_3 + a_3 \delta \alpha L_1 L_2 + a_2 \beta \varepsilon w_1(\lambda) w_2(\lambda) L_1 L_3}$$

IV. Local Stability Analysis

Theorem 4.1: The system is locally asymptotically stable at co-existing state
Proof: Consider the Jacobean matrix for the system (2.3) is

$$J = \begin{bmatrix} -\frac{a_1 x_1}{L_1} & -\alpha x_1 & -\beta w_1(\lambda) x_1 \\ \delta x_2 & -\frac{a_2 x_2}{L_2} & 0 \\ \varepsilon w_2(\lambda) x_3 & 0 & -\frac{a_3 x_3}{L_3} \end{bmatrix} \quad (4.1.1)$$

With the characteristic equation $\lambda^3 + b_1 \lambda^2 + b_2 \lambda + b_3 = 0$
Where

$$b_1 = \left(\frac{a_1 x_1}{L_1} + \frac{a_2 x_2}{L_2} + \frac{a_3 x_3}{L_3}\right)$$

$$b_2 = \left(\frac{a_1 a_2 x_1 x_2}{L_1 L_2} + \frac{a_1 a_3 x_1 x_3}{L_1 L_3} + \frac{a_2 a_3 x_2 x_3}{L_2 L_3} + \alpha \delta x_1 x_2 + \beta \varepsilon w_1(\lambda) w_2(\lambda) x_1 x_2\right)$$

$$b_3 = x_1 x_2 x_3 \left(\frac{a_1 a_2 a_3}{L_1 L_2 L_3} + \frac{a_3 \delta \alpha}{L_3} + \frac{a_2 \beta \varepsilon w_1(\lambda) w_2(\lambda)}{L_2}\right)$$

Calculate the following determinates

$$D_1 = b_1 = \left(\frac{a_1 x_1}{L_1} + \frac{a_2 x_2}{L_2} + \frac{a_3 x_3}{L_3}\right) > 0$$

$$D_2 = b_1 b_2 - b_3 b_0$$

$$= \left(\frac{a_1 x_1}{L_1} + \frac{a_2 x_2}{L_2} + \frac{a_3 x_3}{L_3}\right)\left(\frac{a_1 a_2 x_1 x_2}{L_1 L_2} + \frac{a_1 a_3 x_1 x_3}{L_1 L_3} + \frac{a_2 a_3 x_2 x_3}{L_2 L_3} + \alpha \delta x_1 x_2 w_1(\lambda) + \beta \varepsilon w_1(\lambda) w_2(\lambda) x_1 x_2\right)$$
$$- x_1 x_2 x_3 \left(\frac{a_1 a_2 a_3}{L_1 L_2 L_3} + \frac{a_3 \delta \alpha}{L_3} + \frac{a_2 \beta \varepsilon w_1(\lambda) w_2(\lambda)}{L_2}\right)$$

$$\left(\begin{array}{l} \frac{a_1^2 a_2 x_1^2 x_2}{L_1^2 L_2} + \frac{a_1^2 a_3 x_1^2 x_3}{L_1^2 L_3} + 2\frac{a_1 a_2 a_3 x_1 x_2 x_3}{L_1 L_2 L_3} + \frac{a_1 a_2^2 x_1 x_2^2}{L_1 L_2^2} + \frac{a_2^2 a_3 x_2^2 x_3}{L_2^2 L_3} + \frac{a_1 a_3^2 x_1 x_3^2}{L_1 L_3^2} + \frac{a_2 a_3^2 x_2 x_3^2}{L_2 L_3^2} \\ + \frac{a_1 \alpha \delta x_1^2 x_2 w_1(\lambda)}{L_1} + \frac{a_2 \alpha \delta x_1 x_2^2 w_1(\lambda)}{L_2} + \frac{a_1 \beta \varepsilon w_1(\lambda) w_2(\lambda) x_{1_1}^2 x_2}{L_1} + \frac{a_2 \beta \varepsilon w_1(\lambda) w_2(\lambda) x_1 x_2^2}{L_2} \end{array}\right)$$

$$D_2 > 0 \ at \ (\overline{x_1}, \overline{x_2}, \overline{x_3})$$

$$D_3 = (b_1 b_2 - b_3 b_0) b_3 = b_3 D_2 > 0$$

Clearly $D_2 > 0$ & $b_3 > 0$ the product is also positive.

