$M$**-Projective Curvature Tensor over** $(κ,μ)$**-Contact Riemannian Manifolds**

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**Abstract**: -

In 1995, the concept of $(κ,μ)$-contact Riemannian manifolds was introduced by Blair, Koufogiorgos, and Papantoniou [5]. Subsequently, a comprehensive investigation into the classification of contact metric $(κ,μ)$-spaces was conducted by Boeckx, E. [7] in 2000. Blair explored the $(κ,μ)$-nullity condition in the context of contact Riemannian manifolds and provided various motivations for its study. The current paper focuses on the examination of flatness conditions concerning the $M$-projective curvature tensor within the framework of $(κ,μ)$-contact Riemannian manifolds.

**1. Introduction**-

In 1958, Boothby and Wong first introduced the concept of odd-dimensional manifolds with contact and almost contact structures, primarily approaching it from a topological perspective. Subsequently, in 1961, Sasaki and Hatakeyama re-examined these structures using tensor calculus techniques.

Alternatively, in the work of Pokhariyal and Mishra, a tensor field $W^{\*} $is introduced on a Riemannian manifold as ($θ$ =Riemannian metric$)$

 ‘$W^{\*}\left(S,T,U,V\right)=‘R(S,T,U,V)-\frac{1}{2\left(d-1\right)}×[ρ\left(T,U\right)θ\left(S,V\right)- ρ\left(S,U\right)θ\left(T,V\right)+θ\left(T,U\right)ρ\left(S,V\right)-θ(S,U)ρ(T,V)]$ , (1)

Where ‘$W^{\*}(S,T,U,V)=θ(W^{\*}(S,T)U,V) $and ‘$R(S,T,U,V)=θ(R(S,T)U,V)$. The tensor field $W^{\*} $is referred to as the $M$-projective curvature tensor. Subsequently, Ojha conducted a comprehensive investigation of the properties of this tensor in both Sasakian and Kähler manifolds.

The category of $(κ,μ)$-contact Riemannian manifolds encompasses both Sasakian and non-Sasakian manifolds. Boeckx [7] provided a comprehensive categorization of $(κ,μ)$-contact Riemannian manifolds. These manifolds retain their properties under $D$-homothetic transformations.

In an earlier study [6], Blair, Kim, and Tripathi commenced an inquiry into the concircular curvature tensor of contact Riemannian manifolds. The examination of the pseudo-projective curvature tensor on a contact Riemannian manifold was recorded in [5]. More contemporarily, the investigations carried out by [14] and [15] delved into exploring the quasi-conformal curvature tensor and the E-Bochner curvature tensor on a $(κ,μ)$-contact Riemannian manifold, respectively. In addition to the well-known Riemannian curvature tensor, the Weyl conformal curvature tensor, and the concircular curvature tensor, the $M$-projective curvature tensor emerges as a pivotal tensor within the realm of differential geometry. The curvature tensor serves as a unifying link between the conharmonic curvature tensor, the concircular curvature tensor and the conformal curvature tensor on the one hand while establishing a connection with the $H$-projective curvature tensor on the other.

Recently, the $M$-projective curvature tensor has been a subject of study for various researchers, including Chaubey, Ojha [13], Singh [11], and others.

Expanding upon prior research, our current study investigates the symmetry and flatness characteristics of $(κ,μ)$-contact Riemannian manifolds in the context of the $M$-projective curvature tensor. In Section 3, we review and deduce our initial findings. Subsequently, in Segment 4, we analyze $M$-projectively flat $(κ,μ)$-contact Riemannian manifolds. Segment 5 centers on exploring $ζ$-$M$-projectively Sasakian flat $(κ,μ)$-contact Riemannian manifolds, where we establish the requisite and sufficient conditions for the manifestation of $ζ$-$M$-projective Sasakian flatness in an $(κ,μ)$-contact Riemannian manifold.

