Uniqueness and Value Sharing of Meromomorphic Functions on Annuli

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Abstract. In this paper, we study meromorphic functions that share only one value on annuli and prove the following results. Let $f(z)$ and $g(z)$ two non constant meromorphic functions on annli and For $n \geq 11$, if $f^n f'$ and $g^n g'$ share the same nonzero and finite value a with the same multiplicities on annuli, then $f \equiv dg$ or $g = c_1 e^{cz}$ and $f = c_2 e^{-cz}$, where d is an $(n+1)^{th}$ root of unity, c, c_1 and c_2 being constants.

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1 Introduction and Main results

In this paper, a meromorphic function always mean a function which is meromorphic in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$. Let $f(z)$ and $g(z)$ be non constant meromorphic in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$, $a \in \overline{\mathbb{C}}$. We say that f and g share the value a CM if $f(z) - a$ and $g(z) - a$ have the same zeros with the same multiplicities. We shall use standard notations of value distribution theory in annuli, $T_0(R, f)$, $m_0(R, f)$, $N_0(R, f)$, $\overline{N}_0(R, f)$,...([[4]], [[6]]).

In this paper, we shall show that certain types of differential polynomials on annuli when they share only one value.

Theorem 1.1. *Let f and g be two non constant meromorphic fuctions in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$, $n \geq 11$ an integer and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share the value a CM, then either $f \equiv dg$ or $g = c_1 e^{cz}$ and $f = c_2 e^{-cz}$, where c, c_1 and c_2 are constants and satisfy $(c_1 c_2)^{n+1} c^2 = a^{-2}$.*

Remark 1.1. *The following example shows that $a \neq 0$ is necessary. For $f = e^{e^z}$ and $g = e^z$, we see that $f^n f'$ and $g^n g'$ share 0 CM for any integer n, but f and g do not satisfy the conclusion of Theorem 1.1.*

In order to prove the above result, we shall first prove the following two theorems.

Theorem 1.2. *Let f and g be two non constant meromorphic functions in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$, $n \geq 6$. If $f^n f' g^n g' = 1$, then $f \equiv dg$ or $g = c_1 e^{cz}$ and $f = c_2 e^{-cz}$, where c, c_1 and c_2 are constants and $(c_1 c_2)^{n+1} c^2 = -1$.*

Theorem 1.3. *Let f and g be two non constant entire fuctions in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$, $n \geq 1$. If $f^n f' g^n g' = 1$, then $f \equiv dg$ or $g = c_1 e^{cz}$ and $f = c_2 e^{-cz}$, where c, c_1 and c_2 are constants and $(c_1 c_2)^{n+1} c^2 = -1$.*

2 Some Basic Theorems and Lemmas

Theorem 2.A. [7] *(Lemma on the Logarithmic Derivative). Let f be a non-constant meromorphic function in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$, and $\alpha \geq 0$. Then*
1. In the case, $R_0 = +\infty$,

$$m_0\left(R, \frac{f'}{f}\right) = O\left(\log(RT_0(R, f))\right)$$

for $R \in (1, +\infty)$ except for the set $\triangle_R$ such that $\int_{\triangle_R} R^{\alpha-1} dR < +\infty$;
2. In the case, $R_0 < +\infty$,

$$m_0\left(R, \frac{f'}{f}\right) = O\left(\log\left(\frac{T_0(R, f)}{R_0 - R}\right)\right)$$

for $R \in (1, R_0)$ except for the set $\triangle'_R$ such that $\int_{\triangle'_R} \frac{dR}{(R_0 - R^{\alpha-1})} < +\infty$.

Lemma 2.1. *Let f and g be two non constant entire functions in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$. Then for any $1 < R < R_0$, we have*

$$N_0\left(R, \frac{f}{g}\right) - N_0\left(R, \frac{g}{f}\right) = N_0\left(R, f\right) + N_0\left(R, \frac{1}{g}\right) - N_0\left(R, g\right) - N_0\left(R, \frac{1}{f}\right).$$

In studying on uniqueness theorems of meromorphic functions, the following lemma plays an important role.

Lemma 2.2. *Suppose that $f_1(z), f_2(z), \ldots, f_n(z)$ are linearly independent meromorphic functions in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$ satisfying the following identity*

$$\sum_{j=1}^{n} f_j \equiv 1 \tag{2.1}$$

Then for $1 \leq j \leq n$, we have

$$T_0(R, f) \leq \sum_{k=1}^{n} N_0\left(R, \frac{1}{f_k}\right) + N_0\left(R, f_j\right) + N_0\left(R, D\right) - \sum_{k=1}^{n} N_0\left(R, f_k\right) - N_0\left(R, \frac{1}{D}\right) + S(R, f) \tag{2.2}$$

Where D is the Wronskian determinant $W(f_1, f_2, \ldots, f_n)$, $S(r, f) = o(T_0(R, f))$ and $T_0(R, f) = max_{1 \leq k \leq n}\{T_0(R, f_k)\}$, for every R such that $1 < R < R_0$, $R \notin E$ and E is the set of finite linear measure.

