A Survey Study of Some Graph Labeling Techniques.

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Abstract

Graph theory is a well-known branch of Mathematical sciences (Arithmetic). Mathematics gives the same name to different things. But graph theory creates a graph labeling techniques in which different labeling are given to same graphical structures. Graph is a mathematical structure describing dots, curves, bars, or traces.In Graph labeling natural numbers including zero are used to label the nodes or edges, or all , which fulfill some mathematical constraints. Graphs are used to describe various mathematical models in Operations Research, control, and Engineering, in studies domains of laptop technological know-how which include statistics mining, photograph segmentation, clustering, photograph capturing, networking that's used in structural fashions, and so forth. Graph labeling has many applications in the social community, verbal exchange (communication) community, circuit design, Database management, coding principle, radar, astronomy, and X-ray crystallography. Depending on the trouble scenario a type of graph is used for representing the hassle (problem) and by way of applying appropriate graph labeling techniques the hassle may be solved. Graph labeling is a flourishing as well as application-oriented area of research in Mathematics.
In this chapter, we discuss the different graph labeling techniques related to different graphs.

Keywords- Graph, Graph labeling, Cordial, Magic, Mean, Radio, Power mean, Permutation, combination.

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1 Introduction

Graph concept has been invented in 1735 with the Koinsberg Seven Bridge problem. The phrase graph is derived from the Greek word 'graphein'. A "graph" in this paper consists of "vertices" or"nodes" and lines called edges that connect with vertices. Graph labeling have been introduced within the mid 1960's by Alexander Rosa. Over the past six decades,the research in graph labeling developed very fast. Graph labeling is a flourishing as well as an application-oriented area of research in the theory of Graphs. To date, more than 200 types of graph labeling techniques had been introduced. In this chapter we try to collect some important graph labeling techniques with suitable graphs. Here we consider only simple, finite, connected and non-directed graphs having countable number of edges and vertices. Here we consider the terminology and symbols of graph from Harary's 'Graph Theory'.

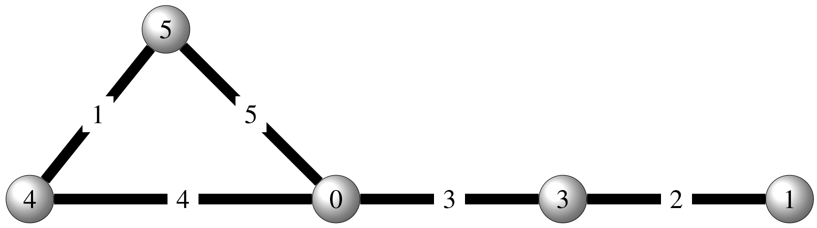
1.1 Graph :

Definition: 1. *A linear graph(simple graph) $G = (V, E, F(G))$,consisting of two sets viz: $V = \{v_1, v_2, --------\}$ whose elements are known as set of vertices, and a set $E = \{e_1, e_2, -----\}$ whose elements are known as edges. The incidence function $F(G)$ which maps every element of E i.e.$e = xy$ to two elements x and y of V.*

Order of a graph: The total count of a vertex set V is called order of a graph.
Size of a graph : The total count of an edge set E is called size of a graph.
Degree of a vertex: The total number of edges incident on a vertex is called a degree of a vertex .
Degree of a graph: The sum of all degrees of each node(vertex) in a graph G is known as degree of a graph.

