Graceful labeling on cycle related graphs

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# Introduction

For an excellent survey on graph labeling we refer to Gallian[4]. All the graphs considered here are finite and undirected. The terms not defined here are used in the sense of Harary [3].

## Graceful labeling on graphs

Initiation of graph labeling were taken in 1960’s. Tremendous work of literature has to been developed around graph labeling over the most recent couple of years. It also provides a mathematical structure with a broad range of application.

The utilization of labeled graph models require imposing of additional constraints which characterize the problem being investigated. To label the graphs, we have several variations for labeling such as graceful, harmonious mean, heron mean, sequential, magic, vertex total magic, cordial, k-equitable, radio, and many other have been introduced by several authors. These all techniques are motivated by real life problems.

The name **”Graceful Labeling”** is by Solomon W. Golomb and this type of labeling was first termed “ tbeta labeling” by Alexander Rosa in 1967.

**Definition 1.1.** *A graph G of order p and size q which admits graceful labeling is called a graceful graph.*

The following results are due to Golomb[15]:

* + 1. A complete graph *Kp* is graceful if and only if *p* ≤ 4.
    2. The following results are due to Rosa[16]:
       - A cycle *Cn* of order n is graceful if and only if *n* ≡ 0 *or* 3(*mod* 4).
       - A friendship graph *Fk* on *k* triangles is graceful if and only if *n* ≡ 0 *or* 1(*mod* 4).
       - If G is a graceful eulerian graph of size *q*, then *q* ≡ 0 *or* 3(*mod* 4).

One of the still unsolved problems on graceful graphs is the now famous Ringel Kotzig Conjecture [19, 38, 48]

* + 1. **Conjecture:** All trees are graceful.

Motivated by the works of graceful labeling of graphs, a labeling called **pronic graceful labeling is discussed** in this work.

# Graceful labeling using pronic numbers

**Definition 2.1.** *Pronic Number:*

*A number of the form n*(*n* + 1) *is called a pronic number. These numbers are also called oblong numbers, heteromecic or rectangular numbers. The sum of the first n even integers is its nth pronic number. All pronic numbers are even(by definition), and the only prime pronic number is 2. Also 2 is the only pronic number in the Fibonacci sequence.*

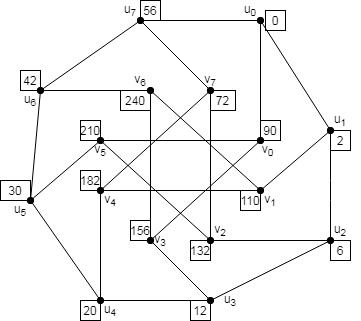
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Figure 1: Mobius Kantor Graph-Pronic Graceful

**Note 2.2.** *A pronic number is squarefree if and only if n and n + 1 are also squarefree. 0, 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, 182, 210, 240, 272, 306, 342, 380, 420, 462 are few among them.*

**Definition 2.3.** *Pronic Graceful Labeling:*

*. A pronic graceful labeling of a graph G with p* ≥ 2 *is a bijection f* : *V* (*G*) → {0*,*2*,*6*,*12*,...,p*(*p*+ 1)} *such that the resulting edge labels obtained by* |*f*(*u*) − *f*(*v*)| *on every edge uv are pairwise disjoint. A graph G is called pronic graceful if it admits pronic graceful labeling.*

**Example 2.4.** *An example for a graph which admits pronic graceful labeling is given in 1*

In this chapter, the pronic graceful labeling on graphs with some graph operations have been discussed.

## Main theorems

**Theorem 2.5.** *Cycle graph Cn, n* ≥ 3 *is a pronic graceful graph*

**Theorem 2.6.** *Star graph K*1*,n, n* ≥ 3 *is a pronic graceful graph.*

**Theorem 2.7.** *Path graph Pn, n* ≥ 3 *is a pronic graceful graph.*

**Theorem 2.8.** *Path graph Pn, n* ≥ 3 *is a pronic graceful graph.*

**Theorem 2.9.** *Complete graph Kn, n* ≥ 4 *does not admit pronic graceful labeling.*

## Wheel related graphs

**Theorem 2.10.** *The wheel graph K*1 + *Cn, n* ≥ 4 *admits pronic graceful labeling.*

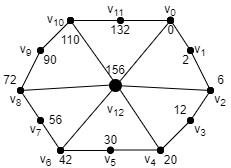
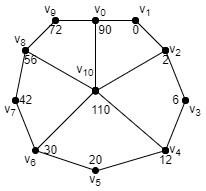


