A STUDY ON GENERALIZED RICCI SOLITONS ON LP-SASAKIAN MANIFOLDS

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Abstract: In the present Chapter we studied LP-Sasakian manifold. At first we introduced historical background of the concern manifold. Next some rudimentary facts and related properties of LP-Sasakian manifold are discussed. After that LP-Sasakian manifold concerning generalized Ricci soliton is studied and investigate main result in the form of theorem that is LP-Sasakian manifold of odd dimension satisfying the generalized Ricci soliton equation is an Einstein manifolds.

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1. INTRODUCTION

An developing area of contemporary mathematics is the geometry of contact manifolds. The mathematical formalisation of classical mechanics has given way to the concept of contact geometry [7]. K- contact manifolds and sasakian manifolds are two significant kinds of contact manifolds [1], [20]. There are various researchers that have analyzed K-contact and Sasakian manifolds ([21], [3], [4], [11], [19], [23]) and many others.

The concept of the LP-Sasakian manifold was initially introduced by Matsumoto [13]. Mihai and Rosca defined the same notion independently in [16]. This type of manifold is also discussed in ([14, [22]). A complete regular contact metric manifold M^{2n+1} carries a K-contact structure (φ,ξ,η,g), which is described in terms of almost kaehler structure (J,G) of the base manifold's M^{2n+1} . If the base manifold (M^{2n+1},J,G) in this case is Kaehlerian, the K-contact structure (φ,ξ,η,g) is Sasakian. If (M^{2n+1},J,G) is only almost Kaehler then (φ,ξ,η,g) is only K-contact [1]. Recent research in [12] has demonstrated the existence of K-contact manifolds that are not Sasakian. Even yet, Sasakian and contact structures are intermediated by K-contact structures. Numerous writers, including [3, [4], [9], [19], [21], [23], have researched K-contact manifolds.

Let us consider a smooth function f on M, the gradient of f is defined by

(1.1) $g(grad\, f, X) = Xf$,

The Hessian of f defined by

(1.2) $(Hess\, f)(X, \Upsilon) = g(\nabla_X grad\, f, \Upsilon)$,

for all smooth vector fields X, Υ. For a smooth vector field X, we have ([15],[18])

(1.3) $X^b(\Upsilon) = g(X, \Upsilon)$.

The generalized Ricci soliton equation in a Riemannian manifold (M, g) is described by [18]

(1.4) $\ell_X g = -2c_1 X^b \cdot X^b + 2c_2 S + 2\lambda g$,

where $\ell_X g$ is the lie derivative of X, defined by

(1.5) $(\ell_X g)(\Upsilon, Z) = g(\nabla_\Upsilon X, Z) + g(\nabla_Z X, \Upsilon)$,

for all vector fields X, Υ, Z and $c_1, c_2, \lambda \in R$. For different values of and equation (1.4) is a generalization of killing equation $(c_1 = c_2 = \lambda = 0)$, for homotheties $(c_1 = c_2 = 0)$, Ricci soliton $(c_1 = 0, c_2 = -1)$, vaccum near-horizon geometry equation $(c_1 = 1, c_2 = \frac{1}{2})$ etc. We suggest the reader for further information ([2], [5], [6], [10], [18]).

If $X = grad\, f$, then the equation for the generalized Ricci soliton is [8]

(1.6) $Hess\, f = -c_1 df \cdot df + c_2 S + \lambda g$.

The work in present Chapter motivated by [8], for the fact that relationship between LP-Sasakian and K-contact manifold, so we studied (2n+1)-dimensional Lorentzian para- Sasakian manifold over generalized Ricci soliton.

2. PRELIMINARIES

A (2n+1)-dimension differentiable manifold will be LP-Sasakian manifold [13] [16], if it admits the (1,1) tensor field φ, vector field ξ, η is a 1 form on M and g is a lorentzian metric, satisfy [14],[17]

(2.1) $\varphi^2 = I + \eta(X)\xi,\ \eta(\xi) = -1,\ g(\varphi X, \varphi \Upsilon) = g(X, \Upsilon) - \eta(X)\eta(\Upsilon)$,

(2.2) $\varphi(\xi) = 0,\ \eta(\varphi X) = 0,\ g(X, \xi) = \eta(X),\ g(\varphi X, \Upsilon) = -g(X, \varphi \Upsilon)$, (2.3) $\nabla_X \xi = \varphi X,$

