# Computation of free solidarity value with its characterization

Vitsono Lungalang *1 and Prem Prakash Mishra †2

1,2Department of Science and Humanities (Mathematics), NIT, Nagaland

**Abstract**

In this paper, we introduce a notion of free solidarity value with partial participation of solidary players that takes into account situations of voluntary formation of solidary group. Adhering to which a unanimity game that forms a basis for games involving the Solidarity and Shapley values is considered. We also add two new axioms, partial positivity and unreserved allocations of solidary players for the characterization of the value.

## 1 Introduction

A basic feature of an evolving society is that of people coming together as group or unions. This leads to better productivity as well as an understanding to share the profit fairly. In particular, we will consider the players volunteer to form a solidary group as highlighted in the working paper [8] titled as "the free solidarity value" of Dhaou and Ziad. Wherein they considered that two types of players; solidary and non-solidary can co-exist without harming the interest of any group. The Shapley value [17] and the solidarity value [12] are employed for the payoffs of the non solidary and solidary players respectively.
Principally, the work attempted to highlight solidarity that can exist in a social structure in situations where players are free to choose to form a solidary group. This voluntary formation of a group of solidary players sets the game in motion. But there also maybe a situation where at least one or more member of the solidary group need to opt out of internal or external factors. In those situations, it will no longer be fair to follow the mode of solidarity value [12] distribution as some of the members need to take the extra burden for the missing ones. In that sense, it is fair to implement the Shapley value [17] distribution if some members of the solidary group get separated.
Dhaou and Ziad characterized the value by engaging the axioms of efficiency, additivity, conditional symmetry, unaffected allocation of non-solidary players and conditional null player. They defined unanimity games to be the basis for the game. But we find that the value function constructed from the unanimity games do not correspond totally with the value function of the free solidarity value as defined in the paper. Whereas for specific unanimity games the equality holds. Evidently, this restricts an efficient payoffs for all the unanimity games.
However, it is reasonable to assume this restriction is valid as the game structure depends on the solidary group of players and so efficiency fails to hold in cases when all solidary players are not present. So, in the light of which we reconsider the payoffs of players when all solidary players are not present.
In particular, the notion of free solidarity do not follow the classic concept of Owen [14] as

*vitsono@gmail.com
†math.prem79@gmail.com

well as the Shapley solidarity value [5], which in fact follows a different approach though solely based on the Shapley and Solidarity distribution.
Owen [14] gave an approach when players organize themselves into groups to better their bargaining position in the game. This notion was incorporated into TU games by means of coalition structure which partitioned players into a set of groups or unions. In the Owen value, players interact at two levels, first among unions and then within unions. In both the levels, the payoffs are given by the Shapley value [Shapley,1953]. Calvo and Gutierrez [5] modelled the Shapley-solidarity value following the same approach as Owen. X.-F. Hu, D.-F. Li [20] proposed another axiomatization of the Shapley-Solidarity value [5] for coalitional structure TU- games. Su, Liang et.al [19] provided cooperative and non-cooperative interpretation of the Shapley-Solidarity value.
In this paper, first, we analyse the unanimity games constructed as a basis game in the working paper of Dhaou and Ziad [2015]. We find that the value function based on the unanimity games agrees partially with the value function employing the Shapley and Solidarity values as defined in their paper. Albeit, for restricted unanimity game equality, holds it fails for the remaining games. We give a restriction that the unanimity games will strictly follow the Shapley distribution when all solidary players are not in a game that allows the solidary players to be treated as non solidary players. And for which we define the notion of partial participation of solidary players. This provides an effective measure to deal with efficiency thereby at the same time allowing free association without making it obligatory. It is relevant to ask how a solidarity group would be sustainable if the interests of some members are provided at the expense of other members. So, our attempt in this paper is to provide fair distribution when some members of the solidary group fail to participate.

We organize the paper as follows. Section 2 emphasizes on the mathematical preliminaries and all related values namely, the Shapley and the Solidarity values. Section 3 discusses on the free solidarity value as defined in their paper and further reconstruct the definition of the unanimity games. Section 4 we define partial participation of solidary players and we provide additional axioms called partial positivity and unreserved allocation of solidary players along with the existing axioms in the characterization of the free solidarity value.

# 2 Preliminaries

Given a finite set $N = \{1, 2, ..., n\}$ of players. $2^N$ is the set of all subsets $S$ of $N$. Any non-empty subset $S$ of $N$ is said to be a coalition. The set $N$ is the grand coalition. A transferable utility $(TU)$ game on a set $N$ is a characteristic function $v : 2^N \to \mathbb{R}^+$ which assigns to each coalition $S$ a real number denoted by $v(S)$, such that $v(\emptyset) = 0$. $v(S)$ is the worth of the coalition $S$. The cardinality of any non empty coalition $T, S, N$ etc can be denoted by $t, s, n$. A cooperative game with transferable utility $(TU)$ is represented by $(N, v)$ or simply $v$. The set of all games for player set $N$ is denoted by $\mathcal{G}^N$ .