Figure 2: Gear Graph Figure 3: *G*5 Graph

**Theorem 2.11.** *Gear graph Gn admits pronic graceful labeling*

**Proof :** Let *vn* be the apex vertex and { *v*0*,v*1*,v*2*...,vn*−1} be the rim vertices of *Gn*, *n* ≥ 3 and {*vivi*+1*,i* = 0*,*1*,...n* − 2*,vn*−1*v*0*,vnvi,i* = 0*,*2*,*4*,...,n* − 2 be the edges of *Gn*.

Define a bijection *f* : *V* (*G*) → {*p*0*,p*1*,...,pn*−1} by

*f*(*vi*) = *pi,i* = 0*,*1*,*2*,...,n* − 1 *f*(*vn*) = *pn.*

For the vertices labeled above, an induced labeling *f*∗ : *E*(*G*) → {2*,*4*,*6*...,pn*−1} is defined by

*f*∗(*vivi*+1) = 2(*i* + 1)*,i* = 0*,*1*,*2*,...,n* − 2;

*f*∗(*vnvi*) = *n*(*n* + 1) − *i*(*i* + 1)*,i* = 0*,*2*,*4*, n* − 2;

*f*∗(*v*0*vn*−1) = (*n* − 1)*n.*

Let *A*1 and *A*2 denote the set of edge labels of {*vivi*+1(0 ≤ *i* ≤ *n* − 2)*,vn*−1*v*0} , {*vivi*+1*,i* = 0*,*1*,*2*, ,n* − 2}*.* Then:

*A*1 = {2*,*4*,*6*, ,*2(*n* − 1)*,n*(*n* − 1)};

*A*2 = {*n*(*n* + 1)*,n*(*n* + 1) − 6*,n*(*n* + 1) − 20*,* 4*n* − 2}*.*

Hence *A*1 ∩ *A*2 = *φ* which results that the gear graph admits pronic graceful labeling

**Theorem 2.12.** *Helm Graph HGn, admits pronic graceful labeling*

**Proof :** Let *vn* be the apex vertex and { *v*0*,v*1*,v*2*...,vn*−1} be the rim vertices of *HGn*, *n* ≥ 3. Let {*vivi*+1*,i* = *n,n* + 1*,...*2*n*

− 2*,vnv*2*n*−1*,v*2*nvi,i* = *n,n* + 1*,...*2*n* − 1 be the edges of *HGn*.

Define a bijection *f* : *V* (*G*) → {*p*0*,p*1*,...,p*2*n*} by

*f*(*vi*) = *pi,i* = 0*,*1*,*2*,...,n* − 1 *f*(*v*2*n*) = *p*2*n.*

For the vertices labeled above, an induced labeling *f*∗ : *E*(*G*) → {2*,*4*,*6*...,pn*−1} is defined by

*f* ∗(*vivi*+1) = 2(*i* + 1)*,i* = *n* − 1*,n,n* + 1*,n* + 2*,...,*2*n* − 2;

*f*∗(*v*2*nvi*) = 3*n*2 − (*n* + 1)*i* − 1*,i* = 0*,*1*,*2*,...n* − 1;

*f*∗(*vivi*+(*n*+1) = *pn*+1 + 2*i*(*n* + 1)*,i* = 0*,*1*,*2*,...,n* − 2;

*f*∗(*vnv*2*n*−1) = 3*n*2 − 3*n.*

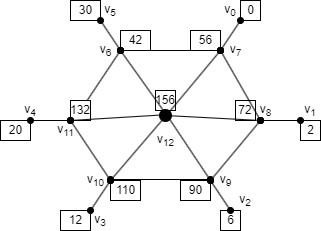


Figure 4: Helm Graph

Let *A*1 and *A*2 denote the set of edge labels of {*vivi*+1(0 ≤ *i* ≤ *n* − 2)*,vn*−1*v*0} , {*vivi*+1*,i* = 0*,*1*,*2*,...,n* − 2}*.* Then:

*A*1 = {2*,*4*,*6*,...,*2(*n* − 1)*,n*(*n* − 1)};

*A*2 = {*n*(*n* + 1)*,n*(*n* + 1) − 6*,n*(*n* + 1) − 20*,...*4*n* − 2}*.*

Hence *A*1 ∩ *A*2 = *φ* which results that the helm graph admits pronic graceful labeling.