Hence $D_3 > 0 \ at \ (\overline{x_1}, \overline{x_2}, \overline{x_3})$

All determinates are positive hence, by using Routh –Hurwitz criteria the co-existing state $E(\overline{x_1}, \overline{x_2}, \overline{x_3})$ is locally asymptotically stable

V. Global Stability

Theorem 5.1: The positive equilibrium point $E(\overline{x_1}, \overline{x_2}, \overline{x_3})$ is globally asymptotically stable

Proof: Let the Lyapunov function be $V(x, y, z) = \left[x_1 - \overline{x_1} - \overline{x_1}\, log\left(\frac{x_1}{\overline{x_1}}\right)\right] + m_1\left[x_2 - \overline{x_2} - \overline{x_2}\, log\left(\frac{x_2}{\overline{x_2}}\right)\right] + m_2\left[x_3 - \overline{x_3} - \overline{x_3}\, log\left(\frac{x_3}{\overline{x_3}}\right)\right]$

(5.1.1)

Clearly $V(\overline{x_1}, \overline{x_2}, \overline{x_3}) = 0 \& V(x_1, x_2, x_3) > 0$

The time derivate of V along the solutions of equations (2.1) is

$$\frac{dV}{dt} = \frac{dx}{dt}\left[1 - \frac{\overline{x_1}}{x_1}\right] + m_1\frac{dy}{dt}\left[1 - \frac{\overline{x_2}}{x_2}\right] + m_2\frac{dz}{dt}\left[1 - \frac{\overline{x_3}}{x_3}\right] \quad (5.1.2)$$

$$= [x_1 - \overline{x_1}]\left[a_1\left(1 - \frac{x_1}{L_1}\right) - \alpha x_2 - \beta\int_{-\infty}^{t} w_1\,(t-u)x_3(u)du\right] + m_1[x_2 - \overline{x_2}]\left[a_2\left(1 - \frac{x_2}{L_2}\right) + \delta x_2\right]$$

$$+ m_2[x_3 - \overline{x_3}]\left[a_3\left(1 - \frac{x_3}{L_1}\right) + \varepsilon\int_{-\infty}^{t} w_2\,(t-u)x_1(u)du\right] \quad (5.1.3)$$

Choose the proper set of values for $a_1, = \frac{a_1\overline{x}}{L_1} + \alpha\overline{x_2} + \beta\int_{-\infty}^{t} w_1\,(t-u)x_3(u)du, a_2 = \frac{a_2\overline{x_2}}{L_2} - \delta\overline{x_2}, a_3 = \frac{a_3\overline{x_3}}{L_3} - \varepsilon\int_{-\infty}^{t} w_2\,(t-u)x_1(u)du$ Then (5.1.3) becomes

$$\frac{dV}{dt} = -\frac{a_1}{L_1}(x_1 - \overline{x_1})^2 - \frac{a_2}{L_2}(x_2 - \overline{x_2})^2 - \frac{a_3}{L_3}(x_3 - \overline{x_3})^2 + (x_1 - \overline{x_1})(x_2 - \overline{x_2})(\delta m_1 - \alpha)$$

choose $m_1 = \frac{\alpha}{\delta}$. $m_2 = 1$

$$\frac{dV}{dt} = -\frac{a_1}{L_1}(x_1 - \overline{x_1})^2 - \frac{a_2}{L_2}(x_2 - \overline{x_2})^2 - \frac{a_3}{L_3}(x_3 - \overline{x_3})^2$$

$$\frac{dV}{dt} = -\left[\frac{a_1}{L_1}(x - \overline{x})^2 + \frac{a_2}{L_2}(y - \overline{y})^2 + \frac{a_3}{L_3}(z - \overline{z})^2\right]$$

Hence $\frac{dV}{dt} \leq 0$

Therefore, the co –existing state $E(\overline{x_1}, \overline{x_2}, \overline{x_3})$ is globally asymptotically stable

Theorem 5.2: The system (1.3) cannot have any periodic orbits in the interior of the quadrant.

