A New Suzuki-Type Fixed Point Result with Applications in Dynamic Programming

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Abstract. The purpose of this study is to investigate a novel mapping for Suzuki-type contraction defined on complete metric space that has at least one fixed point. An example is also provided to emphasize the successes. Some known outcomes are enhanced and expanded by our result. We apply our findings in order to solve functional equations that come up in dynamic programming.

Keywords and Phrases: fixed point; complete metric space; Suzuki-contraction; functional equation; dynamic programming.

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1. Introductions

Mathematical fixed point theory is a fascinating area of study. It has numerous applications in almost all branches of mathematical sciences.

The simplest and most used tool in nonlinear analysis is the classical Banach contraction theorem (cf. Theorem 1.1) (1912) due to Polish mathematician Stefan Banach (1882-1945). There was a huge development on this line which has a tremendous impact on all branches of applicable mathematics and mathematical sciences (see, Agarwal et al. [1], Berinde [5], Geobel and Kirk [12], Rhoades [19] and references thereof).

Definition 1.1. A map E on a metric space Z is a contraction if there exists $c \in [0, 1)$ such that for every $p, q \in Z$,

$$\delta(Ep, Eq) \leq c\delta(p, q). \tag{1.1}$$

Banach states the following theorem popularly known as Banach contraction theorem (Bct).

Theorem 1.1. A contraction map on a complete metric space has a unique fixed point.

There are numerous studies published in the last 70 years that demonstrate various generalizations of the Bct by weakening either the contractive features of the map or by extending the structure of the ambient space (see, for instance, [8], [13] and others).

An expansion of the Banach contraction, Kannan [15] established a fixed point theorem for a map on metric spaces. He was the first, in fact, to put forth a fixed point theorem for a discontinuous map on a metric space.

Definition 1.2. A map E on a metric space Z is said to be a kannan contraction if there exists some $c \in [0, ½)$ such that for every $p, q \in Z$,

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$$\delta(Ep, Eq) \leq c[\delta(p, Ep) + \delta(q, Eq)]. \tag{1.2}$$

Theorem 1.2. A Kannan contraction map on a complete metric space has a unique fixed point.

Definition 1.3. A map E on a metric space Z is said to be a generalized contraction if there exists $c \in [0, 1)$ such that for every $p, q \in Z$,

$$\delta(Ep, Eq) \leq c \max\ \{\delta(p, q), \delta(p, Ep), \delta(q, Eq), [\delta(p, Eq) + \delta(q, Ep)]/2\}. \tag{1.3}$$

Notice that the generalized contraction (1.3) is essentially due to Ciric [9] which is referred as (21´) in a thorough comparison of maps by Rhoades [19]. The Bct and Kannan's own fixed point theorem have both been extensively extended and generalized as a result of his theorem in many different contexts. One of the best generalizations, among contractions for single-valued maps is quasi-contraction given by Ciric [10]:

Definition 1.4. A map E on a metric space Z is said to be a quasi-contraction if there exists $c \in [0, 1)$ such that for every $p, q \in Z$,

$$\delta(Ep, Eq) \leq c \max\{ \delta(p, q), \delta(p, Ep), \delta(q, Eq), \delta(p, Eq), \delta(q, Ep)\}. \tag{1.4}$$

A result due to Ciric [10] popularly called quasi contraction theorem in metric fixed point theory is as follows:

Theorem 1.3. A quasi contraction map on a complete metric space has a unique fixed point.

Generalizing the classical Banach contraction principle, Khojasteh et al.[14] attained the aforementioned outcome.

Theorem 1.4. Let E be a map on a complete metric space Z such that for every $p, q \in Z$,

$$\delta(Ep, Eq) \leq \left(\frac{\delta(p, Eq)+\delta(q, Ep)}{\delta(p, Ep)+\delta(q, Eq)+1}\right)\delta(p, q).$$

Then

(i) there exists at least one $a \in Z$ such that $E(a) = a$;

(ii) for any a, $\{E^r(a)\}$ converges to a fixed point;

(iii) if $E(a) = a, E(b) = b$ with $a \neq b$, then $\delta(a, b) \geq 1/2$.

For an excellent comparison of various contractive conditions for one and two maps, refer to Rhoades [19]. One may consult Boyd and Wong [8], Jachymski [13], Rhoades [19] and references therein for a few basic generalizations of the condition (1.1) and their comparison.