1.2 Graph Labeling :

Definition: 2. *In a Graph, an allotment of positive integers including zero to the vertices or positive integers to the edges, or both, subject to some mathematical criterion is known as Graph labeling.*

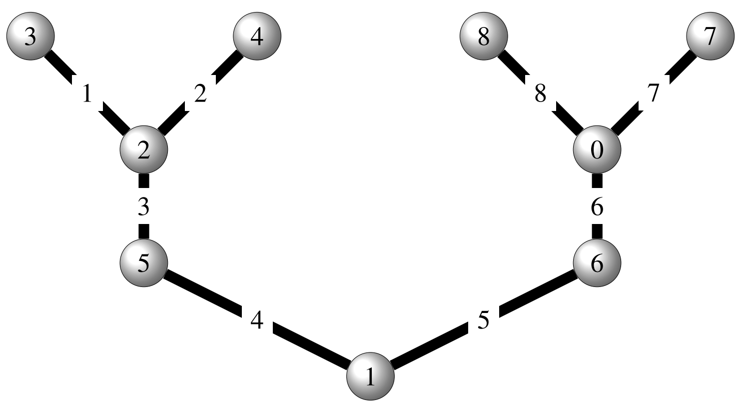


1.3 Different Graph Labeling Techniques :

1.3.1 Graceful Labeling:

Definition: 3. *A graceful labeling of a simple connected graph $G = (V,E)$ with p vertices and q edges is a particular graph labeling in which the vertices are labeled with a subset of distinct integers from 0 to q and the graph edges are labeled with the positive differences between vertex label values. The labeling for edges used are from 1 to q both must be included, the labeling is a graceful labeling and the graph is known as graceful graph.*

Exa. Banana tree ,trees with vertices ≤ 35,Caterpillars,bananas etc. are always graceful.

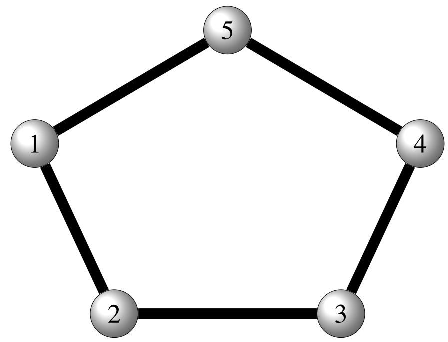


$B_{2,4}$

Examples of graph having Graceful Labeling.

- All symmetrical trees
- Trees with vertices ≤ 35.
- Caterpillars, banana trees etc.

Non-graceful graph: A simple graph G in which we can not make graceful labeling is said to be non-graceful.



$5-cycle$

Result

In graceful graph with q (q is positive integer) edges and $q+1$ vertices the following holds:

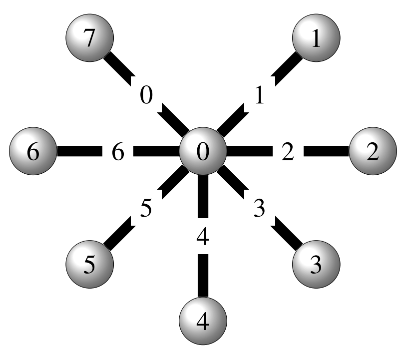
- We can draw distinct graceful labeling for the same graph i.e. it is not unique.
- The vertices labeled with 0 and q are always in neighbouring.
- If the graph has q edges then each graceful labeling must contains vertex label as q.
- Graceful labeling graph may contains a triangle graph.
- The complementarity property for graceful labeling is satisfied. That is for a given graph with graceful labeling if we swap every vertex label q with $q-k$, the resulting labeling is also graceful since the edge labels will not have changed the extreme vertices of an edge. The new labeling for a and b now are $q-a$ and $q-b$.

1.3.2 Harmonious labeling:

Graham and Sloane [33]

Definition: 4. *In a graph $G=(V(G),E(G))$ with q edges, a function f is called as harmonious labeling of graph G if there exists an injective function f from vertex set $V(G)$ to $E(G)$ and the induced function f^* from $V(G)$ to $E(G)$ defined by $f^*(e=xy)=(f(x)+f(y))(modq)$ is both one-one and onto.*
A graph having harmonious labeling is called a harmonious graph.

Exa. $K_{1,7}$



$K_{1,7}$

Examples of graph having Harmonious Labeling.