First of all, we prove a lemma which is a essentially generalization of Borel's theorem.

Lemma 2.3. *Let $g_j(z)$ (j=1,2,...,n) be an entire functions and $a_j(z)$ (j=0,1,2,...,n) be a meromorphic functions in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$, satisfying $T_0(R, aj) = o\left(\sum_{k=1}^{n} T_0(R, e^{gk})\right)$, for every R such that $1 < R < R_0$, $R \notin E$, $(j = 0, 1, 2, ..., n)$. If*

$$\sum_{j=1}^{n} a_j(z)e^{g_j(z)} \equiv a_0(z) \tag{2.3}$$

then there exists constant c_j (j=1,2,...,n) at least one of them is not zero such that

$$\sum_{j=1}^{n} c_j a_j(z)e^{g_j(z)} \equiv 0. \tag{2.4}$$

Lemma 2.4. *Let f and g be two non constant entire functions in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$. If f and g share 1 CM, one of the following three cases holds:*

(i) $T_0(R, f) \leq \overline{N}_0\,(R, f)+\overline{N}_0^{(2}\,(R, f)+\overline{N}_0\,(R, g)+\overline{N}_0^{(2}\,(R, g)+\overline{N}_0\left(R, \frac{1}{f}\right)$

$$+\overline{N}_0^{(2}\left(R, \frac{1}{f}\right) + \overline{N}_0\left(R, \frac{1}{g}\right) + \overline{N}_0^{(2}\left(R, \frac{1}{g}\right) + S(R, f) + S(R, g)$$

the same inequality holding for $T_0(R, g)$;

(ii) $f \equiv dg$;

(iii) $fg \equiv 1$,

where $\overline{N}_0^{(2}\,(R, 1/f) = \overline{N}_0\left(R, \frac{1}{f}\right) - N_0^{1)}\left(R, \frac{1}{f}\right)$ and $N_0^{1)}\left(R, \frac{1}{f}\right)$ is the counting function of the zeros of f in $\{z : |z| \leq R\}$.

3 Proof of Lemmas

1. Proof of Lemma 2.1: By Jensen's formula in annuli, we have

$$N_0\left(R, \frac{1}{f}\right) - N_0\,(R, f) = \int_0^{2\pi} \log\frac{1}{|f(Re^{i\theta})|}\frac{d\theta}{2\pi} + \int_0^{2\pi} \log|f(Re^{i\theta})|\frac{d\theta}{2\pi} - \int_0^{2\pi} \log|f(e^{i\theta})|\frac{d\theta}{\pi}$$

for every R such that $1 < R < R_0$.

Consider,

$$\begin{aligned}
N_0\left(R, \frac{f}{g}\right) - N_0\left(R, \frac{g}{f}\right) &= \int_0^{2\pi} \log\left|\frac{f(Re^{i\theta})}{g(Re^{i\theta})}\right|\frac{d\theta}{2\pi} + \int_0^{2\pi} \log\left|\frac{g(Re^{i\theta})}{f(Re^{i\theta})}\right|\frac{d\theta}{2\pi} + \int_0^{2\pi} \log\left|\frac{g(e^{i\theta})}{f(e^{i\theta})}\right|\frac{d\theta}{\pi} \\
&= \left\{\int_0^{2\pi} \log\left|\frac{1}{g(Re^{i\theta})}\right|\frac{d\theta}{2\pi} + \int_0^{2\pi} \log|g(Re^{i\theta})|\frac{d\theta}{2\pi} - \int_0^{2\pi} \log|g(e^{i\theta})|\frac{d\theta}{\pi}\right\} \\
&\quad -\left\{\int_0^{2\pi} \log\left|\frac{1}{f(Re^{i\theta})}\right|\frac{d\theta}{2\pi} + \int_0^{2\pi} \log|f(Re^{i\theta})|\frac{d\theta}{2\pi} - \int_0^{2\pi} \log|f(e^{i\theta})|\frac{d\theta}{\pi}\right\} \\
&= (R, f) + N_0\left(R, \frac{1}{g}\right) - N_0\,(R, g) - N_0\left(R, \frac{1}{f}\right).
\end{aligned}$$