Consider any two games $v, u \in \mathcal{G}^N$, $\alpha \in \mathbb{R}$. Under the usual operations of addition and scalar multiplication, we can define, $(v + u)(S) = v(S) + u(S)$ and $(\alpha v)(S) = \alpha v(S)$ where $S \in 2^N \setminus \emptyset$. The set $\mathcal{G}^N$ forms a vector space with dimension $2^N - 1$. A value is a mapping $\Phi : \mathcal{G}^N \to \mathbb{R}^n$ defined by $\Phi(v) = (\phi_1(v), ..., \phi_i(v)..., \phi_n(v))$. For any game $v \in \mathcal{G}^N$, the mapping determines a unique payoff vector in $\mathbb{R}^n$. The vector represents the payoffs to each individual player $i \in N$ by taking into account their role in the game.

For any game $v \in \mathcal{G}^N$ and for every $S \subseteq N$ , the marginal contributions of a player $i$ to a coalition $S$ denoted by $m_i(v, S)$ is given by

$$m_i(v, S) = \begin{cases} v(S) - v(S \setminus i) & \text{if } i \in S \\ v(S \cup i) - v(S) & \text{if } i \notin S \end{cases}$$

In Nowak and Radzik [12], the average marginal contribution of a coalition $S$ in a game denoted by $m^{av}(v,S)$ and is given by
$m^{av}(v,S) = \frac{1}{s}\sum_{i\in S}(v(S)-v(S\setminus i)) = \frac{1}{s}\sum_{i\in S} m_i(v,S)$.
In Dhaou and Ziad [8], the average marginal contribution of a coalition $S$ in a game is denoted by $\tilde{m}^{av}(v,S)$ and is given by
$\tilde{m}^{av}(v,S) = \frac{1}{|S\cap S^*|}\sum_{i\in S\cap S^*}(v(S)-v(S\setminus i)) = \frac{1}{|S\cap S^*|}\sum_{i\in S\cap S^*} m_i(v,S)$.
A game $v \in \mathcal{G}^N$ is said to be monotonic if $v(T) \le v(S)$ for any $T, S \subset N$ such that $T \subseteq S$.

In a game $v \in \mathcal{G}^N$, two players are said to be symmetric if $v(S\cup i) = v(S\cup j) \quad \forall S \subseteq N\setminus\{i,j\}$. If $v(S\cup i) = v(S), \ \forall S \subseteq N\setminus i$ then $i \in N$ is a null player. If $\tilde{m}^{av}(v,S) = 0$ for some $i$ which belongs to every coalition $S$ and $S \subseteq N$ then $i$ is an A-null player.
Some of the properties of a value function $\Phi$ in $\mathcal{G}^N$ are as follows:

1. Efficiency: For $v \in \mathcal{G}^N$, $\sum_{i\in N}\Phi_i(N,v) = v(N)$.
2. Additivity: For $v, u \in \mathcal{G}^N$, $\Phi_i(N, v+u) = \Phi_i(N,v) + \Phi_i(N,u)$.
3. Symmetry: If $i, j \in N$ are symmetric players in $v \in \mathcal{G}^N$ then $\Phi_i(N,v) = \Phi_j(N,v)$.
4. Null-player axiom: If $i \in N$ in $v \in \mathcal{G}^N$ is a null player then $\Phi_i(N,v) = 0$.
5. A-null player axioms: If $i \in N$ in $v \in \mathcal{G}^N$ is an A-null player then $\Phi_i(N,v) = 0$.

Players with zero contribution is highlighted in both the Shapley value [17] and the Solidarity value [12] or simply non-productivity is highlighted by null player and A-null player conditions respectively. In the null player axiom, a player will get a zero payoff if all her marginal contributions in a game are zero. That is to label a player as strictly unproductive. In contrast, in the A- null player axiom a player will get a zero payoff when the average productivity of all the coalitions to which she belongs to is zero. This takes into consequence the presence of a player in a coalition before discarding her as unproductive.

## 2.1 The Shapley value

The Shapley value [17], $\Phi_i^{sh}(N,v)$ which gives the payoff for every player $i \in N$ is given by:
$\Phi_i^{sh}(N,v) = \sum_{S\subseteq N, i\in S}\frac{(n-s!)(s-1)!}{n!} m_i(v,S) \quad \forall i \in N$.
The Shapley value was characterized using the axioms of efficiency, additivity, symmetry and null-player axiom. The Shapley value can also be defined by using the unanimity games. A unanimity game $u_T$ with $\emptyset \neq T \subseteq N$ is defined as

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{otherwise} \end{cases}$$

The Shapley value in terms of each vector of the unanimity game $u_T$ is given by:

$$Sh_i(N,u_T) = \begin{cases} \frac{1}{|T|} & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases}$$

The unanimity games $(u_T)_{\emptyset\neq T\subseteq N}$ forms a basis for $\mathcal{G}^N$. It has been established that any game can be written as a linear combination of the unanimity games and coefficients of which is $\Delta_T$. That is, $v = \sum_{\emptyset\neq T\subseteq N}\Delta_T u_T$ where $\Delta_T = \sum_{T\subseteq S}(-1)^{s-t}v(T)$ is the *Harsanyi dividend*. Thus, the Shapley value in the dividend form is defined as
$Sh_i(N,v) = \Delta_T\frac{1}{|T|}$ for every $i \in N$.

**Theorem 1** [17]. A value $\Psi$ on $\mathcal{G}^N$ satisfies efficiency, additivity, symmetry and the null player axiom if, and only if, $\Psi$ is the Shapley value.