## Ladder Graph and Mobius Ladder Graph

**Definition 2.13.** *Ladder Graph Ln,*1

*The ladder graph, denoted by Ln,*1 *is a planar undirected graph which is defined as the cartesian product of two path graphs, one of which has only one edge: Ln,*1 = *Pn* ×*P*2 *with* 2*n vertices and* 3*n*−2 *edges.*

**Theorem 2.14.** *Ladder graph Ln,*1 *is pronic graceful.*

**Proof :** Let *Ln,*1 be the ladder graph with vertex set *V* (*Ln,*1) = {*ui,vi,*0 ≤ *i* ≤ *n* − 1} and edge set

*E*(*Ln,*1) = {*uiui*+1*,vivi*+1*,*0 ≤ *i* ≤ *n* − 2} 𝖴 {*uivi,*0 ≤ *i* ≤ *n* − 1}*.*

Define a bijection *f* : *V* (*G*) → {*p*0*,p*1*,...,p*2*n*−1} by

*f*(*ui*) = *pi,i* = 0*,*1*,*2*,...,n* − 1;*f*(*vi*) = *pi*+*n,i* = 0*,*1*,*2*,...,n* − 1*.*

For the vertices labeled above, an induced labeling *f*∗ : *E*(*G*) → {2*,*4*,*6*...,pn*−1} is defined by

*f* ∗(*uiui*+1) = 2(*i* + 1)*,i* = 0*,*1*,*2*,...,n* − 2;

*f* ∗(*vivi*+1) = 2(*n* + *i* + 1)*,i* = 0*,*1*,*2*,...,n* − 2;

*f*∗(*uivi*) = *n*2 + *n*(1 + 2*i*)*,i* = 0*,*1*,*2*...,n* − 1*.*

Let *A*1, *A*2 and *A*3 denote the set of edge labels of {*uiui*+1(0 ≤ *i* ≤ *n* − 2)} , {*vivi*+1*,i* = 0*,*1*,...n* − 2} and

{*uivi,i* = 0*,*1*,...n* − 1}*.* Then:

*A*1 = {2*,*4*,*6*,...,*2(*n* − 1)};

*A*2 = {2(*n* + 1)*,*2(*n* + 2)*,...,*2(2*n* − 1)}; *A*3 = {*n*2 + *n,n*2 + 3*n,...,n*(3*n* − 1)}*.*

Hence *A*1 ∩ *A*2 = *φ* which results that the ladder graph admits pronic graceful labeling.

**Definition 2.15.** *Mobius Ladder Graph Mn*

*A Mobius ladder graph Mn is a simple cubic graph on* 2*n vertices and* 3*n edges. A Mobius ladder graph Mn is a graph obtained from the ladder PnP*2 *by joining the opposite end points of the two copies of Pn.*

**Theorem 2.16.** *Mobius Ladder Graph Mn is pronic graceful.*

**Proof :** Let *Mn* be the Mobius Ladder graph with vertex set *V* (*Mn*) = {*ui,vi,*0 ≤ *i* ≤ *n* − 1} and edge set

*E*(*Mn*) = {*uiui*+1*,vivi*+1*,*0 ≤ *i* ≤ *n* − 2} 𝖴 {*u*0*vn*−1*,v*0*un*−1}*.*

Define a bijection *f* : *V* (*G*) → {*p*0*,p*1*,...,p*2*n*−1} by

*f*(*ui*) = *pi,i* = 0*,*1*,*2*,...,n* − 1;*f*(*vi*) = *pi*+*n,i* = 0*,*1*,*2*,...,n* − 1*.*

For the vertices labeled above, an induced labeling *f*∗ : *E*(*G*) → {2*,*4*,*6*...,pn*−1} is defined by

*f*∗(*uiui*+1) = 2(*i* + 1)*,i* = 0*,*1*,*2*,...,n* − 2; *f*∗(*u*0*vn*−1) = 2*n*(2*n* − 1);

*f*∗(*vivi*+1) = 2(*n* + *i* + 1)*,i* = 0*,*1*,*2*,...,n* − 2;