**2. Contact Riemannian Manifold-**

An almost contact structure on an $(2d+1)$-dimensional differentiable manifold $E$ is defined by the existence of a tensor field $F$ of type (1, 1), a vector field $ζ$, and a 1-form $η$ such that

 $F^{2}=-I+η⊗ ζ , η(ζ)=1$ (2)

 $Fζ=0$, and $η∘F=0$ (3)

Take into account a consistent Riemannian metric $θ$ in conjunction with an almost contact structure $(F, ζ, η)$

 $θ\left(FS,FT\right)=θ\left(S,T\right)-η(S)η(T)$ (4)

Subsequently, when $E^{2d+1} $undergoes a transformation, it transforms into an almost contact Riemannian manifold by acquiring an almost contact metric structure represented as $(F, ζ, η, θ$). By observing equations (2) and (4), it becomes evident that

 $ θ\left(S,FT\right)=-θ\left(FS,T\right),   θ(S,ζ)=η(S),$ (5)

For any given vector fields $S$ and $T$.

The fact that the tangent sphere bundle of a Euclidean Riemannian manifold possesses a contact metric structure with the property $R(S,T)ζ=0$ is widely acknowledged. Conversely, in the context of a Sasakian manifold, the subsequent assertion is valid:

 $R(S,T) ζ=η(T)S-η(S)T$. (6)

Blair et al. extended the concepts of $R(S,T)ζ=0 $and the Sasakian case by investigating the $(κ, μ)$-nullity condition on a contact Riemannian manifold. They introduced the $(κ, μ)$-nullity distribution $N(κ, μ)$ ([3,5]) to characterize this condition on the contact Riemannian manifold.

$$N\left(κ,μ\right):P \rightarrow  N\_{P}(κ,μ)=\left\{U\in  T\_{P}E:R\left(S,T\right)U=\left(κI+μh\right)\left[θ\left(T,U\right)S-θ\left(S,U\right)T\right]\right\}$$

 …(7)

For any pair of vectors $S$ and $T$ belonging to the tangent space $TE$, where $(κ, μ)$ are elements of the $R^{2}$, a Riemannian manifold $E^{2d+1}$ possessing $ζ$ in the set $N(κ, μ)$ is referred to as a manifold with $(κ,μ)$ characteristic. Specifically, on a manifold with $(κ,μ)$ attributes, the following holds true

 $R(S,T)ζ=κ[η(T)S-η(S)T]+μ [η(T)hS-η(S)hT] $. (8)

On a $(κ, μ)$-manifold, where $κ\leq 1$, the structure becomes Sasakian with $h=0$ and $μ$ remaining indeterminate when $κ=1$. When $κ<1$, the $(κ, μ)$-nullity condition uniquely prescribes the curvature of $E^{2d+1} $Essentially, for a $(κ, μ)$-manifold, the properties of being a Sasakian manifold, a $K$-contact manifold, $κ=1$, and $h=0$ are all interchangeable and equivalent.

In a manifold characterized by the parameters $(κ,μ)$, the following relationships are valid:

$$h^{2}=(κ-1)^{2} F^{2} , κ\leq  1,$$

$$R(ζ,S)T=κ[θ(S,T)ζ-η(T)S]+μ[θ(hS,T)ζ-η(T)hS] , $$

 $ρ(S,ζ)=2dκη (S), $ (9)

$$ρ\left(S,T\right)=\left[2\left(d-1\right)-dμ\right]θ\left(S,T\right)+\left[2\left(d-1\right)+μ\right]θ\left(hS,T\right)$$

 $ +[2(1-d)+ d( 2κ+μ)] η(S)η(T), n\geq 1, $

$$ρ\left(FS,FT\right)=ρ\left(S,T\right)-2dκη\left(S\right)η\left(T\right)-2\left(2d-2+μ\right)θ\left(hS,T\right),$$

Where $ρ$ is the Ricci tensor of type (0, 2), 𝑄 is the Ricci operator, that is, $θ(QS,T)=ρ(S,T).$

Furthermore, the (𝜅, 𝜇)-manifold exhibits the following property:

$$η\left(R\left(S,T\right)U\right)=κ\left[θ\left(T,U\right)η\left(S\right)-θ\left(S,U\right)η\left(T\right)\right]$$

 $+μ\left[θ\left(hT,U\right)η\left(S\right)-θ\left(hS,U\right)η\left(T\right)\right]$ (10)

In the context of Riemannian manifold, the $M$-projective curvature tensor $W^{\*} $can be stated as follows [8].

$$W^{\*}\left(S,T\right)U=R\left(S,T\right)U- \frac{1}{2\left(d-1\right)}×\left[ρ\left(T,U\right)S-ρ\left(S,U\right)T+θ\left(T,U\right)QS-θ\left(S,U\right)QT\right],$$

 …(11)