## 2.2 The Solidarity value

Sprumont [18], defined the solidarity value $\Phi_i^{sd}$ in a recursive manner as follows
$Sd_i(N,v) = \frac{1}{s}m^{av}(v,S) + \sum_{j\in S\setminus i} \frac{1}{s}Sd_i(S\setminus j, v).\ \ i \in S \subseteq N,$
Firstly, $Sd_i(\{i\},v) = v(i)\ \ \forall i \in N$.
Nowak and Radzik [12] gave another version of the solidarity value and characterized the value with the axioms of efficiency, additivity, symmetry and A-null player condition. The difference in the characterization with the Shapley value is in the the axiom of A-null player which replaces the null player axiom.
The Solidarity value in Nowak and Radzik [12] is defined as
$Sd_i(N,v) = \sum_{i\in S} \frac{(n-s)!(n-1)!}{n!} m^{av}(v,S)\ \ \forall i \in N.$
In the authors also constructed a new basis game for $\mathcal{G}^N$. The game is defined as

$$b_T(S) = \begin{cases} \binom{|S|}{|T|}^{-1} & \text{if } T \subseteq S \\ 0 & \text{otherwise} \end{cases}$$

For the game $(N, b_T)_{\emptyset\neq T\subseteq N}$ they showed that all the players in $N \setminus T$ are A-null players. The solidarity value based on this basis game is as follows:

$$Sd_i(N, b_T) = \begin{cases} \frac{1}{|T|} b_T(N) & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases}$$

**Theorem 2** [12]. A value $\Psi$ on $\mathcal{G}^N$ satisfies efficiency, additivity, symmetry and the A-null player axiom if and only if $\Psi$ is the solidarity value.
A coalition structure over a finite set of players, say $N$, is a partition, that is, $P = \{P_1, .., P_m\}$ if it satisfies the following conditions:
(i) $\bigcup_{k=1}^{m} P_k = N$
(ii) $P_k \cap P_l = \emptyset$ when $k \neq l$
(iii) $P_k \neq \emptyset$ for all $k$.
The elements of $P$ are termed as components or unions or blocks. $\{N\}$ and $\{\{i_1\}, \{i_2\}, ..., \{i_n\}\}$ are the trivial coalition structures. $\mathcal{P}(N)$ denote the set of all coalition structures over $N$. A game $(N,v)$ with coalition structure $P \in \mathcal{P}(N)$ is a triple $(N,v,P)$. The family of all games with coalition structure of player set $N$ is denoted by $\mathcal{PG}^N$.
In a coalition structure $(N,v,P)$, a quotient game is a game between unions. For every $(N,v,P) \in \mathcal{PG}^N$ where $P = \{P_1, .., P_m\}$, the quotient game is denoted by $(M, v_P) \in \mathcal{G}^N$ where $M = \{1, 2, ..., m\}$ such that $v_P(T) = v(\cup P_i)$ for every $T \subseteq M$.
The coalition structure is used to model situations where players with similar interests and characteristics form groups as in the case of trade unions, political parties, non-profit organisations etc. Also, groups may be formed due to geographical constraints of players located in cities, states and countries. Groups are formed for bargaining payoffs in cooperative games.

# 3 The Free Solidarity value

Dhaou and Ziad (working paper, [8]) defined the free solidarity value wherein they considered two types of players, solidary and non-solidary players. The group of solidary players is fixed from the onset of the game. The group of solidary players is denoted by $S^*$.
They defined the unanimity game for the Free Solidarity value:

$$B_{(T,S^*)}(S) = \begin{cases} \frac{|T\cap S^*|}{|T|} \binom{|S\cap S^*|}{|T\cap S^*|}^{-1} + \frac{|T\setminus(T\cap S^*)|}{|T|} & \text{if } T \subseteq S \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The authors showed that the unanimity games $(B_{(T,S^*)})_{\emptyset \neq T \subseteq N}$ exhibit the following properties: (i) If $T = S$ then $B_{(T,S^*)}(S) = 1$. (ii) If $T \subset S$ and $S^* = \emptyset$, then $B_{(T,\emptyset)}(S) = 1$. (iii) $B_{(T,S^*)}$ is a basis for $\mathcal{G}^N$.
The free solidarity value based on the unanimity games is defined as follows:

$$\Psi_i(N, B_{(T,S^*)}, S^*) = \begin{cases} \frac{1}{|T|}\binom{|S^*|}{|T \cap S^*|}^{-1} & \text{if } i \in T \cap S^* \\ \frac{1}{|T|} & \text{if } i \in T \setminus (T \cap S^*) \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

**Theorem 3** [8]. A value $\Psi$ on $\mathcal{G}^N$ satisfies efficiency, additivity, conditional symmetry, conditional null player condition, and unaffected allocation of non-solidary players axioms if and only if $\Psi$ is the free solidarity value.
In the free Solidarity value [8], the authors assert that the existence of an exogenous coalition of solidary players promotes fairness in the payoffs of the players. The payoffs are considered according to the players being solidary or non-solidary. So, the authors distinguishes the non-solidary and solidary players by accordingly considering their payoffs to be the Shapley value and the Solidarity value respectively. From which it follows that the contributions of the players in a game are regarded as marginal contribution or average marginal contribution depending on the player being non-solidary or solidary. Thus, if $\forall i \in S \subseteq N \setminus \emptyset, S \cap S^* \neq \emptyset$, then, $m_i(v,S) = v(S) - v(S \setminus i)$. And, if $i \in S \setminus S \cap S^*$ or $\tilde{m}^{av}(v,S) = \frac{1}{|S \cap S^*|} \sum\limits_{i \in S \cap S^*} (v(S) - v(S \setminus i)) = \frac{1}{|S \cap S^*|} \sum\limits_{i \in S \cap S^*} m_i(v,S)\ , if\ i \in S \cap S^*$.
These contributions distinguishes a null player and an A-null player. Null Players will get zero payoffs if their marginal contributions are zero and A-null players will get nothing if average marginal contributions are zero for every coalition the player belongs.