- Cycle $C_n(n \geq 3)$ has harmonious labeling if n is an odd number and vice versa.
- Ladder graph $L_n, n \neq= 2$.
- F_n, Friendship graph excluding n is congruent to 2(mod4).
- The fan graph f_n .

- The graph g_n for $(n \geq 2)$.

Result

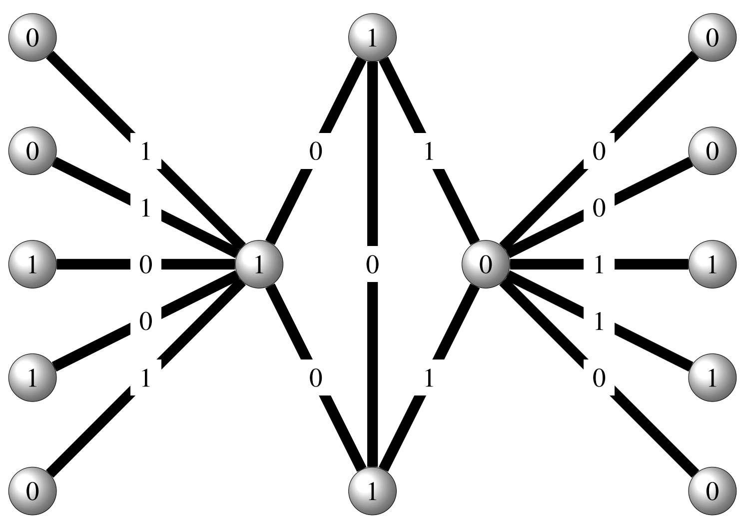
- Harmonious labeling is non distinctive.
- If g is harmonious labeling of a graph then $xg(a)+y$ is harmonious labeling of a graph G , here x is an invertible element of set q and y is any arbirtrary element of q. (i.e. Z_q).
- We can label 0 to any vertex in a graph.
- All trees are harmonious. non-distinctive
- Complete graph C_n is harmonious iff $n \leq 4$.
- The Peterson graph is Harmonious.
- Wheel graph $W_n = C_n + K_1$ is harmonious.
- $K_n^{(2)}$ has harmonious labeling for n=4. $K_n^{(2)}$ has not harmonious labeling for $n = 6$ or when n is an odd number. If $n = 4$, $K_n^{(2)}$ is harmonious, if f or $n = 6$ it is not harmonious..

1.3.3 Cordial Labeling :

Cahit [9]

Definition: 5. *Let $G = G(V,E)$ be a connected graph with n vertices m edges.A graph is called cordial if it is possible to label its vertices with $0's$ and $1's$ such that the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1 and similarly the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1.*

Exa. Cordial labeling graph of Jelly Fish $J_{5,5}$



$J_{5,5}$

Examples of standard graph having Cordial Labeling.

- All trees.
- Complete graph K_n iff n is less than or equal to 3.
- Complete bipartite graph $K_{m,n}$ where m and n are natural numbers.
- Friendship graph $C_3^{(s)}$ iff s is not congruent to 2 modulo 4.
- All fan graph f_n.

- The wheel graph W_n if and only if except n is congruent to 3 modulo 4.
- mK_n for particular values of m and n.
- Every Skolem-graceful graph is cordial.

Types of Cordial Labeling.

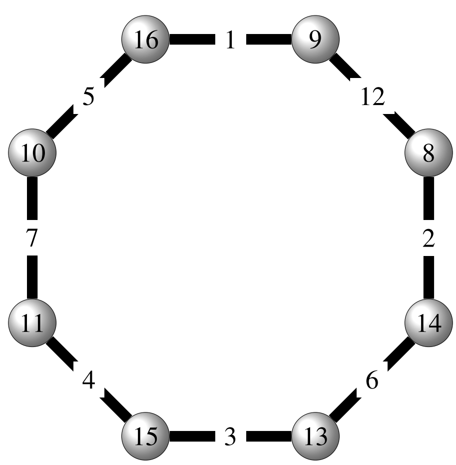
- Difference cordial graph.
- Edge product cordial graph.
- Prime cordial graph.
- Planar grid cordial graph.
- Context of Duplication cordial graphs.
- Second order cordial labeling graph.
- SET cordial graph.
- Integer cordial graph.
- Signed product cordial graph.
- Mean cordial graph.
- Harmonic Mean cordial graph.