*f*∗(*v*0*un*−1) = 2*n*; *f*∗(*uivi*) = *n*2 + *n*(1 + 2*i*)*,i* = 0*,*1*,*2*...,n* − 1*.*

Let *A*1, *A*2, *A*3 and *A*4 denote the set of edge labels of {*uiui*+1(0 ≤ *i* ≤ *n* − 2)} , {*vivi*+1*,i* = 0*,*1*,...n* − 2}, {*uivi,i* = 0*,*1*,...n* − 1} and {*u*0*vn*−1*,v*0*un*−1} Then:

*A*1 = {2*,*4*,*6*,...,*2(*n* − 1)};

*A*2 = {2(*n* + 1)*,*2(*n* + 2)*,...,*2(2*n* − 1)};

*A*3 = {*n*2 + *n,n*2 + 3*n,...,n*(3*n* − 1)}; *A*4 = {2*n*(2*n* − 1)*,*2*n*}*.*

Hence *Ai* ∩ *Aj* = *φ* for all *i* ≠ *j* which results that the mobious ladder graph admits pronic graceful labeling.

## Shell related graphs

From the excellent survey of Gallion, one can find many families of cycle related graphs on which important is the Shell graph family.

**Shell Graph**

**Theorem 2.17.** *A Shell Graph C*(*n,n* − 3)*, for n* ≥ 3 *is a pronic graceful graph.*

**Proof :** Let { *v*0*,v*1*,v*2*...,vn*−1} be the vertices of *C*(*n,n* − 3).

Define a bijection *f* : *V* (*G*) → {*p*0*,p*1*,...,pn*−1} by

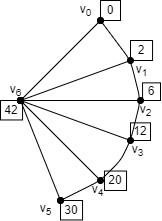
*f*(*vi*) = *pi,i* = 0*,*1*,*2*,...,n* − 1*.*

For the vertices labeled above, an induced labeling *f*∗ : *E*(*G*) → {2*,*4*,*6*...,pn*−1} is defined by

*f*∗(*vivi*+1) = 2(*i* + 1)*,i* = 0*,*1*,*2*,...,n* − 3; *f*∗(*vnvi*) = *n*(*n* + 1) − *i*(*i* + 1)*,i* = 0*,*1*,*2*...,n* − 2*.*

The edge labels are thus {2*,*4*,*8*...,*2(*n* − 2)*,pn*−1*,pn*−1 − 2*,pn*−1 − 6*,...pn*−1 − *pn*−2} and hence shell graph

*C*(*n,n* − 3), for *n* ≥ 3 admits pronic graceful labeling.



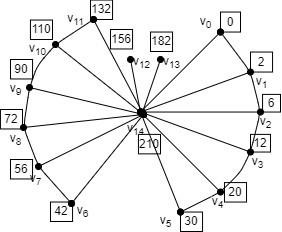
Figure 5: Shell Graph

Figure 6: Shell Butterfly Graph

## Shell Butterfly Graph

J.J. Jesintha, K.E. Hilda[17] defined a Shell -butterfly graph as a double shell in which each shell has any order with exactly two pendant edges at the apex and proved that all shell- butterfly graphs with shells of order *l* and *m*(shell order excludes the apex) are graceful. Note that *G* has *n* = 2*m* + 3 vertices and *q* = 4*m* edges. Here in the following theorem, we consider the shell butterfly graph of same order.

**Theorem 2.18.** *A shell butterfly graph G is a pronic graceful graph.*

**Proof :** Let *G* be a shell-butterfly graph with *n* vertices and *q* edges and have the shell orders as *m*(odd or even) and *l* where *l* = 2*t* + 1. Note that shell orders exclude the apex. Let the shell that is present to the left of the apex be called as the left wing of the *G*. Let the shell that is present to the right of the apex is called the right wing of *G*.