The free solidarity value for an exogeneous coalition $S^*$ is defined as

$$\Phi_i^{fs}(N, v, S^*) := \begin{cases} \sum\limits_{i \in S} \frac{(n-s)!(s-1)!}{n!}[v(S) - v(S \setminus \{i\})] & \text{if } i \notin S^* \\ \sum\limits_{i \in S} \frac{(n-s)!(s-1)!}{n!} \frac{1}{|S \cap S^*|}[\sum\limits_{k \in S \cap S^*} (v(S) - v(S \setminus k))] & \text{if } i \in S^* \end{cases} \tag{3}$$

$$\Phi_i^{fs}(N, v, S^*) := \begin{cases} Sh_i^{fs}(N, v) & \text{if } i \notin S^* \\ Sd_i^{fs} & \text{if } i \in S^* \end{cases} \tag{4}$$

The following relations hold true for equations 3 and 4:

1. If $S^* = \emptyset$ then $\Phi^{fs}(N, v, S^*) = Sh^{fs}(N, v)$.
2. If $S^* = \{i\}$ then $\Phi^{fs}(N, v, S^*) = Sh^{fs}(N, v)$.
3. If $S^* = N$, then $\Phi^{fs}(N, v, S^*) = Sd^{fs}(N, v)$.

We illustrate with an example that the definition (Eq.3) is not in total agreement with the definition (Equ.2) given with respect to the unanimity games.
**Example 1**: The matrix of the unanimity game $(N, B_{(T,S^*)}, S^*)$ based on the example, $N = \{1, 2, 3\}$; $S^* = \{2, 3\}$ is as follows:

| $B_{(T,S^*)}(S)$ | {1} | {2} | {3} | {1,2} | {1,3} | {2,3} | {1,2,3} |
|---|---|---|---|---|---|---|---|
| $B_{\{1\}}$ | 1 | 0 | 0 | 1 | 1 | 0 | 1 |
| $B_{\{2\}}$ | 0 | 1 | 0 | 1 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| {3} | 0 | 0 | 1 | 0 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| {1,2} | 0 | 0 | 0 | 1 | 0 | 0 | $\frac{3}{4}$ |
| {1,3} | 0 | 0 | 0 | 0 | 1 | 0 | $\frac{3}{4}$ |
| {2,3} | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| {1,2,3} | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

The payoff matrix for free solidarity value (Eq.2) in example 1 is as follows:

| $B_{(T,S^*)}$ | $i=1$ | $i=2$ | $i=3$ | $\Psi(N, B_{(T,S^*)}, S^*)$ | $\sum_{i\in N} \Psi_i$ |
|---|---|---|---|---|---|
| {1} | 1 | 0 | 0 | (1,0,0) | 1 |
| {2} | 0 | $\frac{1}{2}$ | 0 | (0, $\frac{1}{2}$,0) | $\frac{1}{2}$ |
| {3} | 0 | 0 | $\frac{1}{2}$ | (0,0, $\frac{1}{2}$) | $\frac{1}{2}$ |
| {1,2} | $\frac{1}{2}$ | $\frac{1}{4}$ | 0 | ( $\frac{1}{2}$, $\frac{1}{4}$,0) | $\frac{3}{4}$ |
| {1,3} | $\frac{1}{2}$ | 0 | $\frac{1}{4}$ | ($\frac{1}{2}$,0,$\frac{1}{4}$) | $\frac{3}{4}$ |
| {2,3} | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | (0,$\frac{1}{2}$,$\frac{1}{2}$) | 1 |
| {1,2,3} | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | ($\frac{1}{3}$,$\frac{1}{3}$,$\frac{1}{3}$) | 1 |

Again we apply the unanimity games on the definition of the free solidarity value (Eq.3) as given in the paper.

| $B_{(T,S^*)}$ | $i=1$ | $i=2$ | $i=3$ | $\Phi^{fs}(N, B_{(T,S^*)}, S^*)$ | $\sum_{i\in N} \Phi_i^{fs}$ |
|---|---|---|---|---|---|
| {1} | 1 | 0 | 0 | (1,0,0) | 1 |
| {2} | 0 | $\frac{1}{2}$ | 0 | (0, $\frac{1}{2}$,0) | $\frac{1}{2}$ |
| {3} | 0 | 0 | $\frac{1}{2}$ | (0,0, $\frac{1}{2}$) | $\frac{1}{2}$ |
| {1,2} | $\frac{5}{12}$ | $\frac{1}{4}$ | 0 | ( $\frac{5}{12}$, $\frac{1}{4}$,0) | $\frac{2}{3}$ |
| {1,3} | $\frac{5}{12}$ | 0 | $\frac{1}{4}$ | ($\frac{5}{12}$,0,$\frac{1}{4}$) | $\frac{2}{3}$ |
| {2,3} | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | (0,$\frac{1}{2}$,$\frac{1}{2}$) | 1 |
| {1,2,3} | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | ($\frac{1}{3}$,$\frac{1}{3}$,$\frac{1}{3}$) | 1 |

