Nonlinear Dynamics of Reaction Diffusion systems: Turing’s Analysis

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ABSTRACT

The target of this chapter is to give a brief outline about Nonlinear Dynamics of reaction diffusion systems. When diffusion is coupled with chemical kinetics some interesting spatio temporal structures arise such as stationary patterns, spatially-varying concentrations of chemical species, spirals, traveling waves, solitons, targets etc. In view of Alan Turings phenomenological theory on Morphogenesis the pattern formation of reaction diffusion systems can be understood. The theoretical prediction of spatio-temporal instability and its experimental demonstration is illustrated in this chapter.

Keywords—stability, Nonlinear dynamics, diffusion, bifurcation, instability.

#  INTRODUCTION

 The study of Non linear dynamics is an important tool for understanding various bio physical and chemical phenomena such as glycolytic oscillations, Ca+2 oscillations, circadian cycles, cell cycles and so on[1-4]. In one of the previous book chapters the emergence of oscillation from Nonlinear dynamical point of view had already been discussed [5]. When reaction kinetics is accompanied with diffusion, nonlinear dynamics of the system becomes more significant. One encounters stationary patterns, spatially-varying concentration of chemical species, spirals, traveling waves, solitons, targets, wave propagations etc. These spatial and spatio-temporal structures can be realized by coupling the reaction part with diffusion at discrete spatial points [6,7]. The dynamics of these systems are governed by reaction-diffusion equations. Alan Turing, in one of his seminal papers on the theory of Morphogenesis [8], had revealed the mystery behind the structural evolution of patterns in biological systems, such as, coat patterns of animals like tigers, zebras, pigmentation patterns of fish, spots, stripes and spiral patterns of various biological species. Turing’s analysis illustrates that the necessary and sufficient condition for generation of pattern is the disparity in diffusivities of the reacting species. Because of the lack of understanding of an open thermodynamic system, experimental observation of Turing pattern remained inconclusive for almost forty years after its theoretical prediction. With gradual development of suitable experimental techniques unambiguous experimental evidences on Turing pattern were clearly explained in the last decade of twentieth century[9,10]. Thus the theoretical prediction of spatio-temporal instability and its experimental demonstration have now opened up a new horizon in the field of reaction-diffusion systems. In this present chapter we will focus on some of the primary features of the spatio temporal structures.

# TURING’S ANALYSIS

Let us think A and B are two reactants.. Without diffusion they react with each other to reach some steady state. Now the stability of this steady state depends upon the parameter space. Turing raised the question whether diffusion can bring in instability to a previously stable steady state. After Turing’s analysis it is concluded that instability is possible provided the rates of diffusion of the two species widely differ among themselves. The idea is novel as it contradicts the well known stabilizing effect of diffusion. Before going through the analytical details let us grasp the details of the theory intuitively. Let us consider an auto-catalytic reaction which involves an ’activator’ and an ‘inhibitor’. Activator diffuses much slowly than inhibitor. Let by any chance a small region of space sees a sudden rise in activator concentration. This subsequently enhances the autocatalytic reaction rate and as a result concentrations of both the species A and B increase. Now the inhibitor diffuses out at a faster rate compared to the activator. The result is that a small region in space becomes richer in activator surrounded by an area richer in inhibitor. Thus any small perturbation of concentration of any reacants grows with time and an inhomogeneity developes. Let us now present the analysis of Turing in a somewhat modified form.

   

Figure 1: Different types of coat patterns generated in animals fish. This is predicted theoretically[10]

 Let us consider a kinetic model of two chemical species in one dimension where one is acting as an activator *u*(*x,t*) and other one is an inhibitor *v*(*x,t*) then the corresponding reaction-diffusion equations are given by

$$u\_{t}=γf\left(u,v\right)+∇^{2}u$$

 $v\_{t}=g\left(u,v\right)+d∇^{2}v$ (1).

where *d* = *Dv/Du* , is the ratio of the diffusion coefficients of the species *v* and *u*, respectively. The parameter space of the system has to be chosen in such a way that without diffusion both *u* and *v* tend to linearly stable steady state which is homogeneous in nature. That means the concentration of the chemical species remains constant throughout the space. In presence of diffusion with variable diffusion coefficients the spatially inhomogeneous patterns may develop which are stationary in time under certain conditions by diffusion-driven instability. Diffusion is usually considered as a stabilizing process. Diffusion makes homogeneity out of inhomogenous nature. But in this case diffusion is responsible for causing the instability. We are in search for the necessary and sufficient conditions for diffusion-driven instability of the homogeneous steady state and the generation of spatial pattern for such a general system given by Eq 1. As per Turing’s analysis the boundary conditions are chosen to be zero flux. The relevant homogeneous steady state ($u\_{0},v\_{0}$) of Eq 1 is given by

$$f\left(u\_{0},v\_{0}\right)=0$$

$$g\left(u\_{0},v\_{0}\right)=0$$

 (2).