1.3.4 Magic labeling of a graph:

Using the magic squares concept in number theory , magic labeling was discovered by Sedlacek in 1963 [41]. **a)Vertex magic labeling of a graph(VML):**

Definition: 6. *Let $G = G(V,E)$ be a simple, connected graph having n vertices m edges.A graph G is said to be vertex magic if there exists a bijective function $F : V \bigcup E \rightarrow \{1,2,...,m+n\}$, such that for all vertices of G, the sum of label on a vertex and the labels of its incident edges is constant such a bijection is called vertex magic labeling of G.*
A graph that admits vertex magic labeling is called a vertex magic graph.

Exa.cycle C_8 having VML with magic constant K=22.

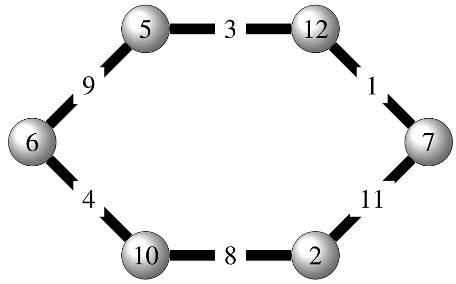


Cycle C_8

b) Edge magic labeling of a graph

Definition: 7. *Let $G = G(V,E)$ be a simple, connected graph having n vertices and m edges.A graph G is said to be edge magic if there is one to one and onto function $F: V \bigcup E \rightarrow \{1,2,...,m+n\}$, such that $F(u)+F(v)+F(uv)$ is constant for all edges uv.*
A graph is an edge magic graph if it has an edge magic labeling .
If the sum of all edges incident on a vertex v is the same for all vertices v then such a connected graph is said to be semi-magic .

Edge magic labeling of cycle graph C_6 with magic constant K=20.



Cycle C_6

Examples of a graph having Magic Labeling.

- For $p = 2$ and p greater than or equal to 5, K_p , the complete graph on p vertices.
- Complete bipartite graph $K_{p,p}$ for all p greater than or equal to 3.
- Fan graph f_n iff n=odd and n greater than or equal to 3.
- The wheel graph on p vertices i.e. W_p for $p \geq 4$.
- A connected (m,n) graph iff $5m/4 < n \leq m(m-1)/2$.

Types of Magic Labeling.

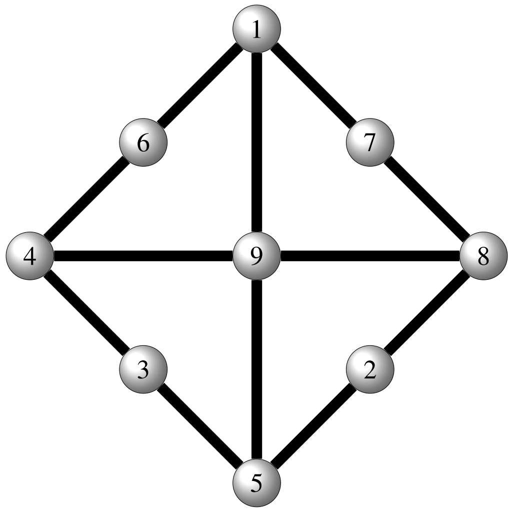
- Semi-Magic
- Super-Magic
- Anti-magic
- Prime-magic
- H-magic
- Sigma labeling/ Distance-magic labeling

1.3.5 Radio labeling:

Chartrand, Erwin, Zhang, and Harary [15]
Radio labeling of graph is used for synchronization of channel allotments to FM radio stations.