In the above payoff matrix, we see that efficiency does not hold for all the unanimity games. In general, the unanimity games are in partial agreement, restricted to those unanimity games which has all solidary players with one or more or all non-solidary players, including only all the solidary players. In order to maintain efficiency for all the unanimity games we need to consider the Shapley value distribution payoffs for the games that do not have all the solidary players. For which we modify the unanimity games by adding an element implementing the Shapley value.
In the light of which, we implement the following Shapley distribution when players from the already formed solidary group move out of the solidary group.

$$\Phi_i^{fs}(N, v, S^*) := \begin{cases} \sum_{i\in S} \frac{(n-s)!(s-1)!}{n!}[v(S) - v(S \setminus \{i\})] & \text{for } i \in N \end{cases} \tag{5}$$

# 4 Definitions

**1. Solidary group:** In a game, when players voluntarily agrees to form a group to promote the common interest with mutual support of its members, we say the players form a solidary group. In this case, players are known as solidary players. We denote this coalition group by $S^*$.
**2. Non-Solidary group:** In a game, when players do not join the solidary group, they naturally give rise to a non-solidary group. In this case, players are known as non-solidary players.
**3. Partial participation of solidary players:** We introduce the notion of partial participation of solidary players in the context of unanimity games where all members of the solidary group do not participate. Then in all those unanimity games the remaining solidary players are treated as non-solidary players.
In other words, if after the voluntary formation of the solidary group, if certain players decide to step out of the solidary group then for the particular game the remaining solidary group members will also be treated as non-solidary players.
Nonetheless, this does not change the spirit of free solidarity which forms the original notion of construction of the unanimity games in their paper. To maintain the solidarity property of solidary group all members of group should fully participate in the game as a solidary player. So, keeping the original axioms we add two axioms that will compensate for the discrepancy that arises in the original construction of the unanimity games.

## 4.1 Axiomatic Characterization of the Free Solidarity value

Let $\Psi^{fs}(N, v, S^*)$ be a value of a cooperative TU-game $\mathcal{G}^N$. The following are the axioms for $\Psi^{fs}$:

1. Efficiency ($\mathcal{E}$): For any $(N, v, S^*) \in \mathcal{G}^N$, we have,
   $\sum\limits_{i\in N} \Psi_i^{fs}(N, v, S^*) = v(N)$.

2. Additivity ($\mathcal{A}$): For any $(N, v, S^*), (N', w, T^*) \in \mathcal{G}^N$
   with $N = N', S^* = T^*$, we have, $\Psi_i^{fs}(N, v+w, S^*) = \Psi_i^{fs}(N, v, S^*) + \Psi_i^{fs}(N, w, S^*)$.

3. Conditional symmetry ($\mathcal{C} - S$): If $i, j \in N$ are symmetric players in $(N, v, S^*)$, then $\Psi_i^{fs}(N, v, S^*) = \Psi_j^{fs}(N, v, S^*)$ if either $i, j \in S^*$ or $i, j \in S \setminus S \cap S^*$.

4. Partial Null Player Property($\mathcal{P} - N$): For any game $(N, v, S^*) \in \mathcal{G}^N$, if $i \in S \setminus S \cap S^*$ has the null player property then $\Psi_i^{fs}(N, v, S^*) = 0$. Moreover, if there is partial participation of solidary players then any $i \in N$ with the null player property has $\Psi_i^{fs}(N, v, S^*) = 0$.

5. Partial positivity($\mathcal{P} - P$): For any game $(N, v, S^*) \in \mathcal{G}^N$, if there is no partial participation of solidary players then $\Psi_i^{fs}(N, v, S^*) > 0$ for every $i \in S^*$.

6. Unaffected allocation of non-solidary players: When players decide freely to be solidary to form a solidary group $S^*$, it will neither affect the allocation of the players who have chosen to stay out of the group nor the value $v$. For every $S \subseteq N$, $v(S, S^*) = v(S, \emptyset)$ and $\forall i \notin S^* \neq \emptyset$, $\Psi_i^{fs}(N, v, S^*) = \Psi_i^{fs}(N, v, \emptyset)$.

7. Unreserved allocation of solidary players: When partial participation of solidary players in a game takes place due to some external or internal factors, all the solidary players are treated as non-solidary and hence their payoffs is given by the Shapley distribution.
   $\Psi_i^{fs}(N, v, S^* \setminus j) = Sh_i(N, v)$ where $j$ denotes the number of non participating solidary players.

**Theorem 4**. A value $\Psi^{fs} : \mathcal{G}^N \to R^n$ satisfies efficiency, additivity, conditional symmetry, partial null player property, partial positivity, unaffected allocation of non-solidary players, and unreserved allocation of solidary players axioms if and only if $\Psi^{fs} = \Phi^{fs}$, that is, $\Psi^{fs}$ is the free solidarity value.

