On pronic gracefulness of graphs

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1 Introduction

Graph labeling is a prospective research area due to its vital applications that could challenge our mind for eventual solutions. A graph labeling is an assignment of integers(values) to the vertices(points) or edges(lines) or both under certain conditions.
There are usually two types of labeling of graphs:

Quantitative Labeling is nothing but an assignment of some numbers to the elements of a graph and this labeling has persuaded research in a wide variety of applications in (synch-set codes)coding theory, radio-astronomy, spectral characterization of materials using crystallography etc., under certain constraints.

The assignment of qualitative nature to the vertices or edges of graph is called **Qualitative Labeling.** These labelings have influenced research in variant areas of human enquiry such as conflict resolutions in social psychology, electrical circuit theory, energy crises etc.,
The factors of graph labeling:

- A set of numbers(values) from which the labels are chosen.
- A rule that assigns a value to each edge.
- A condition that these numbers(values) must satisfy.

1.1 Graceful labeling on graphs

A graph which can be labeled gracefully is said to be a graceful graph. It is done by investigating such a graph with the labeling exists or not. Few results due to Golomb(1972) and Rosa(1967),(1977) are as folllows:

- A necessary condition for a (p, q)-graph $G(V, E)$ to be graceful is that, it is possible to partition its vertex set of $V(G)$ into two subsets V_1 and V_2 such that there are exactly $\lceil \frac{q}{2} \rceil$ edges each of which joins a vertex of V_1 with one of the vertex V_2.
- The necessary and sufficient condition for a complte graph K_n to be graceful is $n \leq 4$
- A cycle C_n of order n is graceful if and only if n is equivalent to 0 or $n \equiv 0(mod 4)$.

Few results on graceful labeling are listed below:

- Vaidya et al.(2009,2010,2011) analysed the gracefulness on certain family of graph.
- Uma and Murugesan(2012) discussed the graceful labeling on graphs and its subgraphs.
- Elumalai(2014) showed that cycle C_n with parellel edge extension admitss graceful labeling.
- Kaneria et al.(2014) analysed the gracefulness of $C_n(C_n)$ and $C_n(K_{m,n})$. Also Elumalai et al.(2015) showed the gracefulness of cycle with chords.

- The Fibonacci gracefulness of the paths, squares of paths P_n^2, Caterpillars are Fibonacci graceful and the bistar $B_{n,n}$ for $n \geq 5$ are showed by Kathiresan et al.(2010).
- Vaidya and Vihol(2011) proved that trees, switching of a vertex in a cycle, Joint Sum of two copies of fanadmits Fibonacci graceful labeling and *wheels and helms are not Fibonacci graceful.*

After going through a number of research works related to graceful labeling, in this chapter we defined and discussed pronic graceful labeling on different graph families.

1.2 Basic definitions

Definition 1.1. ***Shell Graph*** $C(n, n-3)$
A shell graph is a cycle C_n with $(n-3)$ chords sharing a common end point called the apex. Shell graphs is denoted by $C(n, n-3)$.
A multiple shell is defined to be a collection of shell with disjoint edges with a common apex. Hence a double shell consists of two edge disjoint shells with a common apex.

Definition 1.2. ***Wheel Graph*** $K_1 + C_n$, $n \geq 4$
In the mathematical discipline of graph theory, a wheel graph, denoted by $K_1 + C_n$, $n \geq 3$ is a graph formed by connecting a single universal vertex to all vertices of a cycle. A wheel graph with n vertices can also be defined as the 1-skeleton of $(n-1)$-gonal pyramid.

Definition 1.3. ***Gear graph*** G_n
The Gear graph G_n, is the graph obtained from the wheel graph by inserting a vertex between any two adjacent vertices in its cycle C_n.

Definition 1.4. ***Corona Product of two graphs*** H_1 ***and*** H_2
The corona product of two graphs H_1 and H_2 is the graph formed by taking one copy of H_1 and $V(H_1)$ copies of H_2 where i^{th} vertex of H_1 is adjacent to every vertex in the i^{th} copy of H_2.