Given arbitrary vector fields $S,T$ and $U$, where $ρ$ represents the type of Ricci tensor (0, 2) and $Q$ denotes the Ricci operator, $θ$ denotes the Riemannian metric, we have the relation $θ\left(QS,T\right)= ρ\left(S,T\right).$

**Lemma 2.1.** [1] In $(κ,μ)$-contact Riemannian manifolds that are not Sasakian, the conditions that follow are mutually equivalent:

 (i) $η$-Einstein manifold,

 (ii) $QF=FQ$

**Definition 2.1**. An $E$ manifold with a $(k,μ)$-contact metric structure is referred to as $η$-Einstein when the Ricci operator $Q$ fulfills the conditions

 $Q=aI+bη⊗ζ$ (12)

Smooth functions $a$ and $b$ are represented in the given context defined on the manifold. Notably, when $b$ is set to zero, $E$ qualifies as an Einstein manifold.

In the case where an $(2d+1)$-dimensional non-Sasakian $(k,μ)$-contact Riemannian manifold $(E^{2d+1},θ)$ is $η$-Einstein, the expression for the non-zero Ricci tensor $ρ$ takes the following form:

 $ρ\left(S,T\right)=aθ\left(S,T\right)+bη\left(S\right)η\left(T\right).$ (13)

**Lemma 2.2**. On a non-Sasakian $(k, μ)$-contact Riemannian manifold ($E^{2d+1},θ)$, $a+b=2dκ$

Proof. In view of (2)-(5) and (13), we have

 $QS=aS+bη\left(S\right)ζ,$ (14)

such that Ricci operator $Q$ is defined by

 $ρ\left(S,T\right)=θ\left(QS,T\right).$ (15)

Again, contracting (14) with respect to $S$ and using (2)-(5), we have

 $r=\left(2d+1\right)a+b.$ (16)

Now, putting $ζ$ instead of $S$ and $T$ in (13) and then using the equations in (2)-(5) and (9) we get

 $a+b=2dκ.$ (17)

Equations (16) and (17) give

 $a=\left(\frac{r}{2d}-κ\right)$ and $b=\left(\left(2d+1\right)κ-\frac{r}{2d}\right).$ (18)

Equation (18) prove the statement of the Lemma 2.2.

**3. The** $M$**-Projective Curvature Tensor** $W^{\*}$ **for an** $(κ,μ)$**-Contact Riemannian Manifolds-**

The curvature tensor $W^{\*}$ associated with $M$-projective geometry on a $(κ,μ)$-contact Riemannian manifold is expressed as

$W^{\*}\left(S,T\right)ζ=-\frac{κ}{\left(d-1\right)}\left[η\left(T\right)S-η\left(S\right)T\right]+μ\left[η\left(T\right)hS-η\left(S\right)hT\right]- \frac{1}{2\left(d-1\right)}\left[η\left(T\right)QS - η\left(S\right)QT\right],$ (19)

 $η(W^{\*}(S,T) ζ) = 0,$ (20)

$$W^{\*}\left(ζ,T\right)U=-W^{\*}\left(T,ζ\right)U=-\frac{κ}{\left(d-1\right)}\left[θ\left(T,U\right)ζ-η\left(U\right)T\right]+μ\left[θ\left(hT,U\right)ζ-η\left(U\right)hT\right]$$

 $-\frac{1}{2\left(d-1\right)}\left[ρ\left(T,U\right)ζ-η\left(U\right)QT\right],$ (21)

$$η\left(W^{\*}\left(ζ,T\right)U\right)=-η\left(W^{\*}\left(T,ζ\right)U\right) =- \frac{κ}{\left(d-1\right)}\left[θ\left(T,U\right)-η\left(T\right)η\left(U\right)\right]+μ\left[θ\left(hT,U\right)-η\left(U\right)η\left(hT\right)\right]$$

 $-\frac{1}{2\left(d-1\right)}\left[ρ\left(T,U\right)-2dκη\left(T\right)η\left(U\right)\right],$ (22)

$$            η\left(W^{\*}\left(S,T\right)U\right)=-\frac{κ}{\left(d-1\right)}\left[θ\left(T,U\right)η\left(S\right)-θ\left(S,U\right)η\left(T\right)\right]+μ\left[θ\left(hT,U\right)-η\left(U\right)η\left(hT\right)\right]$$