Proof: Using Bendixen -Dulac criterion we establish a dulac function $H(x_1, x_2, x_3) = \frac{1}{x_1x_2x_3}$

And define

$$h_1(x_1, x_2, x_3) = a_1x_1\left[1 - \frac{x_1}{L_1}\right] - \alpha x_1x_2 - \beta x_1x_3w_1(\lambda)$$

$$h_2(x_1, x_2, x_3) = a_2x_2\left[1 - \frac{x_2}{L_2}\right] + \delta x_2x_1 \quad (5.2.1)$$

$$h_3(x_1, x_2, x_3) = a_3x_3\left[1 - \frac{x_3}{L_3}\right] + \varepsilon x_1x_3w_2(\lambda)$$

Clearly function $H(x_1, x_2, x_3) = \frac{1}{x_1x_2x_3}$ is a positive (since the population strengths x_1, x_2, x_3 are positive values) in the interior positive octant of $x_1x_2x_3$ space.

Calculate $\Delta(x_1, x_2, x_3)$ which is given by $\frac{\partial(H\,h_1)}{\partial x_1} + \frac{\partial(H\,h_2)}{\partial x_2} + \frac{\partial(H\,h_3)}{\partial x_3}$

$$\frac{\partial}{\partial x}\left(\frac{1}{x_1x_2x_3}\left\{a_1x_1\left[1 - \frac{x_1}{L_1}\right] - \alpha x_1x_2 - \beta x_1x_3w_1(\lambda)\right\}\right) + \frac{\partial}{\partial y}\left(\frac{1}{x_1x_2x_3}\left\{a_2x_2\left[1 - \frac{x_2}{L_2}\right] + \delta x_2x_1\right\}\right) + \frac{\partial}{\partial z}\left(\frac{1}{x_1x_2x_3}\left\{a_3x_3\left[1 - \frac{x_3}{L_3}\right] + \varepsilon x_1x_3w_2(\lambda)\right\}\right)$$

$$= \frac{\partial}{\partial x_1}\left(\frac{1}{x_2x_3}\left\{a_1\left[1 - \frac{x_1}{L_1}\right] - \alpha x_2 - \beta x_3w_1(\lambda)\right\}\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{x_1x_3}\left\{a_2\left[1 - \frac{x_2}{L_2}\right] + \delta x_1\right\}\right) + \frac{\partial}{\partial x_3}\left(\frac{1}{x_1x_2}\left\{a_3\left[1 - \frac{x_3}{L_3}\right] + \varepsilon x_1w_2(\lambda)\right\}\right)$$

$$= -\left(\frac{a_1}{x_2x_3} + \frac{a_2}{x_1x_3} + \frac{a_3}{x_1x_2}\right) < 0$$

This shows that $\Delta(x_1, x_2, x_3) < 0$

Therefore $\Delta(x_1, x_2, x_3)$ does not change the sign and identically zero in the positive quadrant of $x_1x_2x_3$ space. hence the system (1.3) does not produce any closed orbits and periodic oscillation.

VI. Numerical Simulation:

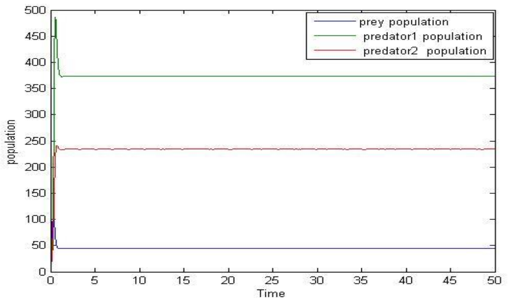
Simulation is carried out for the following set of parametric values with exponential kernel described then the model equation (1.3) with $w_1(\lambda) = w_2(\lambda) = \frac{a}{\lambda+a}$ becomes

$$\frac{dx_1}{dt} = a_1x_1\left[1 - \frac{x_1}{L_1}\right] - \alpha x_1x_2 - \beta x_1x_3\frac{a}{\lambda+a}$$
$$\frac{dx_2}{dt} = a_2x_2\left[1 - \frac{x_2}{L_2}\right] + \delta x_2x_1 \qquad (6.1)$$
$$\frac{dx_3}{dt} = a_3x_3\left[1 - \frac{x_3}{L_3}\right] + \varepsilon x_1x_3\frac{a}{\lambda+a}$$