This completes the proof of Lemma 2.1

2. Proof of Lemma 2.2: Taking the derivative in both sides of identity (2.1), we get

$$\sum_{j=1}^{n} f_j^{(k)} = 0 \qquad (k = 1, 2, ..., n-1) \tag{3.1}$$

Since $f_1(z), f_2(z), \ldots, f_n(z)$ are linearly independent, we see that $D \not\equiv 0$. (2.1) and (3.1) imply

$$D = D_j \quad (j = 1, 2, ..., n), \tag{3.2}$$

where D_j is algebraic cofactor of f_j in D. Hence

$$f_1 = \frac{\frac{D_1}{f_2 f_3 \cdots f_n}}{\frac{D}{f_1 f_2 \cdots f_n}} = \frac{\Delta_1}{\Delta}, \tag{3.3}$$

where $\Delta = \begin{vmatrix} 1 & 1\cdots & 1 \\ \frac{f_1'}{f_1} & \frac{f_2'}{f_2}\cdots & \frac{f_n'}{f_n} \\ \cdots & \cdots\cdots & \cdots \\ \frac{f_1^{(n-1)}}{f_1} & \frac{f_2^{(n-1)}}{f_2}\cdots & \frac{f_n^{(n-1)}}{f_n} \end{vmatrix}$

and Δ is the algebraic cofactor of the elements at the first column and the first row in Δ. From (3.3), we have

$$\begin{aligned} m_0(R, f_1) &\leq m_0(R, \Delta_1) + m_0\left(R, \frac{1}{\Delta}\right) \\ &\leq m_0(R, \Delta_1) + m_0(R, \Delta) + N_0(R, \Delta) - N_0\left(R, \frac{1}{\Delta}\right) \end{aligned} \tag{3.4}$$

since $\Delta = \frac{D}{f_1 f_2 \cdots f_n}$, which leads to

$$N_0(R,\Delta) - N_0\left(R, \frac{1}{\Delta}\right) = \sum_{k=1}^{n} N_0\left(R, \frac{1}{f_k}\right) - \sum_{k=1}^{n} N_0(R, f_k) + N_0(R, D) - N_0\left(R, \frac{1}{D}\right) \tag{3.5}$$

Note that $m_0\left(R, \frac{f_j^{(k)}}{f_j}\right) = S(R, f_j) = S(R, f)$, (j=1,2,...,n and k=1,2,...,n-1). We have

$$m_0(R, \Delta_1) + m_0(R, \Delta) = S(R, f) \tag{3.6}$$

From (3.4), (3.5) and (3.6), we get

$$\begin{aligned} &T_0(R, f_1) = m_0(R, f_1) + N_0(R, f_1) \\ &\leq \sum_{k=1}^{n} N_0\left(R, \frac{1}{f_k}\right) + N_0(R, f_1) + N_0(R, D) - \sum_{k=1}^{n} N_0(R, f_k) - N_0\left(R, \frac{1}{D}\right) + S(R, f) \end{aligned} \tag{3.7}$$

By the same method, we can prove other results similar to (3.7) for f_j, $(2 \leq j \leq n)$. Hence (2.2) holds.

3. Proof of Lemma 2.3: If $a_0(z) \equiv 0$, Lemma 2.3 is obviously true. In the following, we assume that $a_0(z) \not\equiv 0$. From (2.3), we have $\sum_{j=1}^{n} \frac{a_j(z)}{a_0(z)} e^{g_j(z)} \equiv 1$.
Let $G_j(z) = \frac{a_j(z)}{a_0(z)} e^{g_j(z)}$ (j=1,2,...,n). Then $\sum_{j=1}^{n} \equiv 1$.

If $G_1(z), G_2(z), \ldots, G_n(z)$ are linearly independent, then from Lemma 2.1 we have

$$T_0(R,G) \leq \sum_{j=1}^{n} N_0\left(R, \frac{1}{G_j}\right) + N_0(R,D) + S(R,f), \tag{3.8}$$

where D is Wronskian $W(G_1, G_2, ..., G_n)$, and $S(r,f) = o(T_0(R,f))$ and $T_0(R,f) = max_{1 \leq k \leq n}\{T_0(R,f_k)\}$, as $1 < R < R_0$, $R \notin E$. E is the set of finite linear measure.
Note that

$$\begin{aligned} N_0\left(R, \frac{1}{G_j}\right) &\leq N_0\left(R, \frac{1}{a_j}\right) + N_0(R, a_0) \leq T_0(R, a_j) + T_0(R, a_0) \\ &= o\left(\sum_{k=1}^{n} T_0(R, e^{gk})\right), \quad (1 < R < R_0,\ R \notin E). \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} N_0(R, G_j) &\leq N_0(R, a_j) + N_0\left(R, \frac{1}{a_0}\right) \leq T_0(R, a_j) + T_0(R, a_0) \\ &= o\left(\sum_{k=1}^{n} T_0(R, e^{gk})\right), \quad (1 < R < R_0,\ R \notin E). \end{aligned}$$