Definition 1.5. ***Generalized Peterson graph*** $P(n, k)$
For natural numbers n and k, where $n > 2k$, a generalized Peterson graph $P(n, k)$ is the graph whose vertex set is $\{u_1, u_2, ..., u_n\} \cup \{ v_1, v_2, ..., v_n\}$ and its edge set is $\{u_i u_{i+1}, u_i v_i, v_i v_{i+k}, 1 \leq i \leq n\}$, where the subscript arithmetic is done modulo n using the residues $0, 1, 2, ..., n-1$.

Note: In $P(n, k)$,
⇒ For $n = 6$ and $k = 2$, the graph $P(6, 2)$ is said to be Durer Graph.
⇒ For $n = 8$ and $k = 3$, the graph $P(8, 3)$ is said to be Mobius Kantor Graph.
⇒ For $n = 10$ and $k = 2$, the graph $P(10, 2)$ is said to be Dodecahedran or a Dodecahedral Graph.
⇒ For $n = 10$ and $k = 3$, the graph $P(10, 3)$ is said to be Desargues Graph.
⇒ For $n = 12$ and $k = 5$, the graph $P(12, 5)$ is said to be Nauru Graph.
⇒ For $n = 24$ and $k = 5$, the graph $P(24, 5)$ is said to be Cubic Symmetric Graph.

Definition 1.6. ***Graceful Labeling***
Let G be a graph of order p and size q. A *graceful labeling* of G is an injection $f : V \to 0, 1, ..., q$ such that while each edge uv is assigned the label(absolute difference of the corresponding vertex labels), the induced edge labels are all distinct. Such a function g_f is called the induced edge function and a graph which admits such a labeling is called a graceful graph.

2 Graceful labeling using pronic numbers

Definition 2.1. ***Pronic Number:***
A pronic number is a number which is the product of two consecutive integers, that is, a number of the form $n(n+1)$. The study of these numbers dates back to Aristotle. They are also called oblong numbers, heteromecic or rectangular numbers. The n^{th} pronic number is the sum of the first n even integers. From the definition, it is seen that all pronic numbers are even, and the only prime pronic number is 2. Also the only pronic number in the Fibonacci sequence is 2.

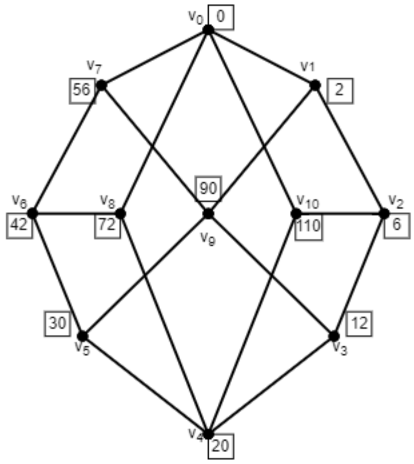

Figure 1: Herschel Graph-Pronic Graceful

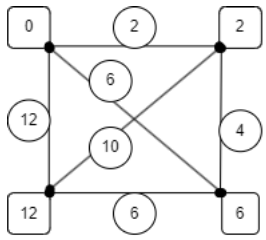

Figure 2: Not Pronic Graceful

Note 2.2. *A pronic number is squarefree if and only if n and n + 1 are also squarefree. The number of distinct prime factors of a pronic number is the sum of the number of distinct prime factors of n and $n+1$. 0, 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, 182, 210, 240, 272, 306, 342, 380, 420, 462 are few among them.*

Definition 2.3. ***Pronic Graceful Labeling:****[21][23][25]*

Let $G(p,q)$ be graph with $p \geq 2$. A **pronic graceful labeling** *of G is a bijection $f : V(G) \to \{0, 2, 6, 12, ..., p(p+1)\}$ such that the resulting edge labels obtained by $|f(u) - f(v)|$ on every edge uv are pairwise disjoint. A graph G is called pronic graceful if it admits pronic graceful labeling.*

Example 2.4. *An example for a graph which admits pronic graceful labeling is given in 1*

Example 2.5. *An example for a graph which does not admits pronic graceful labeling is given in 2*

2.1 Main theorems

Theorem 2.6. *Path graph P_n, $n \geq 3$ admits pronic graceful labeling.*

Theorem 2.7. *Cycle graph C_n, $n \geq 3$ admits pronic graceful labeling.*

Theorem 2.8. *Star graph $K_{1,n}$, $n \geq 3$ admits pronic graceful labeling.*

Theorem 2.9. *Path graph P_n, $n \geq 3$ admits pronic graceful labeling.*

Theorem 2.10. *Complete graph K_n, $n \geq 4$ does not admit pronic graceful labeling.*

Proof : If $n = 3$, the complete graph is nothing but the cycle graph of order 3 and it admits pronic graceful labeling is which is shown in previous theorem.
Assume the graph for $n \geq 4$.