Denote the apex of *G* be *v*2*m*+2 and the vertices of right wing of the graph from top to bottom as *v*0*,v*1*,...,vm*−1. Similarly the left wing vertices by {*vm,vm*+1*,...,v*2*m*−1}. Let {*v*2*m,v*2*m*+1} be the two pendant vertices of *G.*

Define a bijection *f* : *V* (*G*) → {*p*0*,p*1*,...,pn*−1} by *f*(*vi*) = *pi, i* = 0*,*1*,* 2 *,...,* 2*n* + 2*.*

For the vertices labeled above, an induced labeling *f*∗ : *E*(*G*) → {2*,*4*,*6*...,pn*−1} is defined by

*f* ∗(*vivi*+1) = 2(*i* + 1)*,i* = 0*,*1*,*2*,...,m* − 1*,m,m* + 1*,...,*2*m* − 2;

*f*∗ (*v*2*m*+2*vi*) = (2*m* + 2)(2*m* + 3) − *i*(*i* + 1)*,i* = 0*,*1*,*2*...,m* − 1*,m,m* + 1*,...,*2*m* + 2*.*

The edge labels are thus {2*,*4*,*8*...,*2(*m*−1)*,*2(*m*+1)*,*2(*m*+2)*,...,*2(2*m*−1)}. The labels of the edges *v*2*m*+2*vi* are of the form (2*m*+2)(2*m*+3)−*i*(*i*+1)*,i* = 0*,*1*,*2*...,m*−1*,m,m*+1*,...,*2*m*−1 and begins with *p*2*m*+2 and the difference of each label is of the form 2*i, i* = 1*,*2*,...m* − 1*,m* + 1*,m* + 2*,...,*2*m* + 1*.* and hence shell butterfly graph admits pronic graceful labeling.

**2.5.1 PGL on corona product and joint sum of graphs Definition 2.19.** *Corona Product of Cn and mK*1

*The corona product of Cn and mK*1*, denoted by Cn* ◦ *mK*1 *is the graph with the vertex set*

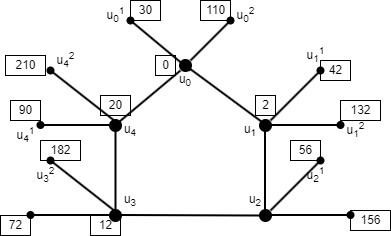


Figure 7: Corona graph *C*5 ◦ 2*K*1

*V* (*Cn* ◦ *mK*1) = {*x ,y j* : 1 ≤ *i* ≤ *n,*1 ≤ *j* ≤ *m*} *and the edge set*

*i i*

*E*(*Cn* ◦ *mK*1) = {*xi,xi*+1 : 1 ≤ *i* ≤ *n* − 1} 𝖴 {*x ,y j* : 1 ≤ *i* ≤ *n,*1 ≤ *j* ≤ *m*} 𝖴 {*xn,x*1}*.*

*i i*

**Theorem 2.20.** *Corona product Cn* ◦ *mK*1 *is a pronic graceful graph.*

**Proof :** Let {*u*0*,u*1*,u*2*,...,un*−1} be the vertices of the cycle *Cn* and {  *,j* = 1*,*2*,...,m*} be the corresponding pendant vertices attached to the *u*0*,u*1*,u*2*,...,un*−1.

Define a bijection *f* : *V* (*G*) → {0*,*2*,*6*...*(*nm* + *n*)(*nm* + *n* − 1)} by

*f*(*ui*) = *pi,i* = 0*,*1*,...,n* – 1;

*f*(*uj* ) = *p ,i* = 0*,*1*,*2*,...,n* − 1*,j* = 1*,*2*,...,m.*

*i nj*+*i*

And the induced edge labeling *f*∗ : *E*(*G*) → *N* is defined by

*f*∗(*uiui*+1) = 2(*i* + 1)*,i* = 0*,*1*,*2*...,n* − 2; *f*∗(*u*0*un*−1) = *n*(*n* − 1);

*f*∗(*u u*( *j*)) = *n*[2*ij* + *j*(*nj* + 1)]*,i* = 0*,*1*,*2*...,n* − 1*,j* = 1*,*2*,...,m.*

*i i*

Let *A*1, *A*2, *A*3 denote the set of edge labels of {*uiui*+1*,i* = 0*,*1*,...n*−2}, {*un*−1*u*0} and {*uiu*(*ij*)*,i* = 0*,*1*,*2*...,n* − 1*,j*

= 1*,*2*,...,m*} respectively.

Clearly the labels of the edges for the above sets are of the form as follows:

*A*1 contains the edges of the form 2*k, k* = 1*,*2*,...*(*n* − 1) and each label differs by 2 and hence they are distinct.

*A*2 contains the edge of the form *n*(*n* − 1) and is differed from the above labeling by *pn*−1*.*

Consider the labels of *A*3

For *j* = 1, the set contains edges of the form {*pn,pn* + 10*i,...,n*(2*i* + (*n* + 1)} For *j* = 2, the set contains edges of the form {*p*2*n,p*2*n* + 20*i,...,n*(4*i* + 2(2*n* + 1)}

.....