We have

$$N_0(R,D) \leq n \sum_{j=1}^{n} N_0(R, G_j) = o\left(\sum_{k=1}^{n} T_0(R, e^{gk})\right), \quad (1 < R < R_0,\ R \notin E). \tag{3.10}$$

From (3.8), (3.9) and (3.10), we get

$$T_0(R, G_j) < o\left(\sum_{k=1}^{n} T_0(R, e^{gk})\right) + S(R,f), \quad (1 < R < R_0,\ R \notin E),\ j = 1, 2, ..., n.$$

On the other hand, we have

$$T_0(R, G_j) = T_0(R, e^{g_k}) + o\left(\sum_{k=1}^{n} T_0(R, e^{gk})\right) \quad (R \notin E),$$

$$S(R,f) = o\left(\sum_{k=1}^{n} T_0(R, e^{gk})\right) \quad (R \notin E).$$

Hence for $j = 1, 2, ..., n$ we have

$$T_0(R, e^{g_k}) = o\left(\sum_{k=1}^{n} T_0(R, e^{gk})\right) \quad (R \notin E).$$

Therefore

$$\sum_{k=1}^{n} T_0\left(R, e^{g_k}\right) = o\left(\sum_{k=1}^{n} T_0(R, e^{gk)})\right) \quad (R \notin E).$$

This is a contradiction. Hence $G_1(z), G_2(z), \ldots, G_n(z)$ are linearly dependent. This completes the proof of Lemma 2.3.

4. Proof of Lemma 2.4: Set

$$\phi = \frac{f''}{f'} - 2\frac{f'}{f-1} - \frac{g''}{g'} + 2\frac{g'}{g-1} \tag{3.11}$$

Since f and g share 1 CM, a simple computation on local expansions shows that $\phi(z_0) = 0$ if z_0 is a simple zero of $f-1$ and $g-1$. Next we consider two cases $\phi \not\equiv 0$ and $\phi \equiv 0$.

If $\phi \not\equiv 0$, then

$$\begin{aligned} N_0^{1)}\left(R, \frac{1}{f-1}\right) &= N_0^{1)}\left(R, \frac{1}{g-1}\right) \le N_0\left(R, \frac{1}{\phi}\right) \\ &\le T_0\left(R, \phi\right) + O(1) \le N_0\left(R, \phi\right) + S(R,f) + S(R,g) \end{aligned} \tag{3.12}$$

where $N_0^{1)}\left(R, 1/f-1\right)$ is the counting function of the simple zeros of $f-1$ in $\{z : |z| \le R\}$. Since f and g share 1 CM, any root of $f(z) = 1$ can not be a pole of $\phi(z)$. In addition, we can easily see from (3.11) that any simple pole of f and g is not a pole of ϕ. Therefore, by (3.11), the poles of ϕ only occur at zeros of f' and g' and the multiple poles of f and g. If $f'(z_0) = f(z_0) = 0$, then z_0 is a multiple zero of f. We denote by $N_0(R, 1/f')$ the counting function of those zeros of f' but not that of $f(f-1)$. From (3.11), (3.12) and the above observation that

$$\begin{aligned} N_0^{1)}\left(R, \frac{1}{f-1}\right) &\le \overline{N}_0^{(2}(R,f) + \overline{N}_0^{(2}(R,g) + N_0\left(R, \frac{1}{f'}\right) + N_0\left(R, \frac{1}{g'}\right) + N_0^{(2}\left(R, \frac{1}{f'}\right) \\ &\quad + N_0^{(2}\left(R, \frac{1}{g'}\right) + S(R,f) + S(R,g) \end{aligned} \tag{3.13}$$

On the otherhand, by the second fundamental theorem we have

$$T_0(R,f) \le \overline{N}_0(R,f) + N_0\left(R, \frac{1}{f}\right) + \overline{N}_0\left(R, \frac{1}{f-1}\right) - \overline{N}_0\left(R, \frac{1}{f'}\right) + S(R,f) \tag{3.14}$$

and by the first fundamental theorem we have

$$\begin{aligned} N_0\left(R, \frac{1}{g'}\right) - N_0\left(R, \frac{1}{g}\right) &= N_0\left(R, \frac{g}{g'}\right) \le T_0\left(R, \frac{g}{g'}\right) + O(1) \\ &= \overline{N}_0(R,g) + \overline{N}_0\left(R, \frac{1}{g}\right) + S(R,g). \end{aligned}$$

This implies that

$$N_0\left(R,\frac{1}{g'}\right)=\overline{N}_0(R,g)+\overline{N}_0\left(R,\frac{1}{g}\right)+S(R,g).$$