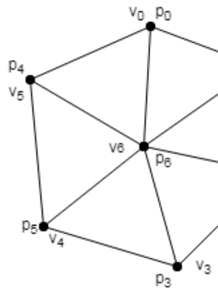

Figure 3: Pronic graceful labeling of Wheel graph $K_1 + C_6$

Let $\{v_0, v_1, v_2..., v_{n-1}\}$ be the vertices of K_n, $n \geq 4$ and are assigned the pronic numbers $p_0, p_1, ...p_{n-1}$. It is to be noted that the number "6" appears for the absolute difference of two pairs of pronic numbers (p_0, p_2) and (p_2, p_3).

Now as the given graph is complete, all edges of it are adjacent. Thus there exists two adjacent edges for which they are assigned by the label "6". Hence the complete graph does not admit pronic graceful labeling.

□

2.1.1 Wheel and shell related graphs

Theorem 2.11. *The wheel graph $K_1 + C_n$, $n \geq 4$ admits pronic graceful labeling.*

Proof : Let v_n be the apex vertex and $\{v_0, v_1, v_2..., v_{n-1}\}$ be the rim vertices of $K_1 + C_n$, $n \geq 4$.
Case (i): $n \neq 6, 10$
Define a bijection $f : V(G) \to \{p_0, p_1, ..., p_n\}$ by

$$f(v_i) = p_i, i = 0, 1, 2, ..., n-1 \qquad\qquad f(v_n) = p_n.$$

For the vertex labeling above, an induced edge function $f^* : E(G) \to N$ is given by

$$f^*(v_i v_{i+1}) = 2(i+1), i = 0, 1, 2, ..., n-2;$$
$$f^*(v_n v_i) = n(n+1) - i(i+1), i = 0, 1, 2..., n-1;$$
$$f^*(v_0 v_{n-1}) = (n-1)n.$$

The edges are hence labeled as follows:
(i)the labels of the edges $\{v_i v_{i+1},\ i = 0, 1, 2, ..n-2, v_0 n_{n-1}\}$ are $\{2, 4, 6, ..., 2(n-1), n(n-1)\}$.
(ii)the labels of the edges $\{v_n v_i, i = 0, 1, 2, ..., n-1\}$ are $\{n(n+1), (n+2)(n-1), (n+3)(n-2), ..., 2n\}$.
Hence the wheel graph admits pronic graceful labeling in this case.
Case (ii)$n = 6, 10$
Define a bijection $f : V(G) \to \{p_0, p_1, ..., p_{n-1}\}$ by

$$f(v_i) = \begin{cases} p_i, i = 0, 1, 2, ..., n-3; \\ p_{i-1}\ i = n-1; \\ p_{i+1}\ i = n-2; \\ p_n\ i = n \end{cases}$$

For the vertex labeling above, an induced edge function $f^* : E(G) \to N$ is given by

$$f^*(v_i v_{i+1}) = 2(i+1), i = 0, 1, 2, ..., n-2; \qquad f^*(v_0 v_{n-1}) = (n-1)(n-2);$$
$$f^*(v_n v_{n-2}) = 2n; \qquad f^*(v_n v_{n-1}) = 4n-2;$$
$$f^*(v_n v_i) = n(n+1) - i(i+1), i = 0, 1, 2, ..., n-3.$$

The edges are hence labeled as follows:
(i)the labels of the edges $\{v_i v_{i+1},\ i = 1, 2, ..n-2, v_0 n_{n-1}\}$ are $\{2, 4, 6, ..., 2(n-1), (n-2)(n-1)\}$.
(ii)the labels of the edges $\{v_n v_i, i = 0, 1, 2, ...n-3\}$ and $\{v_n v_{n-2}, v_n v_{n-1}\}$ are $\{n(n+1), (n+2)(n-$