(2.4) $g(R(\xi, X)\Upsilon, \xi) = \eta(R(\xi, X)\Upsilon) = -g(X, \Upsilon) - \eta(X)\eta(\Upsilon),$

(2.5) $R(\xi, X)\xi = X + \eta(X)\xi,$

(2.6) $S(X, \xi) = (n-1)\eta(X),$

(2.7) $(\nabla_X \varphi)\Upsilon = [g(X, \Upsilon) + \eta(X)\eta(\Upsilon)]\xi + [X + \eta(X)\xi]\eta(\Upsilon),$

as any vector fields, X, Υ on $\chi(M)$.

Additionally, a manifold is considered Einstein if its Ricci tensor has the following form:

(2.8) $S(X, \Upsilon) = ag(X, \Upsilon),$

for vector fields X, Υ.

Substituting $X = \Upsilon = \xi$ in (2.6) and then (2.4) and (2.2), we get

(2.9) $a = (n-1),$

Take in account (2.9) we obtain from (2.8)

(2.10) $S(X, \Upsilon) = (n-1)g(X, \Upsilon),$

similarly from (2.10) we infer

(2.11) $QX = (n-1)X,$

3. GENERALIZED RICCI SOLITON ON LP-SASAKIAN MANIFOLD

Theorem 3.1. Let $(M, \varphi, \xi, \eta, g)$ be a LP-Sasakian manifold then

(3.1) $(\ell_\xi(\ell_X g))(\Upsilon, \xi) = -g(X, \Upsilon) + g(\nabla_\xi \nabla_\xi X, \Upsilon) + \Upsilon g(\nabla_\xi X, \xi),$

for smooth vector fields X, Υ with Υ orthogonal to ξ.

Proof: It is known that

(3.2) $(\ell_\xi(\ell_X g))(\Upsilon, \xi) = \xi((\ell_X g)(\Upsilon, \xi)) - (\ell_X g)(\ell_\xi \Upsilon, \xi),$

using (1.5) in (3.2) yields

$$
\begin{aligned}
(3.3)\qquad &(\ell_\xi(\ell_X g))(\Upsilon,\xi)=\xi(g(\nabla_\Upsilon X,\xi)+g(\nabla_\xi X,\Upsilon)-g(\nabla_{[\xi,\Upsilon]}X,\xi)\\
&-g(\nabla_\xi X,[\xi,\Upsilon])=g(\nabla_\xi\nabla_\Upsilon X,\xi)+g(\nabla_\Upsilon X,\nabla_\xi\xi)+g(\nabla_\xi\nabla_\xi X,\Upsilon)\\
&+g(\nabla_\xi X,\nabla_\xi\Upsilon)-g(\nabla_{[\xi,\Upsilon]}X,\xi)-g(\nabla_\xi X,\nabla_\xi\Upsilon)+g(\nabla_\xi X,\nabla_\Upsilon\xi)\\
&=g(\nabla_\xi\nabla_\Upsilon X,\xi)+g(\nabla_\Upsilon X,\nabla_\xi\xi)+g(\nabla_\xi\nabla_\xi X,\Upsilon)\\
&-g(\nabla_{[\xi,\Upsilon]}X,\xi)+g(\nabla_\xi X,\nabla_\Upsilon\xi),
\end{aligned}
$$

by definition of Riemannian curvature tensor, from (3.3) it follows that

(3.4) $(\ell_\xi(\ell_X g))(\Upsilon,\xi)=\mathrm{g}(R(\xi,\Upsilon)X,\xi)+g(\nabla_\xi\nabla_\xi X,\Upsilon)(\ell_X g)+\Upsilon g(\nabla_\xi X,\xi),$

using (2.4) in (3.4) and with Υ orthogonal to ξ, we get

(3.5) $\mathrm{g}(R(\xi,\Upsilon)X,\xi)=-g(X,\Upsilon),$

so, (3.4) may be expressed as

(3.6) $(\ell_\xi(\ell_X g))(\Upsilon,\xi)=-\mathrm{g}(X,\Upsilon)+g(\nabla_\xi\nabla_\xi X,\Upsilon)+\Upsilon g(\nabla_\xi X,\xi),$

Lemma 3.2: Let M be a Riemannian manifold and let f be a smooth function. Then [15]

(3.7) $(\ell_\xi(df\cdot df))(\Upsilon,\xi)=\Upsilon(\xi(f))\xi(f)+\Upsilon(f)\xi(\xi(f)),$

for every vector field Υ.