It is easy to see from the definition of $N_0^{(0)}(R,1/g')$ that

$$\overline{N}_0^{(0)}\left(R,\frac{1}{g'}\right)+\overline{N}_0^{(2}(R,\frac{1}{g-1})+\overline{N}_0^{(2}\left(R,\frac{1}{g}\right)-\overline{N}_0^{(2}\left(R,\frac{1}{g}\right)\leq N_0\left(R,\frac{1}{g'}\right).$$

The above two inequalities yield

$$\overline{N}_0^{(0)}\left(R,\frac{1}{g'}\right)+\overline{N}_0^{(2}(R,\frac{1}{g-1})\leq N_0\left(R,g\right)+N_0\left(R,\frac{1}{g}\right)+S(R,g). \quad (3.15)$$

Since f and g share 1 CM, we have

$$\overline{N}_0\left(R,\frac{1}{f-1}\right)\leq \overline{N}_0^{1)}\left(R,\frac{1}{f-1}\right)+\overline{N}_0^{(2}\left(R,\frac{1}{g-1}\right). \quad (3.16)$$

Combining (3.13) to (3.16), we obtain (i). If $\phi(z)\equiv 0$, we deduce from (3.11) that

$$f\equiv\frac{Ag+B}{Cg+D}, \quad (3.17)$$

where A, B, C and D are finite complex numbers satisfying $AD-BC\neq 0$. Then, by the first fundamental theorem,

$$T_0(R,f)=T_0(R,g)+S(R,f). \quad (3.18)$$

Next we consider three respective subcases.

Subcase 1. $AC\neq 0$. Then

$$f-\frac{A}{C}=\frac{B-AD/C}{Cg+D}.$$

By the second fundamental theorem, we have

$$\begin{aligned} T_0(R,f) &\leq \overline{N}_0(R,f)+\overline{N}_0\left(R,\frac{1}{f-(A/C)}\right)+\overline{N}_0\left(R,\frac{1}{f}\right)+S(R,f)\\ &= \overline{N}_0(R,f)+\overline{N}_0(R,g)+\overline{N}_0\left(R,\frac{1}{f}\right)+S(R,f). \quad (3.19)\end{aligned}$$

we get (i).

Subcase 2. $A\neq 0$, $C=0$ Then $f\equiv(Ag+B)/D$. If $B\neq 0$, by the second main theorem

$$\begin{aligned} T_0(R,f) &\leq \overline{N}_0(R,f)+\overline{N}_0\left(R,\frac{1}{f}\right)+\overline{N}_0\left(R,\frac{1}{f-(B/D)}\right)+S(R,f)\\ &= \overline{N}_0(R,f)+\overline{N}_0\left(R,\frac{1}{f}\right)+\overline{N}_0\left(R,\frac{1}{g}\right)+S(R,f). \quad (3.20)\end{aligned}$$

we get (i). If $B=0$, then $f\equiv Ag/D$. If $A/D=1$, then $f\equiv g$; this is (ii). If $A/D\neq 1$, then by the assumption that f and g share 1 CM, it is easy to see that $f\neq 1$ and $g\neq 1$, which yields $f\neq 1,\ A/D$. By the second fundamental theorem we have

$$T_0(R,f)\leq \overline{N}_0(R,f)+S(R,f),$$

and (i) follows.

Subcase 3. $A=0,\ C\neq 0$ Then $f\equiv B/(Cg+D)$. if $D\neq 0$, by the second fundamental theorem we have

$$\begin{aligned} T_0(R,f) &\leq \overline{N}_0(R,f)+\overline{N}_0\left(R,\frac{1}{f}\right)+\overline{N}_0\left(R,\frac{1}{f-(B/D)}\right)+S(R,f) \\ &= \overline{N}_0(R,f)+\overline{N}_0\left(R,\frac{1}{f}\right)+\overline{N}_0\left(R,\frac{1}{g}\right)+S(R,f). \end{aligned} \tag{3.21}$$

we get (i). If $D=0$, then $f\equiv B/Cg$. If $B/C=1$, then $fg\equiv 1$ and we obtain (iii). If $B/C\neq 1$, by the assumption that f and g share 1 CM, we have $f\neq 1,\ B/C$. By the second fundamental theorem we get

$$T_0(R,f)\leq \overline{N}_0(R,f)+S(R,f).$$

This implies (i). Thus the proof of Lemma 2.4 is complete.

4 Proof of Theorems

1. **Proof of Theorem 1.2:** We prove the theorem step by step as follows.
   Step 1. We prove that

$$f\neq 0,\qquad g\neq 0. \tag{4.1}$$

In fact, suppose that f has a zero z_0 with order m. Then z_0 is a pole of g (with order p, say) by

$$f^nf'g^ng'=1. \tag{4.2}$$

Thus, $nm+m-1=np+p+1$, i.e.,$(m-p)(n+1)=2$. This impossible since $n\geq 6$ and $m,\ p$ are integers.