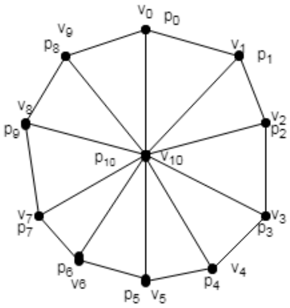

Figure 4: Pronic graceful labeling of Wheel graph $K_1 + C_{10}$

$1), (n+3)(n-2), ..., 6(n-1)\}$ and $\{2n, 4n-2\}$. Hence the wheel graph admits pronic graceful labeling in this case.
Thus the wheel graph admits its pronic gracefulness in both the cases. Thus the Wheel graph $K_1 + C_n$, for $n \geq 4$ admits pronic graceful labeling.

$\square$

Theorem 2.12 (4). *Gear graph G_n admits pronic graceful labeling*

Theorem 2.13 (4). *Helm Graph HG_n, admits pronic graceful labeling*

Theorem 2.14 (12). *A Shell Graph $C(n, n-3)$, for $n \geq 3$ admits pronic graceful labeling.*

Theorem 2.15 (12). *A Shell Butterfly Graph G admits pronic graceful labeling.*

2.1.2 PGL on corona product and joint sum of graphs

Theorem 2.16 (5). *Corona product $C_n \circ mK_1$ admits pronic graceful labeling.*

Theorem 2.17 (29). *Barycentric subdivision of cycle $C_n(C_n)$ admits pronic graceful labeling.*

Theorem 2.18 (16). *The joint sum of cycle C_m and C_n, $m, n \geq 3$ admits pronic graceful labeling.*

Proof : Let C_m and C_n, $m, n \geq 3$ be the cycles of order m and n.
Case(i):$m = n$**.**
Subcase(i):$m = n \geq 5$**.**
Let the vertices of the joint sum be $\{v_0, v_1, ...v_{m-1}, v_m, v_{m+1}, v_{m+2}, ...v_{2m-1}\}$ and the edges be $\{v_iv_{i+1}, i = 0, 1, 2, ..., m-1, m, m+1, m+2, ...2m-1\} \cup \{v_0v_{m-1}, v_mv_{2m-1}\}$. Let us connect the two graphs by the new edge $v_{m-1}v_m$ so that
$\{v_0, v_1, v_2..., v_{m-1}, v_m, v_{m+1}, v_{m+2}, ..., v_{2m-2}\}$ forms a spanning path in G.
Define a bijection $f : V(G) \to \{p_0, p_1, ..., p_{2m-1}\}$ by

$$f(v_i) = p_i, \ i = 0, 1, 2, ..., 2m-1.$$

For the vertex labeling above, an induced edge function $f^* : E(G) \to N$ is given by

$$f^*(v_iv_{i+1}) = 2(i+1), i = 0, 1, 2, ..., m-1, m, m+1, m+2, ...2m-2;$$
$$f^*(v_0v_{m-1}) = m(m-1);$$
$$f^*(v_mv_{m+n-1}) = 3m(m-1).$$

The edge labels are thus $\{2, 4, 6, ..., 2(m-1), 2m, 2(m+1), 2(m+2), ..., 2(2m-1), m(m-1), 3m(m-1)\}$ and hence in this case the joint sum of cycles admits pronic graceful labeling.
Subcase(ii):$m = n = 3, 4$**.**

Case(ii)$m \neq n$**.**
Let the vertices of C_m be $\{v_0, v_1, ...v_{m-1}\}$ and C_n be $\{v_m, v_{m+1}, v_{m+2}, ...v_{m+n-1}\}$.
Let the edges of the C_m and C_n are $\{v_iv_{i+1}, i = 0, 1, 2, ..., m-1, m, m+1, m+2, ..., m+n-2\} \cup$