Now in absence of diffusion *u* and *v* the dynamical equations become

$$u\_{t}=γf\left(u,v\right)$$

$$v\_{t}=g\left(u,v\right)$$

 (3).

The equations are linearised around the steady state $\left(u\_{0},v\_{0}\right)$. (*u,v*) is changed to $u\_{0}+δu$ and $v\_{0}+δv$. where $δu $and$ δv$ are the small amount of perturbations applied. We ultimately arrive at the dynamical equations for the perturbations $δu$ and $δv$as,

 (4)

where *A* is the stability matrix . The matrix elements are the partial derivatives of *f* and *g* with respect to *u* and *v* respectively. The partial derivatives are evaluated at the steady state ($u\_{0},v\_{0}$) . *y* has the form

 (5)

The solutions of the above dynamical equations Eq 4 must be of the form $y\~e^{λt}$ where *λ* is the eigenvalue. For the steady state to be linearly stable the real part of the eigenvalue should be negative which assumes that the perturbation *y→*0 as t tend*s* to infinity. That results the stability of the steady state in absence of diffusion. We proceed in the same way and obtain the following algebraic equation for the eigenvalues

$ λ^{2}- γ\left(f\_{u}+g\_{v}\right)λ+γ^{2}(f\_{u}g\_{v}-g\_{u}f\_{v})=0$ (6)

From the above equation Eq (6) $λ$ will be negative that means the linear stability of the steady state is possibleif $f\_{u}+g\_{v}$< 0 and $f\_{u}g\_{v}-g\_{u}f\_{v}>0. $Since the sability of a steady state of a system is determined by the kinetic parameters the above mentioned inequalities define the stability regions for the homogeneous steady state . We now proceed for the analysis of full reaction-diffusion system and linearize it about the steady state to obtain

  (7)

Here the small perturbations in *u* and *v* i.e. *du* and *dv are* allowed to grow both in space and time around steady state as *du∼*$ e^{λt}$cos*kx* and dv accordingly. Here *k* is the wavenumber. The significant inclusion of both spatial and temporal part in the trial wave function is notable. Application of zero flux boundary condition then results in *k* = $\frac{nπ}{a}$. *n* is an integer and *a* determines the domain size in one dimension. Putting the aforesaid form of the perturbations in Eq (7) ultimately the following equation is obtained

  (8)

Where .

Note that as a function of k^2 h is quadratic that opens upward, that is it has minimum. Since |A| >0, we know that h can not be less than zero or equal to zero for any positive value of k^2 unless

 $df\_{u}+ g\_{v}>0$ (9)

Because if Eq (9) does not hold, all the terms in Eq (8) would be positive for any k^2>0. This Eq (9) is a necessary condition for Turing instability.

Eq (9) provides an important physical insight. We know from the stability analysis that the sum of the Jacobain elements that is $f\_{u}+g\_{v}$< 0. If both the terms are individually negative and on the above condition will not held. Therefore both the condition to hold true i.e. $f\_{u}+g\_{v}$< 0 and $df\_{u}+ g\_{v}>0$ one must be negative and one positive element. That is one species enhances the rate of its own production and the other decreases the rate production as its production grows. If one species is called the activator, other is inhibitor and the model is called activtator-inhibitor model. This type of model plays an important role in investigating Turing pattern.

Inorder o hold $f\_{u}+g\_{v}$< 0 |$f\_{u}|$ < |$g\_{v}$|. Eq (9) provides an important restriction on the diffusion coefficients of the activator and inhibitor in any system that permits Turing pattern formation. The inhibitor must diffuse more rapidly that the activator. This condition is referred to as ‘local activation and lateral inhibition’.

 Thus we can conclude if the requisite conditions hold then the initial infinitesimal perturation will ultimately evolve to a stable spatially inhomogeneous structure, the Turing pattern.

It is also possible to extend the treatment for two and three dimensional reaction-diffusion systems and depending on the nature of nonlinearity one may observe various types of patterns ranging from stripes to spots for the systems obeying the conditions discussed above.