Step 2. We claim that

$$N_0(R,f)+N_0(R,g)\leq 2m_0\left(R,\frac{1}{fg}\right)+O(1). \tag{4.3}$$

By step 1 and (4.2) we deduce that

$$(n+1)N_0(R,g)+\overline{N}_0(R,g)=N_0\left(R,\frac{1}{f'}\right). \tag{4.4}$$

From Lemma 2.1 we have

$$\begin{aligned} N_0\left(R,\frac{f}{f'}\right)-N_0\left(R,\frac{f'}{f}\right) &= N_0\left(R,f\right)+N_0\left(R,\frac{1}{f'}\right)-N_0\left(R,f'\right)-N_0\left(R,\frac{1}{f}\right) \\ &= N_0\left(R,\frac{1}{f'}\right)-\overline{N}_0\left(R,f\right). \end{aligned}$$

By the first fundamental theorem, the left side is $m_0(R,f'/f)-m_0(R,f/f')+O(1)$, so we have

$$N_0\left(R,\frac{1}{f'}\right)=\overline{N}_0\left(R,f\right)+m_0\left(R,\frac{f}{f'}\right)-m_0\left(R,\frac{f'}{f}\right)+O(1). \tag{4.5}$$

Now we rewrite (4.2) in the form $g'/g=(f'/f)(1/fg)^{n+1}$. Then

$$m_0\left(R,\frac{f}{f'}\right)\geq m_0\left(R,\frac{g'}{g}\right)-(n+1)m_0\left(R,\frac{1}{fg}\right)-O(1).$$

combining this, (4.4) and (4.5), we get

$$(n+1)N_0\left(R,g\right)+\overline{N}_0\left(R,g\right)\leq\overline{N}_0\left(R,f\right)+m_0\left(R,\frac{f'}{f}\right)-m_0\left(R,\frac{g'}{g}\right)+(n+1)m_0\left(R,\frac{1}{fg}\right)+O(1).$$

By symmetry,

$$(n+1)N_0\left(R,f\right)+\overline{N}_0\left(R,f\right)\leq\overline{N}_0\left(R,g\right)+m_0\left(R,\frac{g'}{g}\right)-m_0\left(R,\frac{f'}{f}\right)+(n+1)m_0\left(R,\frac{1}{fg}\right)+O(1).$$

By adding above two inequalities we obtain (4.3).

Step 3. We prove that fg is constant. Let $h=1/fg$. Then h is entire by Step 1, and (4.2) can be written as

$$\left(\frac{g'}{g}+\frac{1}{2}\frac{h'}{h}\right)^2=\frac{1}{4}\left(\frac{h'}{h}\right)^2-h^{n+1}.$$

Let

$$\alpha=\frac{g'}{g}+\frac{1}{2}\frac{h'}{h}$$

The above equation becomes

$$\alpha^2=\frac{1}{4}\left(\frac{h'}{h}\right)^2-h^{n+1}. \tag{4.6}$$

If $\alpha\equiv 0$, then $h^{n+1}=\frac{1}{2}\left(h'/h\right)^2$. Combining this with Step 1 we obtain $T_0(R,h)=m_0(R,h)=S(R,h)$; thus h is a constant. Next we assume that $\alpha\not\equiv 0$. Differentiating (4.6) yields

$$2\alpha\alpha'=\frac{1}{2}\frac{h'}{h}\left(\frac{h'}{h}\right)'-(n+1)h'h^n.$$

From this and (4.6) it follows that

$$h^{n+1}\left((n+1)\frac{h'}{h}-2\frac{\alpha'}{\alpha}\right)=\frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)'-\frac{\alpha'}{\alpha}\frac{h'}{h}\right) \tag{4.7}$$

If $(n+1)\frac{h'}{h}-2\frac{\alpha'}{\alpha}\equiv 0$, then there exists a constant c such that $\alpha^2=ch^{n+1}$. This and (4.6) give

$$(c+1)h^{n+1}=\frac{1}{4}\left(\frac{h'}{h}\right)^2.$$

If $c=-1$, then $h'\equiv 0$, and so h is constant. If $c\neq -1$, we have $T_0(R,h)=S(R,h)$, and h is constant. Next we suppose that

$$(n+1)\frac{h'}{h}-2\frac{\alpha'}{\alpha}\not\equiv o.$$

Then, by (4.7) and the fact that h is entire,

$$\begin{aligned}(n+1)T_0(R,h) &= (n+1)m_0(R,h)\\ &\le m_0\left(R,h^{n+1}\left((n+1)\frac{h'}{h}-2\frac{\alpha'}{\alpha}\right)\right)+m_0\left(R,\frac{1}{(n+1)h'/h-2\alpha'/\alpha}\right)+O(1)\\ &\le m_0\left(R,\frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)'-\frac{h'}{h}\right)\right)+T_0\left(R,(n+1)\frac{h'}{h}-2\frac{\alpha'}{\alpha}\right)\\ &\le \overline{N}_0(R,f)+\overline{N}_0(R,g)+\overline{N}_0\left(R,\frac{1}{\alpha}\right)+S(R,h)+S(R,\alpha).\end{aligned}$$