Definition: 8. *Let $G = G(V,E)$ be a connected graph with n vertices and m edges.A radio labeling of a connected graph G is an injective function c from the vertex set V to the set of natural numbers such that $d(u,v)+|c(u)-c(v)| \geq 1+diam(G)$ for every two distinct vertices u and v of G.*

The radio number denoted by $rn(c)$ is the greatest number alloted by radio labeling c to any vertex of G by radio labeling c.
The smallest value of radio numbers $rn(c)$ taken from radio labeling c of G is the radio number of $G$and is denoted by $rn(G)$.



Cycle C_8

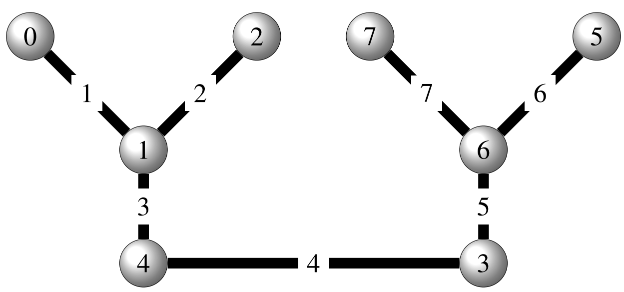
Examples of Radio Labeling graph.

- The $rn(G)$ of a sunflower graph SF_n is equal to its order.
- The $rn(G)$ of a Helm graph H_n is an odd no. i.e.$2n+1$.
- The $rn(G)$ of a gear graph G_n is also an odd no. i.e.$2n+1$.

1.3.6 Mean labeling of graphs:

Somasundaram and Ponraj [40].

Definition: 9. *Let $G = G(V,E)$ be a connected graph with n vertices and m edges.A graph G is said to be a mean graph if there is an injective mapping M from the vertices of G to $\{0,1,2,-----,m\}$ such that when each edge (u,v) is labeled with $(M(u)+M(v))/2$ if $M(u)+M(v)$ is even and $(M(u)+MF(v)+1)/2$ if $M(u)+M(v)$ is odd, the resultant edge labels are easily distinguishable.*



$F_{2,4}$

Examples of Mean Labeling graph.

- The graph P_n, C_n, the bipartite graph $K_{2,n}$, triangular snakes, quadrilateral snakes etc. all are mean graphs..
- K_p if and only if p is less than 3.
- $K_{1,p}$ iff p is less than3.
- The friendship graph $C_3^{(t)}$ iff t less than 2.
- Bistars $B_{p,q}(p > q)$ iff $p < q+2$

Types of Mean Labeling graph.

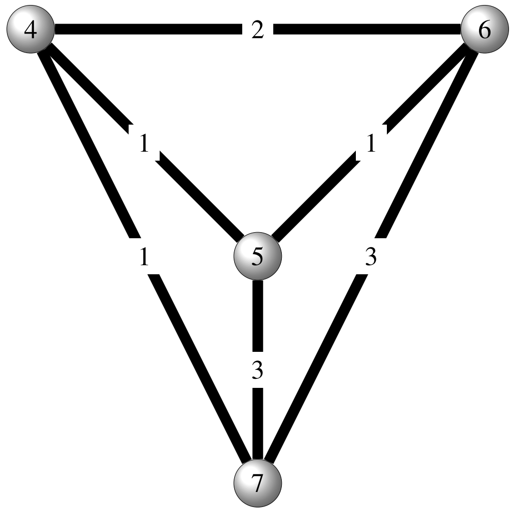
- Vertex even and odd mean graph.
- Super graph.