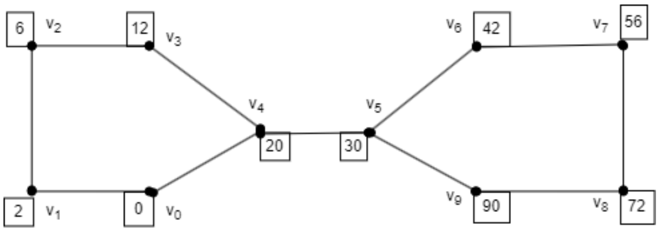

Figure 5: PGL of joint sum of two copies of C_3 and C_4

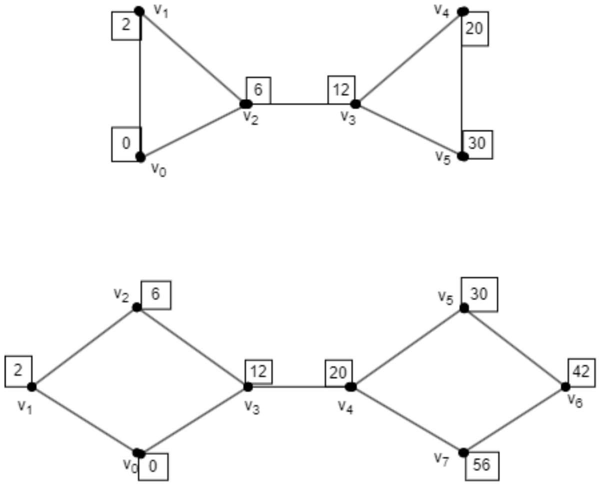

Figure 6: PGL of joint sum of two copies of C_3 and C_4

$\{v_0v_{m-1}, v_mv_{m+n-1}\}$. Let us connect the two graphs by the new edge $v_{m-1}v_m$ so that $\{v_0, v_1, v_2..., v_{m-1}, v_m, v_{m+1}, v_{m+2}, ..., v_{m+n-2}\}$ forms a spanning path in G.
Define a bijection $f : V(G) \to \{p_0, p_1, ..., p_{m+n-1}\}$ by
$f(v_i) = p_i,\ i = 0, 1, 2, ..., m + n - 1.$
For the vertex labeling above, an induced edge function $f^* : E(G) \to N$ is given by

$$f^*(v_iv_{i+1}) = 2(i + 1), i = 0, 1, 2, ..., m - 1, m, m + 1, m + 2, ...m + n - 2;$$
$$f^*(v_0v_{m-1}) = m(m - 1);$$
$$f^*(v_mv_{m+n-1}) = (n - 1)(2m + n).$$

The edge labels are thus $\{2, 4, 6, ..., 2(m - 1), 2m, 2(m + 1), 2(m + 2), ..., 2(m + n - 1), m(m - 1), (n-1)(2m+n)\}$ and hence in this case the joint sum of cycles admits pronic graceful labeling. Hence the joint sum of C_m and C_n, $m, n \geq 3$ admits pronic graceful labeling. □

Note 2.19. *(m,n)-tadpole and n-pan graphs admit pronic graceful labeling.*

2.2 Pronic Graceful Labeling of Bipartite Graphs

In this section, the labeling for complete bipartite graphs have been investigated.

Theorem 2.20. *The complete bipartite graph $K_{2,n}$ admits pronic graceful labeling.*

Proof : Let X and Y be the partition of vertices of $K_{2,n}$ and let $V(X) = \{u_0, u_1\}$ and $V(Y) = \{v_0, v_1, v_2, ..., v_{n-1}\}$. Hence $|V(X)| = 2; |V(Y)| = n \Rightarrow |V(K_{2,n})| = n + 2$.
Define a $f : V(G) \to \{p_0, p_1, ..., p_{n+1}\}$ by

$$f(u_i) = p_i, i = 0, 1;$$
$$f(v_i) = p_{i+2}, i = 0, 1, 2, ..., n - 1.$$

Clearly f is a bijection.