 **III. SOME REACTION DIFFUSION SYSTEMS EXHIBITING TURING’S PATTERN**

## **The pigmentation fish model**

It is a two variable reaction diffusion sytem. Bario et al first proposed this model where pattern in fish arises due to physical interaction between cells with external surrounding leading to cell aggregation and differentiation. The equations are as follows



where *α*, *β*, *γ*, $r\_{1}$, $r\_{2}$ are the given parameters of the dynamics. *d* is the length scale. When the diffusion not present *v* = *-*( *α* + *γ*)*u/*(1 + *β*) is the another steady state. apart from the steady state (0,0) can be ensured as the only uniform steady state by setting the parameter *α* = *-γ*.



Fif 2: Generation of patterns in Fish as a result of different activator inhibitor concentrations [9]

## **Gierer Meinherdt model**

To take into consideration the two central features of pattern forming phenomena, viz. the
local self-enhancement and the long range inhibition, Gierer and Meinhardt introduced the
two species (A and B are the concentrations of the two reacting species) model.



In this model the species B is an antagonist and consequently *DB >> DA* which is the condition for formation of spatial instabilities. The coefficients $μ\_{A}$and $μ\_{B}$are the removal rates. The basic production terms are given by $σ\_{A}$and $σ\_{B}$, the cross reaction coefficients are given by $ρ\_{A}$and $ρ\_{B}$. The constant *KA* is called the saturation constant and is believed not that necessary for pattern forming instability to develop, rather it determines the shape of
the pattern.

There are some experimental systems which have been studied to illustrate the instabilities but we will elaborate those in some other reviews rather we will focus on some different tyes of instabilities eg spirals.

**IV** **Spirals**

Spirals are fascinating spatio-temporal patterns in reaction diffusion system where circular symmetry is disrupted. A wave created from a point due the infinitesimal perturbation of any one of the reactants can form circles having unique centre. It is due to fact that waves can propagate uniformly in all directions. It can form spots, stripes, labyrinth patterns and so on. But local disruption of concentric waves lead to curl and results spirals.

Spirals have been ubiquitous in several areas in complex media, involving living
systems as well. This is because physical heterogeneity in these spatially extended excitable systems is so common that most often spirals are born out of targets. Spirals have been detected in aggregating slime molds , carbon monoxide oxidation on single crystals of platinum, developing frog eggs, heart muscle, ferrocin-catalyzed BZ reaction, Gierer-Meinhardt model and other related systems [13,14]. However, the most studied of all these systems is Beluosov-Zhabotinskii reaction.

  

Fig 3: Theoretical prediction of spiral patterns generated in CDIMA system which has quite relevance with experimental observations[13].

Before going to review some of the mathematical techniques that are in frequent use it is
necessary, however, to make it clear what precisely we mean by the term spiral. In the case of BZ reaction, it is a rotating, time-periodic, spatial structure of reactant concentrations as noted by Murray[6,11]. If one stands at the center of a spiral he would see a periodic wave train is passing by him since every time the spiral turns a wave front moves past him. A simple rotating spiral is described by a periodic function of the phase *φ* with

 *φ* = ψ (t) + m (θ) ± ψ (r)

where ψis the frequency, *m* is the number of arms on the spiral and ψ (r) is a function which
describes the type of spiral. The *±* in the m (θ) term determines the sense of rotation. Numerous authors have investigated spiral wave trains of general reaction diffusion mode. These involved analyses usually make use of asymptotic methods. The *λ-w* system has been very extensively used as a model system because of the relative algebraic simplicity of the analysis. In this section we present some representative solutions for the *λ -w* system for illustration, following Murray, keeping in mind the direct relevance to real reaction-diffusion mechanism. The *λ -w* reaction diffusion mechanism for two reactants is



where *w*(*A*) and *λ* (*A*) are real functions of A.

If we consider w as a complex function of A and re do the analysis we will get the following dynamical equations as a solution of u and v.

 

Another way to deal with the problem of spiraling is to arrive at a linear amplitude equation from the solvability criterion applied at first order in a perturbation expansion of the model involved using multiple scales. It can then be shown that a general solution of a spiral exists for the amplitude equation in a region of phase space where a homogeneous oscillatory state is stable. Targets and stars are some special cases.

 **V CONCLUSIONS**

So far we have discussed the generation of different patterns in animal, fish etc. It is an analytical support to morphogenesis. Non linear dynamical studies have utmost strength to describe the pattern generations. Now to differentiate between spots, stripes and spirals we have to study the effect of different nodes on the reaction diffusion system. We can also study the effect of external noise, electric field, magnetic field , stochasticity on the systems. Worthmentioned will be to tune the pattern generations above any critical threshold through any of the external applications. In an upcoming issue it is planned to elaborate the effects on reaction diffusion systems.

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