We define the reconstructed unanimity game as follows:
**Definition 4**

$$B_{(T,S^*)}(S) = \begin{cases} \frac{|T\cap S^*|}{|T|}\binom{|S\cap S^*|}{|T\cap S^*|}^{-1} + \frac{|T\setminus(T\cap S^*)|}{|T|} & \text{if } T \subseteq S \text{ when all solidary players are in } T \\ 1 \quad \text{if } T \subseteq S \text{ when all solidary players are not in } T \\ 0 \qquad \text{otherwise} \end{cases} \tag{6}$$

**Lemma 4.1:** The unanimity game $B_{(T,S^*)}(S)$, $\forall T \subseteq S$ with $T \neq \emptyset$, has the following properties:

1. If $T = S$, then $B_{(T,S^*)}(S) = 1$.
2. If $T \subset S$ with $S^* = \emptyset$, then $B_{(T,\emptyset)}(S) = 1$.
3. If $T \subset S$ with $T \subset S^*$, then $B_{(T,S^*)}(S) = 1$.
4. The unanimity games with $B_{(T,S^*)}$ with $T$ containing all the solidary players has strictly $\tilde{m}^{av}(B_{(T,S^*)}, S) \neq 0$.
5. In the unanimity games, if $i \in S^*$ and $i \in N \setminus T$ with $T$ not containing all the solidary players and $T \subseteq S$, then we have, $\tilde{m}^{av}(B_{(T,S^*)}, S) \neq 0$ for every solidary player $i \in T$, that is,

$B_{(T,S^*)},(S) \neq \frac{1}{|S \cap S^*|} \sum_{j \in (S \cap S^*)} (B_{(T,S^*)},(S) - B_{(T,S^*)},(S \backslash j))$. This imply that $m_i(B_{(T,S^*)},S) = 0$ will hold for any $i \in N \setminus T$ having the null player property.

**Proof:** The proof follows immediately from the definition of the unanimity games. The parts 1, 2, 3 are direct. Also, the games $B_{(T,S^*)}$ with $T$ containing all the solidary players has strictly $\tilde{m}^{av}(B_{(T,S^*)},S) \neq 0$ for every solidary player $i \in T$. Hence, in this case none of the solidary players will have the A-null player property. Again, by the definition of games $B_{(T,S^*)}$ with $T$ containing all the solidary players will exhibit the null-player property if $i \in N \setminus T$ that is $m_i(B_{(T,S^*)},S) = 0$.

**Lemma 4.2:** Let $(N,v,S^*)$ be a game and $S^*$ be the coalition group of solidary players with $|S^*| \geq 2$. If there is no partial participation of solidary players in a game, then no player in $S^*$ is an A-null player.

**Proof:** Clearly, it follows from part 4 of the lemma 3.3.

**Remark 1.** We conclude that, the A-null player condition will not hold for the solidary players and for which we imply that a player will only have the null player property. So all players regardless of being solidary or non solidary will only possess the null player property.

**Lemma 4.3:** The set $\mathcal{B} = \{B_{(T,S^*)}(S) | T \neq \emptyset\}$ of games is a basis for the vector space $\mathcal{G}^N$.

We follow the proof as given in Nowak and Radzik [12]. Firstly, $\mathcal{G}^N$ is a K-dimensional vector space. $K = 2^n - 1$ is the number of possible coalitions of the player set $N$ excluding the empty set $\emptyset$. Suppose that $T_1, T_2, ..., T_k$ be a fixed sequence of non-empty coalitions of the player set $N$ satisfying $1 = |T_1| \leq |T_2| \leq ... \leq |T_k| = n$. We define a $KXK$ matrix $A = [a_{ij}]$ as $a_{ij} = B_{(T_i,S^*)}(S_j)$, for $i,j = 1,2,...,k$. By the definition of unanimity game (*Equation* 6), it follows that matrix $A$ is a triangular matrix with all diagonal entries equal to 1. This implies $|A| \neq 0$. Hence, the set $\{B_{(T_i,S^*)} : i = 1,2,...,k\}$ forms $k$ linearly independent vectors in the vector space $\mathcal{G}^N$. So, the set $\mathcal{B}$ is a basis.

**Lemma 4.4:** If the value $\Psi^{fs}$ satisfies efficiency, additivity, symmetry, partial null player property, partial positivity, unaffected allocation of non-solidary players and unreserved allocation of solidary players axioms, then for every player $i \in N$, non-empty coalition $T$, and a real constant $\beta$ we have the following two cases:

**Case 1: When all solidary players are in T**

$$\Psi_i^{fs}(N, \beta B_{(T,S^*)}, S^*) = \begin{cases} \frac{\beta}{|T|}\binom{|S^*|}{|T \cap S^*|}^{-1} & \text{if } i \in T \cap S^* \\ \frac{\beta}{|T|} & \text{if } i \in T \setminus (T \cap S^*) \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

**Case 2: When all solidary players are not in T**

$$\Psi_i^{fs}(N, \beta B_{(T,S^*)}, S^*) = \begin{cases} \frac{\beta}{|T|} & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

**Proof:** Suppose that $\emptyset \neq T \subseteq N$. As $\beta$ is any real number, it follows that $\beta B_{(T,S^*)}$ is a game. Consider $\beta = 0$, then the lemma follows from efficiency and symmetry axioms. Assume that $\beta \neq 0$. For $T \subseteq N$, in both the cases of $T$ consisting of all solidary and not all solidary players, every player $i \in N \setminus T$ will be a null player in the game $\beta B_{(T,S^*)}$ by lemma 3.4. Hence $\Psi_i^{fs}(N, \beta B_{(T,S^*)}, S^*) = 0$ for each $i \in N \setminus T$. By efficiency, the two cases can be derived.