For the vertex labeling above, the induced edge labeling $f^* : E(G) \to N$ is given by

$$f^*(u_0v_i) = (i+2)(i+3), i = 0,1,2,3,...,n-1;$$
$$f^*(u_1v_i) = (i+1)(i+4), i = 0,1,2,3,...,n-1.$$

The distinct labels thus obtained for the edges u_0v_i and u_1v_i for $i = 0,1,2,...,n-1$ are $\{p_2, p_3, p_4, ..., p_{n+1}\}$ and $\{p_2 - 2, p_3 - 2, p_4 - 2, ..., p_{n+1} - 2\}$ which results the graph$K_{2,n}$ admits pronic graceful labeling. □

Theorem 2.21. *The complete bipartite graph $K_{3,n}$ admits pronic graceful labeling.*

Proof : Let X and Y be the partition of vertices of $K_{3,n}$ and let $V(X) = \{u_0, u_1, u_2\}$ and $V(Y) = \{v_0, v_1, v_2, ..., v_{n-1}\}$. Hence $|V(X)| = 3; |V(Y)| = n \Rightarrow |V(K_{3,n})| = n + 3$.
Define a $f : V(G) \to \{p_0, p_1, ..., p_{n+2}\}$ by

$$f(u_i) = p_i, i = 0,1,2;$$
$$f(v_i) = p_{i+3}, i = 0,1,2,...,n-1.$$

Clearly f is a bijection.

For the vertex labeling above, the induced edge labeling $f^* : E(G) \to N$ is given by

$$f^*(u_0v_i) = (i+3)(i+4), i = 0,1,2,3,...,n-1;$$
$$f^*(u_1v_i) = (i+2)(i+5), i = 0,1,2,3,...,n-1;$$
$$f^*(u_1v_i) = (i+1)(i+6), i = 0,1,2,3,...,n-1.$$

The distinct labels thus obtained for the edges $\{u_0v_i,\ u_1, v_i\}$ and $\{u_2v_i, i = 0,1,2,...,n-1$ are $\{p_3, p_4, ..., p_{n+2}\}$, $\{p_3 - 2, p_4 - 2, ..., p_{n+2} - 2\}$ and $\{p_3 - 6, p_4 - 6, ..., p_{n+2} - 6\}$ which results that $K_{3,n}$ admits pronic graceful labeling. □

Observation 2.22. The complete bipartite graph $K_{4,4}$ does not admit pronic graceful labeling.

For, the pronic number p_3, while commutes with the pronic numbers $\{p_0, p_1, p_2, p_5\}$ induces a label "k" which occurs twice for two different edges.i.e., the same label is assigned for different edges. The pairs are listed below:

$$(12,6) = (0,6); \qquad (12,2) = (20,30);$$
$$(12,0) = (30,42); \qquad (12,30) = (20,2).$$

Hence the above mentioned pronic numbers including p_3 must be assigned to same partition of vertices. Such a labeling is not possible since only 4 vertices are in one partition. Hence the $K_{4,4}$ does not admit pronic graceful labeling.

Problem 2.23. *Does there exist any n other than $n = 2,3$ for which the complete bipartite graph $K_{n,n}$ admits pronic graceful labeling?*

2.3 Pronic Graceful labeling of Generalized Peterson Graph $P(n,1)$

In graph theory, the **generalized Petersen graphs** are a family of cubic graphs formed by connecting the vertices of a regular polygon to the corresponding vertices of a star polygon. The Peterson Graph is the complement of the line graph of the complete graph K_5.

Alice Steimle and William Staton(2009) analysed the isomorphism classes of the generalized Petersen graphs. Zehui Shao et al(2017) proposed a backtracking algorithm with a specific static variable ordering and dynamic value ordering to find graceful labeling for generalized Petersen graphs and that algorithm is able to find gracefulness of generalized $P(n,k)$ with the number of vertices greater than or equal to 75 within several seconds.