1.3.7 Irregular labeling of graph:

Ruiz, Chartrand, Saba, Lehel, Oellermann and Jacobson. [14]

Definition: 10. *Let $G = G(V,E)$ be a simple, connected graph having n vertices and m edges.An irregular labeling of a graph G without isolated vertices is an allotment of natural numbers as weights to the edges of G in such a way that the sums of all the weights of the edges incident at every vertex are distinct.*

The lowest of the greatest weight of an edge over all irregular labeling of a particular graph is known as the irregularity strength of G and is denoted by $s(G)$. If no such weight exists for an iiregular labeling then $s(G) = \infty$.

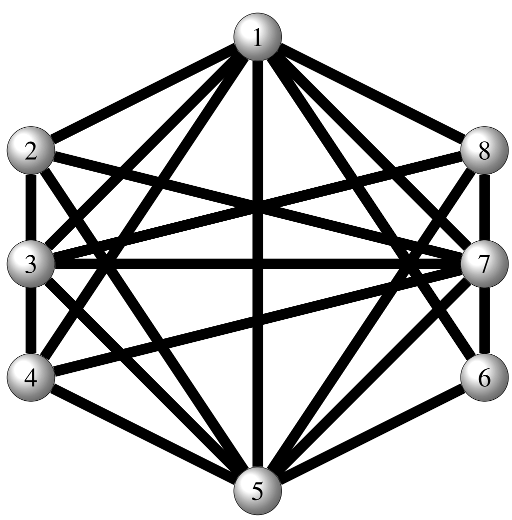


An irregular 3-labeling of the wheel W_3

1.3.8 Prime labeling of graph:

Tout, Dabboucy, and Howalla.[20]

Definition: 11. *Let $G = G(V,E)$ be a simple, connected graph having n vertices and m edges.A graph G is said to posses a prime labeling if there is a bijection $P\colon V \to \{1,2,...,m\}$ such that for each edge(u,v), gcd $(P(u),P(v)) = 1$ i.e $(P(u)$ and $P(v)$ are relatively prime.*



R_8

Examples of Prime Labeling of graph.

- Caterpillar graph with maximum degree 5.
- Path graph, star graph, complete binary trees, spider graph.

1.3.9 Power Mean Labeling of graph:

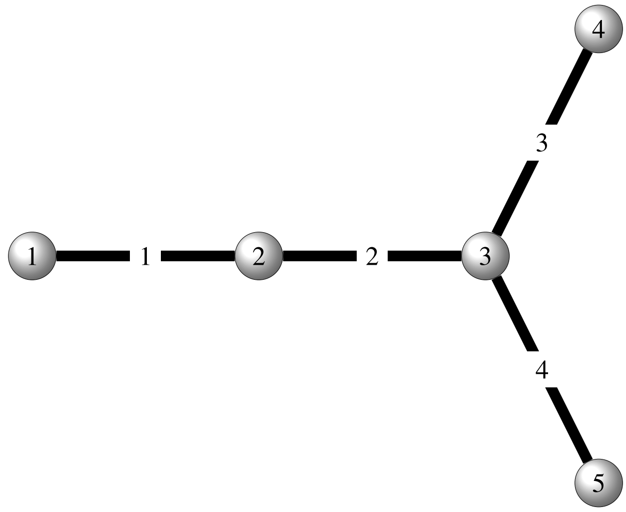
Mercy P.[37]

Definition: 12. *Let $G = G(V,E)$ be a simple, connected graph with n vertices and m edges.A graph G is said to be a Power Mean Graph if it is feasible to label the vertices $v \in V$ with distinct labels $F()$ from $\{1,2,3,...,m+1\}$i.e. there exists an injective function $F : V \to \{1,2,...,m\}$in such a way that when each edge $e = rs$ is labeled with*

$F(e = rs) = \lfloor (F(r)^{F(s)} * F(s)^{F(r)})^{\frac{1}{F(r)+F(s)}} \rfloor$

$F(e = rs) = \lceil (F(r)^{F(s)} * F(s)^{F(r)})^{\frac{1}{F(r)+F(s)}} \rceil$ *then the resulting edge labels are distinct and are from* $\{1,2,3,...,m\}$.