Theorem 3.2: Let (M,φ,ξ,η,g) is a LP-Sasakian manifold which satisfies the generalized Ricci soliton equation. Then

(3.8) $\nabla_\xi grad\, f=(\lambda+(n-1)c_2 n)\xi-c_1\xi(f)grad\, f.$

Proof: Using (2.6) we have

(3.9) $\lambda\eta(\Upsilon)+c_2 S(\xi,\Upsilon)=[\lambda+(n-1)]\eta(\Upsilon).$

Making use of (1.6) and (3.9) implies

(3.10) $(Hess\, f)(\xi,\Upsilon)=-c_1\xi(f)g(grad,\Upsilon)+[\lambda+(n-1)]\eta(\Upsilon).$

The lemma thus follows from (3.5) and (1.6), which gives the Hessian definition.

Next, Suppose that is Υ orthogonal to ξ. From Lemma 3.1, and taking $X=grad\, f$, we get

(3.11) $2(\ell_\xi(Hess\, f)(\Upsilon,\xi)=\Upsilon(f)+g(\nabla_\xi\nabla_\xi grad\, f,\Upsilon)+\Upsilon g(\nabla_\xi grad\, f,\xi),$

by Lemma (3.2) and above equation, we obtain

$$(3.12)\quad \begin{aligned} 2(\ell_{\xi}(Hess\, f)(\Upsilon,\xi) &= \Upsilon(f)+(\lambda+(n-1)c_2)g(\nabla_{\xi}\xi,\Upsilon)-c_1 g(\nabla_{\xi}(\xi(f)grad\, f),\Upsilon) \\ &\quad +(\lambda+(n-1)c_2))\Upsilon g(\xi,\xi)-c_1\Upsilon(\xi(f)^2), \end{aligned}$$

since and from equation (2.10), we obtain

$$(3.13)\quad 2(\ell_{\xi}(Hess\, f)(\Upsilon,\xi)=\Upsilon(f)-c_1\xi(\xi(f)\Upsilon(f)-c_1\xi(f)g(\nabla_{\xi}(grad\, f,\Upsilon)-2c_1(\xi(f)\Upsilon(\xi(f)),$$

Note that, from equation (2.3) , we have $\ell_{\xi}g=0$ it implies . Using the above fact and taking the Lie derivative to the generalized Ricci soliton equation (1.6) yields

$$(3.14)\quad 2(\ell_{\xi}(Hess\, f)(\Upsilon,\xi)=-2c_1(\ell_{\xi}(df\circ df))(\Upsilon,\xi).$$

Using (3.13), (3.14) and Lemma (3.2) we infer that

$$(3.15)\quad \Upsilon(f)[1+c_1\xi\xi(f)+c_1\xi(f^2)]=0.$$

According to Lemma 3.2 we have

$$(3.16)\quad \begin{aligned} c_1\xi(\xi(f)) &= c_1\xi g(\xi,grad\, f) \\ &= c_1 g(\xi,\nabla_{\xi}grad\, f) \\ &= c_1(\lambda+(n-1)c_2)-c_1^{\,2}\xi(f)^2, \end{aligned}$$

by equation (3.15) and (3.16), we obtain

$$(3.17)\quad \Upsilon(f)[1+c_1(\lambda+(n-1)c_2)]=0.$$

Which implies $\quad\Rightarrow\quad \Upsilon(f)0.$

Provided $1+c_1(\lambda+(n-1)c_2\neq 0$. Therefore $grad\, f$ is parallel to ξ. Hence $grad\, f$ as $d=\ker\eta$ is nowhere integrable, that is, f is a constant function. Thus the manifold is an Einstein one follows from (1.6), so we concluded that

Theorem 3.3: If (M,φ,ξ,η,g) is a LP-Sasakian manifold of odd dimension, which satisfies the generalized Ricci soliton equation with $c_1(\lambda+(n-1)c_2\neq -1$. Then f has a constant value. Additionally, manifold is an Einstein manifold if $c_2\neq 0$. The lemma thus follows from (3.5) and (1.6), which gives the Hessian definition.

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