Theorem 2.24. *Peterson graph $P(5,2)$ admits pronic graceful labeling.*

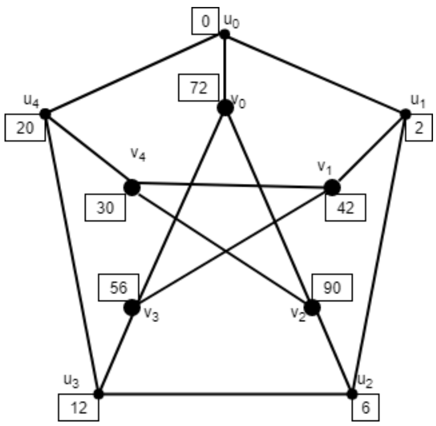

Figure 7: Pronic graceful labeling of Peterson graph $P(5,2)$

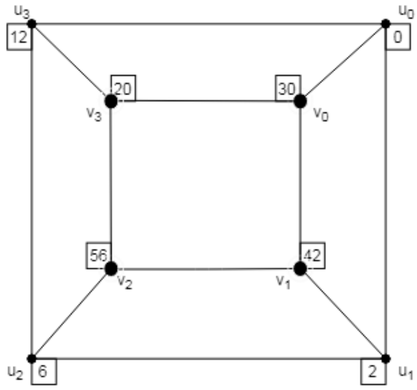

Figure 8: Pronic Graceful Labeling of Cubical Graph $P(4,1)$

Proof : Let $\{v_0, v_1, v_2, v_3, v_4\}$ be the inner vertices and $\{u_0, u_1, u_2, u_3, u_4\}$ be the outer vertices of $P(5,2)$. Define a bijection $f : V(G) \to \{p_0, p_1, p_2, ..., p_9\}$ by

$$f(u_i) = p_i, i = 0, 1, 2, 3, 4; \qquad f(v_{2i+1}) = p_{i+6}, i = 0, 1;$$
$$f(v_4) = p_5; \qquad f(v_{2i}) = p_{i+8}, i = 0, 1.$$

Clearly f is a bijection.

For the vertex labeling above, the induced edge labels are as follows:
(i) Consider the path in $\{u_0, u_1, u_2, u_3, u_4, v_4, v_1, v_3, v_0, v_2\}$ in $P(5,2)$. The edges of the path $\{u_iu_{i+1}, (0 \leq i \leq 3), u_4v_4, v_4v_1, v_1v_3, v_3v_0, v_0v_2\}$ are consecutively labeled by the numbers $\{2, 4, 6, ..., 2(2n-1)\}$.
(ii) the remaining edges are labeled as follows:

$$f^*(u_0u_{n-1}) = (n-1)n; \qquad f^*(u_{2i}v_{2i}) = 72 + 12i, i = 0, 1;$$
$$f^*(u_{2i+1}v_{2i+1}) = 40 + 4i, i = 0, 1; \qquad f^*(v_{n-3}v_{n-1}) = 20(n-2).$$

Hence the labels are $\{20, 72, 84, 40, 44, 60\}$ respectively. Thus the edge labels are distinct which results that the Peterson graph admits pronic graceful labeling. □

Theorem 2.25. *n-prism $P(n,1)$ for $n > 3$ admits pronic graceful labeling.*

Proof : Let $\{v_0, v_1, v_2, v_3, ..., v_{n-1}\}$ be the inner vertices and $\{u_0, u_1, u_2, u_3, ..., u_{n-1}\}$ be the outer vertices of $P(n,1)$.
Define a function $f : V(G) \to \{p_0, p_1, p_2, ..., p_{2n-1}\}$ as follows:

$$f(u_i) = p_i, i = 0, 1, 2, ...n-1;$$
$$f(v_i) = p_{i+1}, i = n-1;$$
$$f(v_i) = p_{i+(n+1)}, i = 0, 1, 2, ..., n-2.$$

Clearly f is a bijection.

Let A_1, A_2 and A_3 denote the set of edge labels of $\{u_iu_{i+1}(0 \leq i \leq n-2), u_{n-1}v_{n-1}, v_{n-1}v_0, v_iv_{i+1}, (0 \leq i \leq n-3\}$, $\{u_iv_i, (0 \leq i \leq n-2)\}$ and
$\{u_{n-1}u_0, v_{n-2}v_{n-1}\}$. Clearly the labels of the edges for the above sets are as follows:

$A_1 = \{2, 4, 6, ...4n-2\}$;
$A_2 = \{p_{n+1}, p_{n+1} + 2(n+1), p_{n+1} + 4(n+1), ..., p_{n+1} + 2(n-2)(n+1)\}$;
$A_3 = \{n(n-1), n[3(n-4)+9]\}$.

In the view of above defined labeling, it is observed that $A_1 \cap A_2 \cap A_3 = \phi$ and hence the Peterson graph $P(n, 1)$ admits its pronic gracefulness. $\square$

Example 2.26. 1.The pronic graceful labeling for Peterson graphs $P(10, 15)$ and $P(4, 1)$ are given in Figure 7 and Figure 8. The graph $P(4, 1)$ is also called as the cubical graph.

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