F in called Power mean labeling of G .



Examples of Power Mean Labeling of a graph.

- Tadpoles $T(n,k)$.

Result:

- When two cycles C_p and C_q are joined by a pathP_q, the resultant graph is a Power mean graph
- $T(P_m)$ for m greater than or equal to 2 is a Power mean graph.
- Subdivision of any P_n for n greater than or equal to 2 is a Power mean graph.
- Subdivision of any C_n for n greater than or equal to 3 is a Power mean graph.

1.3.10 Combination and Permutation labeling of graph:

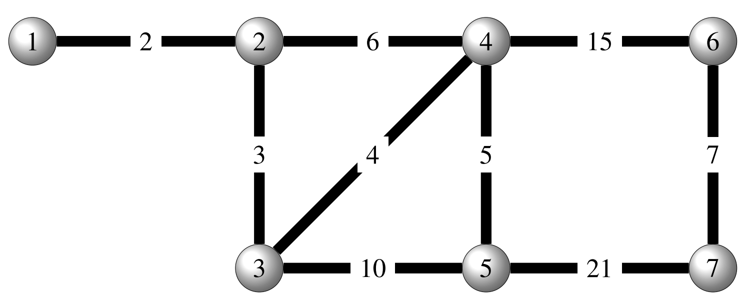
Hegde and Shetty [20]

a)Combination labeling of graph:

Definition: 13. *Let $G = G(V,E)$ be a connected graph having n vertices and m edges.A function $F : V \to \{1,2,...,m\}$ is called as combination labeling of graph G if every edge (r,s) is labeled with*

$(F(r))!/[F(r) - F(s)]!(F(s))!$ *where* $F(r) > F(s)$.

The resulting edge labels are all well-defined.

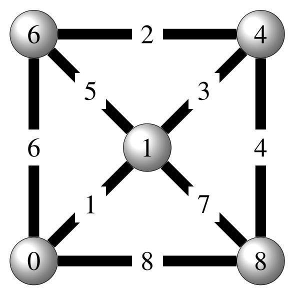


Examples of Combination Labeling of a graph.

- K_n if and only if n is less than or equal to 5.
- C_n, n greater than 3.
- $K_{n,n}$ iff n is less equals to 2. .

b) Permutation Labeling of graph:

Definition: 14. *Let $G = G(V,E)$ be a simpke connected graph having n vertices and m edges. A function $F : V \to \{1,2,...,m\}$ is called as permutation labeling of graph G if every edge (r,s) is labeled with $(F(r))!/[F(r)-F(s)]!$ where $F(r) > F(s)$.*
The resulting edge labels all are well-defined.



grap **Examples of Permutation Labeling of a graph.**

- K_p iff p is less equals 5.

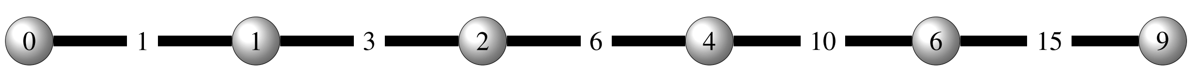
Result:

- In a permutation graph G if we remove any edge from this graph then the resultant graph is again a permutation graph.
- The graph k-wheel , for all integer $n \geq 2$.
- The graph k-fan $F_{n,k}$.
- The gear graph G_{2n} is also a permutation grph.

1.3.11 Triangular sum labeling of graph:

Hegade and Shankaran[36]

Definition: 15. *Let $G = G(V,E)$ be a simple.connected graph accompanying n vertices and m edges. A labeling of the graph with m edges is called a triangular sum labeling if the vertices can be determined with distinct not-negative integers in such a way that, when an edge whose end vertices are labeled with r and s is labeled with the value $r+s$.*
All vertex and edge labels are different.



P_6

Examples of Triangular sum Labeling of graph.

- Star graph, Path graph, n-ary complete trees.
- K_n iff $n = 1$ or $n = 2$

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