Now by (4.6) and (4.3) we have

$$T_0(R,\alpha)\le\frac{1}{2}(n+3)T_0(R,h)+S(R,h),$$

and

$$N_0(R,f)+N_0(R,g)\le 2m_0(R,h)+O(1).$$

Combining the above three inequalities we obtain

$$\frac{1}{2}(n-5)T_0(R,h)\le S(R,h).$$

Thus h must be a constant.

Step 4. We prove our conclusion. By Step 3, h is constant. Then, by (4.2),

$$\frac{g'}{g}=c,\qquad c=ih^{(n+1)/2.}$$

Thus

$$g(z)=c_1e^{cz},\qquad f=c_2e^{-cz}$$

where c, c_1 and c_2 are constants and satisfy $(c_1c_2)^{n+1}c^2=-1$ by (4.2). This completes the proof of the theorem.

2. Proof of Theorem 1.3: From

$$f^n f' g^n g' = 1$$

and the assumption that f and g are entire we immediately see that f and g have no zeros. Thus there exists two entire functions $\alpha(z)$ and $\beta(z)$ such that

$$f(z) = e^{\alpha(z)}, \qquad g(z) = e^{\beta(z)}.$$

Inserting these in the above equality, we get

$$\alpha' \beta' e^{(n+1)(\alpha+\beta)} \equiv 1.$$

Thus α' and β' have no zeros and we may set

$$\alpha' = e^{\delta(z)}, \qquad \beta' = e^{\gamma(z)}.$$

Differentiating this gives

$$(n+1)(e^{\delta} + e^{\gamma}) + \delta' + \gamma' \equiv 0.$$

By Lemma 2.3, $\delta = \gamma + (2m+1)\pi i$ for some integer m. Inserting this in the above equality we deduce that $\delta' \equiv \gamma' \equiv 0$, and so δ and γ are constants, i.e., α' and β' are constants. From this we can easily obtain the desired result.

3. Proof of Theorem 1.1: Let $F = f^{n+1}/a(n+1)$ and $G = g^{n+1}/a(n+1)$. Then condition that $f^n f'$ and $g^n g'$ share the value a CM implies that F' and G' share the value 1 CM. Obviously,

$$\begin{aligned} N_0(R, F') &= (n+1)N_0(R, f) + \overline{N}_0(R, f), \\ N_0(R, G') &= (n+1)N_0(R, g) + \overline{N}_0(R, g), \qquad (4.8) \\ \overline{N}_0(R, F') &= \overline{N}_0^{(2}(R, F') = \overline{N}_0(R, f) \le \frac{1}{n+2} T_0(R, F') + O(1), \qquad (4.9) \end{aligned}$$

$$\begin{aligned} \overline{N}_0\left(R, \frac{1}{F'}\right) + \overline{N}_0^{(2}\left(R, \frac{1}{F'}\right) &= 2\overline{N}_0\left(R, \frac{1}{f}\right) + \overline{N}_0\left(R, \frac{1}{f'}\right) + \overline{N}_0^{(2}\left(R, \frac{1}{f'}\right) \\ &\le 2\overline{N}_0\left(R, \frac{1}{f}\right) + \overline{N}_0\left(R, \frac{1}{f'}\right) \qquad (4.10) \\ &\le 2T_0(R, f) + \overline{N}_0\left(R, \frac{1}{f'}\right) + O(1). \end{aligned}$$

Since

$$\begin{aligned} n\,m_0(R,f) &= m_0\left(R, a\frac{F'}{f'}\right) \le m_0\left(R,F'\right) + m_0\left(R,\frac{1}{f'}\right) + O(1) \\ &= m_0\left(R,F'\right) + T_0(R,f) - N_0\left(R,\frac{1}{f'}\right) + O(1) \\ &\le m_0\left(R,F'\right) + T_0(R,f) + \overline{N}_0(R,f) - N_0\left(R,\frac{1}{f'}\right) + m_0\left(R,\frac{f'}{f}\right) + O(1) \\ &\le m_0\left(R,F'\right) + T_0(R,f) + \overline{N}_0(R,f) - N_0\left(R,\frac{1}{f'}\right) + m_0\left(R,\frac{F'}{F}\right) + O(1), \end{aligned}$$

it follows from this,(4.8), and Theorem 2.A that

$$(n-1)T_0(R,f) \le T_0(R,F') - N_0(R,f) - N_0\left(R,\frac{1}{f'}\right) + S(R,F').$$