 $-\frac{1}{2\left(d-1\right)}\left[ρ\left(T,U\right)η\left(S\right)-ρ\left(S,U\right)η\left(T\right)\right].  $ (23)

**4.** $M$**-Projectively Flat** $(κ,μ)$**-Contact Riemannian Manifolds-**

The class of $(κ,μ)$-contact Riemannian manifolds known as $M$-projectively flat manifolds is a distinctive category within contact Riemannian manifold where the geometry is such that the curvature tensor satisfies certain conditions related to the $M$-projective flatness property. The parameters $κ$ and $μ$ are involved in the definition of the curvature conditions and can affect the geometry of the manifold.

**Theorem 4.1**. A $(κ,μ)$-contact Riemannian manifold $E^{2d+1}$that is $M$-projectively flat exhibits the property of being an Einstein manifold.

Proof. Let $W^{\*}(S,T,U,V)=0$. Subsequently, utilizing equation (11), we derive the following outcome:

 $‘R(S,T,U,V)=\frac{1}{2\left(d-1\right)} [ρ(T,U)θ(S,V)-ρ(S,U)θ(T,V)+θ(T,U)ρ(S,V)$

 $-θ(S,U)θ(T,V)] $ (24)

Considering $e\_{i} $as an orthonormal basis of the tangent space at any point, if we set $T=U=e\_{i}$ in the given equation and then sum up over $i$, where $1 \leq i \leq 2d + 1$, we arrive at the same result,

 $ρ(S,T)=-rθ(S,T),$ (25)

Where $r$-Scalar curvature of the manifold and $r=2d\left(2d-2+κ-dμ\right)$.

This indicates that $E^{2d+1}$ is a manifold that satisfies the Einstein condition. This completes the proof.

**5.** $ζ$**-**$M$**-Projectively Sasakian Flat** $(κ,μ)$**-Contact Riemannian Manifolds-**

$ζ$-$M$-Projectively Sasakian flat $(κ, μ)$-contact Riemannian manifolds likely refer to a specific class of contact Riemannian manifolds that satisfy curvature conditions related to $M$-projective flatness and these manifolds also have a distinguished Reeb vector field ($ζ$) and Sasakian geometry. This indicates a very specialized and intricate geometric structure where various curvature conditions, contact structures, and vector fields are intertwined.

**Definition 5.1**. An (2d+1) (with d > 1)-dimensional $(κ,μ)$-contact Riemannian manifold is classified as $ζ$-$M$-projectively Sasakian flat when the condition $W^{\*}(S,T)ζ=0$ holds for all $S$ and $T$ belonging to the tangent space $TE$.

**Theorem 5.1.** An (2d+1)-dimensional (d>1) $(κ,μ)$-contact Riemannian manifold exhibits $ζ$-$M$-projective Sasakian flatness iff it possesses the characteristic of being an $η$-Einstein manifold.

Proof. Let $W^{\*}(S,T)ζ=0. $Then, in view of (11), we have

 $R(S,T)ζ=\frac{1}{2(d-1)}[ρ(T,ζ)S-ρ(S,ζ)T+θ(T,ζ)QS-θ(S,ζ)QT] $ (26)

Due to the presence of (5), (8), and (9), the equation above can be simplified to

$$κ\left[η\left(T\right)S-η\left(S\right)T\right]+μ\left[η\left(T\right)hS-η\left(S\right)hT\right]$$

 $ = \frac{dκ}{d-1}\left[η\left(T\right)S-η\left(S\right)T\right]$+$\frac{1}{2(d-1)}\left[η\left(T\right)QS-η\left(S\right)QT\right] $ (27)

which by putting $T=ζ$, gives

 $QS=2κ \left[-S+\left(d+1\right)η\left(S\right)ζ\right]+2(d-1)μ(hS)$ (28)

In the case of Sasakian manifolds, $κ=1,$ (and hence $h=0$)

Now, taking the inner product of above equation with $V$, we get

 $ρ(S,V)=2[-θ(S,V)+(d+1) η(S)η(V)]$ (29)

Furthermore, it can be proved that a $(κ,μ)$-contact Riemannian manifold represents an $η$-Einstein manifold. Conversely, assume that condition (29) is fulfilled. As a result of the implications of (28) and (19), we can deduce $W^{\*}(S,T)ζ=0$. Thus, the proof is concluded.

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