**Illustration using example 1:**

The matrix of the new unanimity game $(N, B_{(T,S^*)}, S^*)$ based on example 1,
$N = \{1,2,3\}$; $S^* = \{2,3\}$ is as follows:

| $B_{(T,S^*)}(S)$ | {1} | {2} | {3} | {1,2} | {1,3} | {2,3} | {1,2,3} |
|---|---|---|---|---|---|---|---|
| {1} | 1 | 0 | 0 | 1 | 1 | 0 | 1 |
| {2} | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| {3} | 0 | 0 | 1 | 0 | 1 | 1 | 1 |
| {1,2} | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| {1,3} | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| {2,3} | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| {1,2,3} | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

Then, the payoff matrix for free solidarity value (Eq. 7 and 8) in example 1 is as follows:

| $B_{(T,S^*)}$ | $i=1$ | $i=2$ | $i=3$ | $\Psi^{fs}(N, B_{(T,S^*)}, S^*)$ | $\sum_{i\in N} \Psi_i^{fs}$ |
|---|---|---|---|---|---|
| {1} | 1 | 0 | 0 | (1,0,0) | 1 |
| {2} | 0 | 1 | 0 | (0,1,0) | 1 |
| {3} | 0 | 0 | 1 | (0,0,1) | 1 |
| {1,2} | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $(\frac{1}{2}, \frac{1}{2}, 0)$ | 1 |
| {1,3} | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $(\frac{1}{2}, 0, \frac{1}{2})$ | 1 |
| {2,3} | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $(0, \frac{1}{2}, \frac{1}{2})$ | 1 |
| {1,2,3} | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ | 1 |

Again we apply the unanimity games on the definition of the free solidarity value (Eq. 3) as given in the paper.

| $B_{(T,S^*)}$ | $i=1$ | $i=2$ | $i=3$ | $\Phi^{fs}(N, B_{(T,S^*)}, S^*)$ | $\sum_{i\in N} \Phi_i^{fs}$ |
|---|---|---|---|---|---|
| {1} | 1 | 0 | 0 | (1,0,0) | 1 |
| {2} | 0 | 1 | 0 | (0,1,0) | 1 |
| {3} | 0 | 0 | 1 | (0,0,1) | 1 |
| {1,2} | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $(\frac{1}{2}, \frac{1}{2}, 0)$ | 1 |
| {1,3} | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $(\frac{1}{2}, 0, \frac{1}{2})$ | 1 |
| {2,3} | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $(0, \frac{1}{2}, \frac{1}{2})$ | 1 |
| {1,2,3} | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ | 1 |

**Remark:** By axiom of additivity, lemma 4.3 and lemma 4.4, we have the following lemma.

**Lemma 4.5:** Any value that satisfies axioms A1-A7 is a linear mapping from $\mathcal{G}^N$ into $R^n$.

## Proof of the theorem:

Existence: We first prove the existence of a value which satisfies the axioms 1 to 7. Firstly, the efficiency of $\Phi^{fs}$ is justified by the fact that $\Phi^{fs}$ is efficient for any unanimity game $B_{(T,S^*)}$. Let $v \in \mathcal{G}^N$. Then, there exist constants $\lambda_T, \emptyset \neq T \subset N$, such that $v = \sum_{\emptyset \neq T \subset N} \lambda_T B_{(T,S^*)}$

To show that $\Phi^{fs}$ is efficient, we use the linearity of $\Phi^{fs}$.
$\sum_{i\in N} \Phi_i^{fs}(v) = \sum_{\emptyset \neq T \subset N} \lambda_T \sum_{i\in N} \Phi_i^{fs}(B_{(T,S^*)}) = \sum_{\emptyset \neq T \subset N} \lambda_T B_{(T,S^*)}(N) = v(N)$. Thereby, showing that $\Phi^{fs}$ is an efficient value.
It also follows that the value function $\Phi^{fs}$ satisfies conditional symmetry, partial null player condition, partial positivity, unaffected allocation of non-solidary players, and unreserved allocation of solidary players. $\Phi^{fs}$ is additive as it is a linear mapping.