This and Theorem 2.A imply that

$$\begin{aligned} 2T_0(R,f) + N_0\left(R,\frac{1}{f'}\right) &= \frac{2}{n-1}\left\{(n-1)T_0(R,f) + N_0\left(R,\frac{1}{f'}\right)\right\} + \frac{n-3}{n-1}N_0\left(R,\frac{1}{f'}\right) \\ &\le \frac{2}{n-1}\left\{T_0(R,F') + N_0\left(R,f\right)\right\} + \frac{n-3}{n-1}\left\{T_0(R,f) + \overline{N}_0\left(R,f\right)\right\} + m_0\left(R,\frac{f'}{f}\right) + O(1) \\ &\le \left(\frac{2}{n-1} + \frac{n-3}{(n-1)^2}\right)T_0(R,F') + \left(\frac{n-5}{n-1} + \frac{n-3}{(n-1)^2}\right)N_0\left(R,f\right) + S(R,F'). \end{aligned}$$

combining this (4.9), and (4.10), we obtain

$$\overline{N}_0\left(R,\frac{1}{F'}\right) + \overline{N}_0^{(2}\left(R,\frac{1}{F'}\right) \le \frac{4n^2-6n-2}{(n-1)^2(n+2)}T_0(R,F') + S(R,F'). \quad (4.11)$$

We similarly derive for G' that

$$\overline{N}_0(R,G') = \overline{N}_0^{(2}(R,G') = \overline{N}_0(R,g) \le \frac{1}{n+2}T_0(R,G') + S(R,G'), \quad (4.12)$$

$$\overline{N}_0\left(R,\frac{1}{G'}\right) + \overline{N}_0^{(2}\left(R,\frac{1}{G'}\right) \le \frac{4n^2-6n-2}{(n-1)^2(n+2)}T_0(R,G') + S(R,G'). \quad (4.13)$$

Without loss of generality, we suppose that there exists a set $I \subset [0,\infty)$ such that $T_0(R,G') \le T_0(R,F')$. Next we apply Lemma 2.4 to F' and G', it follows that there are three cases to be considered.

Case (i).

$$\begin{aligned} T_0(R,F') \le\ & \overline{N}_0(R,F') + \overline{N}_0^{(2}(R,F') + \overline{N}_0(R,G') + \overline{N}_0^{(2}(R,G') + \overline{N}_0\left(R,\frac{1}{F'}\right) \\ & + \overline{N}_0^{(2}\left(R,\frac{1}{F'}\right) + \overline{N}_0\left(R,\frac{1}{G'}\right) + \overline{N}_0^{(2}\left(R,\frac{1}{G'}\right) + S(R,F') + S(R,G'). \end{aligned}$$

Setting (4.9), (4.11), (4.12), and (4.13) into the above inequality and keeping in mind that $T_0(R, G') \leq T_0(R, F')$, we get

$$\frac{n^3 - 12n^2 + 17n + 2}{(n+1)^2(n+2)} T_0(R, F') \leq S(R, F'). \tag{4.14}$$

We denote by $p(n)$ the numerator of the coefficient on the left hand side above. Then $p'(n) = 3n^2 - 24n + 17 > 0$ for $n \leq 8$. Note that $p(11) = 68$; thus $p(n)$ is positive for $n \leq 11$. It follows from (4.14) that F' must be rational function. But then, by the above derivatives, $S(R, F') = O(1)$. Using (4.14) again, F' must be a constant, which is impossible.

Case (ii). $F' = G'$. Then we deduce that $f^{n+1} = g^{n+1} + c$ ($c \in \mathbb{C}$). Let $f = hg$, and we have

$$(h^{n+1} - 1)g^{n+1} = c. \tag{4.15}$$

If $h^{n+1} \equiv 1$, then h is $(n+1)^{th}$ unit root and we obtain the desired result. If $h^{n+1} \not\equiv 1$, then by (4.15),

$$g^{n+1} = \frac{c}{h^{n+1} - 1.}$$

Thus h is not constant. We write this in the form

$$g^{n+1} = \frac{c}{(h - u_1) \ldots (h - u_{n+1})},$$

where $u_1, \ldots, u_{n+1}$ are different $(n+1)^{th}$ roots of unity. Thus h has at least $n + 1(\geq 14)$ multiple values. However, from Nevanlinna's second fundamental theorem we know that h has at most 4 multiple values, a contradiction.

Case (iii). $F'G' \equiv 1$, i.e., $a^{-2}f^n f' g^n g' \equiv 1$. Let $\widehat{f} = a^{-1/(n+1)f}$ and $\widehat{g} = a^{-1/(n+1)g}$. Then $\widehat{f^n}\, f' \widehat{g^n}\, g' = 1$. The conclusion follows follows from Theorem 1.2.

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