In each graph, figure (a) represents Time series responses and (b) represents Phase portraits

Example:5.1 $a_1 = 12, a_2 = 3, a_3 = 4, \alpha = 0.01, \beta = 0.1, \delta = 0.02, \varepsilon = 0.05, L_1 = 150, L_2 = 150,$
$L_3 = 150, x_1 = 20, x_2 = 10, x_3 = 10$

with the above parametric values, the simulation is carried out for the system of equations (6.1) without impose delay arguments converging to fixed equilibrium point E (45,374,234) shown in the graphs 6.1(A) & 6.1 (B) respectively



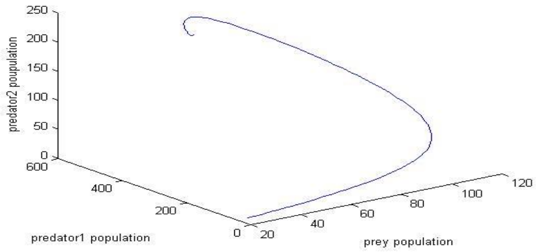
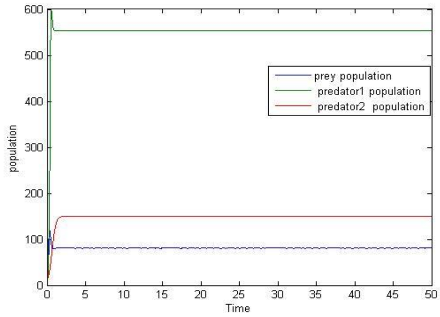


Fig. 6.1(A) **Fig. 6.1 (B)**

Defined as follows with different kernel strengths as

Case (1) for a = 0.01, λ = 5.



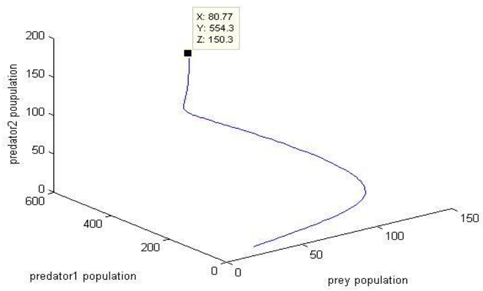


Fig. 6.1.1(A) **Fig. 6.1.2 (B)**

Converging to fixed equilibrium point E(81,554,150)

Case (ii) a = 0.1, λ = 5 converging to fixed equilibrium point E (80,551,152)

Case (iii) a = 0.5, λ = 5 converging to fixed equilibrium point E (78,542,163)

Case (iv) a = 5, λ = 5 converging to fixed equilibrium point E (65,472,211)

Case(v) : a = 50, λ = 50 converging to fixed equilibrium point E (65,472,211)

As on the weight kernel strength increase from $0.01 < a < 50$ and $0.5 < \lambda < 50$ the prey population & predator1 population decreases and predator2 population increase when compared with the dynamics of the system without delay arguments.

VI. Conclusion:

In this work we investigated the stability analysis between one prey and two predators in which the two predators neutral to each other. Distributed type of delay is incorporated in the interaction of prey and second predator species. The mathematical model was described by a couple integro differential equations. Co-existing state is identified, and prove that the system is locally and globally asymptotically stable. The system does not admit any closed orbits and periodic solutions. Numerical simulation is performed with suitable parametric values and exponential type delay kernel and shown that the system is stable for different types of delay kernel strengths and the weight are significant in influencing the population dynamics.

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