However in all these fixed point theorems, the contractive or contractive conditions are required to hold for all points p, q of the domain. Therefore, it makes sense to anticipate the day when this stipulation is significantly eased without compromising a theorem's conclusions.

Definition 1.5. Define a nonincreasing function $\theta : [0, 1) \rightarrow (1\ 2\ , 1]$ by

$$\theta(c) = \begin{cases} 1 & \text{if } 0 \leq c \leq \frac{1}{2}(\sqrt{5}-1), \\ \frac{1-c}{c^2} & \text{if } \frac{1}{2}(\sqrt{5}-1) \leq c \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{1+c} & \text{if } \frac{1}{\sqrt{2}} \leq c < 1. \end{cases}$$

A map E on a metric space Z is said to be Suzuki contraction if there exists c ∈ [0, 1) such that for every $p, q \in Z$,

$$\theta(c)\, \delta(p, Ep) \leq \delta(p, q) \text{ implies } \delta(Ep, Eq) \leq c\, \delta(p, q). \quad (1.5)$$

The following extraordinary generalization of the Bct was recently demonstrated by Suzuki in [23].

Theorem 1.5. A Suzuki contraction map on a complete metric space has a unique fixed point and the sequence of Picard iterates $\{ E^r p\}$ converges to the fixed point for any $p \in Z$.

Definition 1.6. Define a nonincreasing function ψ : [0, 1) → (1 2 , 1] by

$$\psi(c) = \begin{cases} 1 & \text{if } 0 \leq c \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{1+c} & \text{if } \frac{1}{\sqrt{2}} \leq c < 1 \end{cases}$$

A map E on a metric space Z is said to be Kikkawa Suzuki Kannan contraction if there exists c ∈ [0, 1/2) such that for every $p, q \in Z$,

$$\psi(c)\, \delta(p, E\,p) \leq \delta(p, q) \text{ implies } \delta(Ep, Eq) \leq c[\delta(p, Ep) + \delta(q, Eq)]. \quad (1.6)$$

The following outcome is made possible by Kikkawa and Suzuki [16].

Theorem 1.6. A map satisfying (1.6) has a unique fixed point on a complete metric space.

For some extensions and generalizations of the above theorem, we may refer to [11], [17] and others.

Forceful nature of the Suzuki contraction theorem has inspired many researchers to present some beautiful and interesting extensions and generalizations during a small span of five years (see, for instance, [2], [3], [11], [17], [18], [21], [22], [24] and others).

This paper is devoted to the concepts of Suzuki contraction. By combining the idea of Suzuki contraction [23] and Khojasteh et al. contraction [14], we obtain a new type of fixed point theorem generalizing the results of Banach, Khojasteh et al. [14] and others. An illustrative example, highlighting the realizing improvements, is also discussed. We apply our key finding to find solutions to some functional equations that occur during dynamic programming under much weaker condition than those in [4], [6] and [7].

2. Main Results

Theorem 2.1. Let (Z, δ) be a complete metric space and let E: $Z \rightarrow Z$ with the condition that for every $p, q \in Z$,

$$\frac{1}{2}\,\delta(p, Ep) \le \delta(p, q) \tag{2.1}$$

implies

$$\delta(Ep, Eq) \le \left(\frac{\delta(p,Eq)+\delta(q,Ep)}{\delta(p,Ep)+\delta(q,Eq)+1}\right)\delta(p, q) \tag{2.2}$$

Then

(i) E has atleast one fixed point $a \in Z$.
(ii) For all $p \in Z$,{ $E^r p$ } converges to a fixed point.
(iii) If $a \neq b$ are fixed points of E, then $\delta(a, b) \ge 1/2$.

Proof : Let $p_0 \in Z$ and choose $\{p_r\}$ such that $p_{r+1} = Ep_r$, we have

$$\frac{1}{2}\,\delta(p_{r-1}, Ep_{r-1}) = \frac{1}{2}\,\delta(p_{r-1}, p_r)$$

$$\le \delta(p_r, p_{r-1})$$

So (2.1) holds, therefore from (2.2)

$$\delta(p_{r+1}, p_r) = \delta(Ep_r, Ep_{r-1}) \le \left(\frac{\delta(p_r,p_r)+\delta(p_{r-1},p_{r+1})}{\delta(p_r,p_{r+1})+\delta(p_{r-1},p_r)+1}\right)\delta(p_r, p_{r-1})$$

$$= \left(\frac{\delta(p_{r-1},p_{r+1})}{\delta(p_r,p_{r+1})+\delta(p_{r-1},p_r)+1}\right)\delta(p_r, p_{r-1})$$

$$\le \left(\frac{\delta(p_{r-1},p_r)+\delta(p_r,p_{r+1})}{\delta(p_r,p_{r+1})+\delta(p_{r-1},p_r)+1}\right)\delta(p_r, p_{r-1}) \tag{2.3}$$