.....

.....

For *j* = *m*, the set contains edges of the form {*pmn,pmn* + 10*mi,...,n*[2*im* + *j*(*mn* + 1)]}

It is observed that the labels in the above sets are distinct, that is *A*1 ∩ *A*2 ∩ *A*3= *φj* and hence *Cn* ◦ *mK*1 is a pronic graceful graph.

## Barycentric Subdivision of a graph

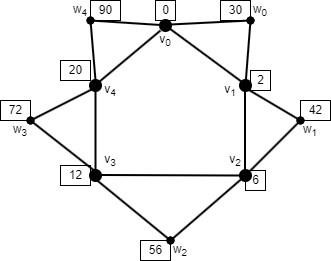
**Definition 2.21.** *Creating a barycentric subdivision is a recursive process. In this section we consider the concept of barycentric subdivision of cycles introduced by Vaidya et al. An edge e* = *uv of a graph G is said to be subdivided when it is deleted and replaced by path of length 2 . Let Cn* = *u*1*...un be a cycle on n vertices. Subdivide each edge uiui*+1 *of Cn and let the new vertex be ui,*1 ≤ *i* ≤ *n. Join ui with ui*+1*,*1 ≤ *i* ≤ *n . All suffixes are taken modulo n. The resulting graph is denoted as* (*Cn*)2*. This graph is called the barycentric subdivision of Cn and it is denoted by Cn*(*Cn*) *as it look like Cn inscribed in Cn. The barycentric subdivision subdivides each edge of the graph.*

Figure 8: *C*5(*C*5)

**Theorem 2.22.** *Barycentric subdivision of cycle Cn*(*Cn*) *is a pronic graceful graph.*

**Proof :** Let {*v*0*,v*1*,...,vn*−1} be the vertices of *n*− cycle and {*w*0*,w*1*,w*2*...,wn*−1} such that *wi* connected to *vi* and *vi*+1 for 0 ≤ *i* ≤ *n* − 2 and *wn*−1 is connected to *vn*−1 and *v*1.

Define a bijection *f* : *V* (*G*) → {*p*0*,p*1*,p*2*,...,p*2*n*−1} by

*f*(*vi*) = *pi,i* = 0*,*1*,*2*,...,n* − 1; *f*(*wi*) = *pn*+*i,i* = 0*,*1*,*2*,...,n* − 1*.*

Clearly f is a bijection. For the above vertices labeled above, the edge labeling *f*∗ : *E*(*G*) → *N* is defined by

|  |  |
| --- | --- |
| *f*∗(*vivi*+1) = 2(*i* + 1)*,i* = 0*,*1*,*2*...,n* − 2; | *f* ∗(*wivi*+1) = *pn* − 2 + *i*(2*n* − 2)*,i* = 0*,*1*,*2*,...,n* − 2; |
| *f*∗(*v*0*vn*−1) = *n*(*n* − 1); *f*∗(*viwi*) = *pn* + 2*ni,i* = 0*,*1*,*2*,...,n* − 1*.* | *f*∗(*v*0*wn*−1) = 2*n*(2*n* − 1); |

Let *A*1, *A*2, *A*3 and *A*4 denote the set of edge labels of {*vivi*+1*,i* = 0*,*1*,...n*−2},{*viwi,i* = 0*,*1*,*2*,...,n*−1},

{*viwi*−1*,i* = 0*,*1*,*2*,...,n* − 1} and {*v*0*vn*−1*,v*0*wn*−1 respectively. Then:

*A*1 = {2*,*4*,*6*,...,*2(*n* − 1)}; *A*2 = {*n*(*n* + 1)*,n*(*n* + 3)*,n*(*n* + 5)*...,n*(3*n* − 1)};

*A*3 = {(*n* − 1)(*n* + 3)*,*(*n* − 1)(*n* + 5)*,*(*n* − 1)(*n* + 5)*,...,*(*n* − 1)(3*n* − 2)}; *A*4 = {2*n*(2*n* − 1)*,*2*n*}*.*

Hence *A*1 ∩ *A*2 = *φ* which results that the barycentric subdivision of cycle *Cn*(*Cn*) admits pronic graceful labeling.

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