Uniqueness: Let $\Psi^{fs}(N, B_{(T,S^*)})$ be a value function on $\mathcal{G}^N$ satisfying the axioms A1-A7. As $\beta$ is any real number, it follows that $\beta B_{(T,S^*)}$ is a game.
If $i \notin T$ and $T \not\subseteq S$ then $\Psi_i^{fs}(N, B_{(T,S^*)}) = 0$. If $i \notin T$ and $T \subseteq S$, that is, for any $i \notin N \setminus T$, $\beta B_{(T,S^*)}(S) = \beta B_{(T,S^*)}(S \cup i)$ which implies that $m_i(\beta B_{(T,S^*)}, S) = 0$.
Let $T_1 = T \setminus (T \cap S^*)$ and $T_2 = (T \cap S^*)$. For any two players $i, j \in T \setminus (T \cap S^*)$ or $i, j \in T \cap S^*$ with $i, j \notin S$, we have $T_1, T_2 \not\subset S$ which implies $T_1, T_2 \not\subset S \cup \{i\}$ and $T_1, T_2 \not\subset S \cup \{j\}$. It follows that for $T = T_1 \cup T_2$, $\beta B_{(T,S^*)}(S \cup i) = \beta B_{(T,S^*)}(S \cup j)$. Then, by conditional symmetry $\Psi_i^{fs}(N, \beta B_{(T,S^*)}) = \Psi_j^{fs}(N, \beta B_{(T,S^*)})$ for every $i, j \in T$. Moreover, $\sum_{i \in T} \Psi_i^{fs}(N, \beta B_{(T,S^*)}) = |T| \Psi_i^{fs}(N, \beta B_{(T,S^*)})$ for any $i \in T$.
If there is no partial participation of solidary players then
$\tilde{m}^{av}(B_{(T,S^*)}, S) \neq 0$ for every solidary player $i \in T$. This implies by the partial positivity axiom that $\psi_i^{fs}(N, B_{(T,S^*)}) > 0$.
As the unaffected allocation holds for non solidary players, in a similar way the unreserved allocation of solidary players hold. By definition of the unanimity game, if in a game solidary players withdraw from the formed solidary group, they are to be treated as non solidary. Hence no reservation is set for them as well. Hence by the axiom, partial participation of the solidary players will make it necessary for their payoffs to follow the Shapley distribution which is indicated to be $\Psi_i^{fs}(N, \beta B_{(T,S^*)}, S^* \setminus j) = Sh_i(N, \beta B_{(T,S^*)})$ where $j$ denotes the number of non participating solidary players. More precisely, in terms of unanimity game $\beta B_{(T,S^*)}$ with $T$ containing not all the solidary players.
By lemma 4.5, $\Psi^{fs}$ is a linear mapping. Applying lemma 4.4 to both $\Psi^{fs}$ and $\Phi^{fs}$, we observe that $\Psi^{fs}(B_{(T,S^*)}) = \Phi^{fs}(B_{(T,S^*)})$ for each unanimity game $B_{(T,S^*)}$. Hence, $\Psi^{fs}(v) = \Phi^{fs}(v)$ for every game $v \in \mathcal{G}^N$.

# References


[1] Aumann,R,J., Dreze,J,H., 1974. Cooperative games with coalition structures.International Game Theory, Vol.3, Issue 4, page 217-237, Physica-Verlag, Vienna.

[2] Calvo, E., 2008, Random marginal and random removal values, Internat. J. Game Theory 37, 533-564.

[3] Calvo, E., E. Gutierrez, 2010. Solidarity in games with a coalition structure, Mathematical Social Sciences, Elsevier, Vol. 60, 196-203.

[4] Calvo, E., E. Gutierrez, 2011. A value for cooperative games with coalition structure, Discussion Papers on Economic Behavior 03/11, ERICES, Universitat de Valencia.

[5] Calvo, E., E. Gutierrez, 2013. The Shapley-Solidarity value for games with a coalition structure. International Game Theory Review Vol.15, No.1 (2003) 1350002 (c) World Scientific Publishing Company.

[6] Calvo, E., E. Gutierrez, 2014. Axiomatic characterizations of the weighted solidarity values. Mathematical Social Sciences, Elsevier, vol. 71(C), 6-11.

[7] Hart, S., Kurz, M., 1983. Endogenous formation of coalitions. Econometrica, Vol.51, No.4.

[8] Dhaou B., Ziad, A. (2015). The Free Solidarity Value, Economics Working Paper Archive, Center for Research in Economics and Management (CREM).

[9] Hausken, K., 2020. The Shapley value of coalitions to other coalitions. Humanities and Social Sciences Communications.

[10] Kalai, E., Samet, D., 1987. On Weighted Shapley values. Internat. J. Game Theory 16, 205-222.

[11] Kamijo, Y., 2009. A Two-Step Shapley value for cooperative games with coalition structures. International game theory, Vol.11, No.2 (2009) 207-214 (C) world Scientific Publishing Company.

[12] Nowak, A.S., Radzik. T., 1994. A solidarity value for n-person transferable utility games. Internat. J. game Theory 23, 43-48.

[13] Nowak, A.S., Radzik. T., 1995. On axiomatization of the weighted Shapley values. Games Economic. Behav. 8, 389-405.

[14] Owen, G. (1977). Values of games with a priori unions. Mathematical Economics and Game Theory: Essays in Honor of Oskar Morgenstern (Eds.: O. Moeschlin R. Hein). Springer, New York.

[15] Radzik, T., 2012. A new look at the role of player's weights in the weighted Shapley value. European J. Oper. Res. 223, 407-416.

[16] Shapley, L.S., 1953a. Additive and non-additive Set functions (Ph.D. Thesis). Department of Mathematics, Princeton University.

[17] Shapley, L.S., 1953b. A value for n-person Games. In: Kuhn, H.W., Tucker., A.W. (Eds.), Contributions to the theory of games II. In: Annals of Mathematics Studies, vol. 28. Princeton University Press, Princeton, pp. 307-317.

[18] Sprumont, Y., 1990. Population monotonic allocation schemes for cooperative games with transferable utility. Games Econom. Behav. 2, 378-394.

[19] Su,J, Liang,Y et.al, 2020. Characterizations, Potential, and an Implementation of the Shapley-Solidarity Value. MDPI.

[20] X.-F. Hu, D.-F. Li, 2017. A new axiomatization of the Shapley-Solidarity value for games with a coalition structure. Operations Research Letters.