Given $\omega_r = \left(\frac{\delta(p_{r-1},p_r)+\delta(p_r,p_{r+1})}{\delta(p_r,p_{r+1})+\delta(p_{r-1},p_r)+1}\right)$, (2.4)

we have

$$\delta(p_{r+1}, p_r) \le \omega_r \delta(p_r, p_{r-1})$$

Similarly, $\delta(p_r, p_{r-1}) \le \omega_{r-1}\,\delta(p_{r-1}, p_{r-2})$

So $\delta(p_{r+1}, p_r) \le \omega_r \delta(p_r, p_{r-1})$

$$\le \omega_r\,\omega_{r-1}\,\delta(p_{r-1}, p_{r-2})$$

$$\le \omega_r\,\omega_{r-1}\ldots\ldots\,\omega_1\,\delta(p_1, p_0). \tag{2.5}$$

Keep in mind that ω_r is non-increasing and has positive terms, so $\omega_1\,\omega_2\ldots\ldots\,\omega_r \le \omega_1^{\,r}$ and $\omega_1^{\,r} \to 0$. It follows that

$$\lim_{r\to\infty}(\omega_1\omega_2\,\ldots\ldots\,\omega_r) = 0. \tag{2.6}$$

Thus, it is verified that

$$\lim_{r\to\infty}\delta(p_{r-1},p_r) = 0. \quad (2.7)$$

Now, for all r, s $\in N$ with $r < s$, we have

$\delta(p_s, p_r) \leq \delta(p_r, p_{r+1}) + \delta(p_{r+1}, p_{r+2}) + \ldots\ldots\ldots + \delta(p_{s-1}, p_s)$

$$\leq [(\omega_r\omega_{r-1} \ldots\ldots\omega_1) + (\omega_{r+1}\omega_r\omega_{r-1} \ldots\ldots\omega_1) + \ldots\ldots + (\omega_{m-1} \ldots\ldots\omega_1)]\delta(p_1,p_0)$$

$$= \sum_{k=r}^{s}(\omega_k,\omega_{k-1} \ldots\ldots\omega_1)\delta(p_1,p_0) \quad (2.8)$$

Suppose that $a_k = (\omega_k,\omega_{k-1} \ldots\ldots\omega_1)$. Since

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = 0. \quad (2.9)$$

$\sum_{k=1}^{\infty} a_k < \infty$. It means that

$$\sum_{k=n}^{s}(\omega_k,\omega_{k-1} \ldots\ldots\omega_1) \to 0 \quad (2.10)$$

as s, $r\to\infty$. Thus, $\{p_r\}$ converges to $a \in Z$.

We assert that $a = E(a)$.

Now $\frac{1}{2}\,\delta(p_r, Ep_r) = \frac{1}{2}\,\delta(p_r, p_{r+1}) \leq \frac{1}{2}\,[\delta(p_r, a) + \delta(a, p_{r+1})]$

$$\leq \frac{1}{2}\,[\delta(p_r, a) + \delta(a, p_r)] = \delta(p_r, a)$$

So $\delta(Ep_r, Ea) \leq \left(\frac{\delta(p_r,Ea)+\delta(a,Ep_r)}{\delta(a,Ea)+\delta(p_r,Ep_r)+1}\right)\delta(p_r, a)$

That is, $\delta(Ep_r, Ea) \leq \left(\frac{\delta(p_r,Ea)+\delta(a,p_{r+1})}{\delta(a,Ea)+\delta(p_r,p_{r+1})+1}\right)\delta(p_r, a)$

Upon imposing a limit on both sides of the aforementioned equation, we have

$$\delta(a, Ea) \leq \left(\frac{\delta(a,Ea)+\delta(a,a)}{\delta(a,Ea)+\delta(a,a)+1}\right)\delta(a, a)$$

So, $\delta(a, Ea) \leq 0$

i.e. $\delta(a, Ea) = 0$

Thus $a = Ea$.

If $a \neq b$ are fixed points of E, then

$\frac{1}{2}\,\delta(a, Ea) = 0 \leq \delta(a, b)$, so from (2)

$$\delta(a, b) = \delta(Ea, Eb) \leq [\delta(a, Eb) + \delta(Ea, b)]\,\delta(a, b) = 2[\delta(a, b)]^2$$

Therefore $\delta(a, z') \leq 2[\delta(a, b)]^2$

i.e. $1 \leq 2\,\delta(a, b)$

i.e. $\delta(a, b) \geq 1/2$ and we find the desired result.

Theorem 2.1's generality over Theorem 1.4 is demonstrated in the next example.

Example 2.1. Let $Z = \left\{0, 1, \frac{1}{2}\right\}$ and $\delta: Z \times Z \rightarrow R^{+}$ defined as

$\delta(0,0) = \delta(1,1) = \delta\left(\frac{1}{2}, \frac{1}{2}\right) = 0,$

$\delta\left(0, \frac{1}{2}\right) = \delta\left(\frac{1}{2}, 0\right) = \delta(0, 1) = \delta(1, 0) = 3,$

$\delta\left(1, \frac{1}{2}\right) = \delta\left(\frac{1}{2}, 1\right) = 1.$

Clearly, δ is a metric on Z.

Define $E: Z \rightarrow Z$ by

$E(0) = 0,\ E\left(\frac{1}{2}\right) = \frac{1}{2},\ E(1) = 0.$

Now, $\delta(E0, E1) = \delta(0,0) = 0.$

$\delta\left(E0, E\left(\frac{1}{2}\right)\right) = \delta\left(0, \frac{1}{2}\right) = 3,$

$\delta\left(E1, E\left(\frac{1}{2}\right)\right) = \delta\left(0, \frac{1}{2}\right) = 3,$

and we have

$$\delta\left(E0, E\left(\frac{1}{2}\right)\right) = \delta\left(0, \frac{1}{2}\right) = 3 \leq \left(\frac{\delta\left(0, E\left(\frac{1}{2}\right)\right) + \delta\left(\frac{1}{2}, E(0)\right)}{\delta(0, E0) + \delta\left(\frac{1}{2}, E\left(\frac{1}{2}\right)\right) + 1}\right)\delta\left(0, \frac{1}{2}\right)$$

$$= \left(\frac{\delta\left(0, \frac{1}{2}\right) + \delta\left(\frac{1}{2}, 0\right)}{\delta(0,0) + \delta\left(\frac{1}{2}, \frac{1}{2}\right) + 1}\right)\delta\left(0, \frac{1}{2}\right) = \left(\frac{3+3}{0+0+1}\right) \times 3 = 18,$$

but

$$\delta\left(E1, E\left(\frac{1}{2}\right)\right) = \delta\left(0, \frac{1}{2}\right) = 3 > \left(\frac{\delta\left(1, E\left(\frac{1}{2}\right)\right) + \delta\left(\frac{1}{2}, E(1)\right)}{\delta(1, E1) + \delta\left(\frac{1}{2}, E\left(\frac{1}{2}\right)\right) + 1}\right)\delta\left(1, \frac{1}{2}\right)$$

$$=\left(\frac{\delta\left(1,\frac{1}{2}\right)+\delta\left(\frac{1}{2},0\right)}{\delta(1,0)+\delta\left(\frac{1}{2},\frac{1}{2}\right)+1}\right)\delta\left(0,\frac{1}{2}\right)=\left(\frac{1+3}{3+0+1}\right)\times 1=1,$$

Thus we note that $\delta(Ep, Eq) \leq \left(\frac{\delta(p,Eq)+\delta(q,Ep)}{\delta(p,Ep)+\delta(q,Eq)+1}\right)\ \delta(p, q)$ if $(p, q) \neq \left(1, \frac{1}{2}\right)$,

since at $\left(1,\frac{1}{2}\right)$

$$\frac{1}{2}\ \delta\ (1, \mathrm{T}1)= \frac{1}{2} \times 3 > 1 = \delta\left(1,\frac{1}{2}\right).$$

Thus E satisfy Theorem 2.1 but not Theorem 1.4.

3. Applications

We take for granted that Y and Z are Banach spaces throughout this section and $C \subseteq X$ and $D \subseteq Y$. Let R denote the field of reals, $g: C \times D \to R$and $G: C \times D \times R \to R$. Viewing C and D as the state and decision spaces respectively, dynamic programming's problem reduces functional equations problem:

$$u = \sup_{q\in D}\{g(p,q) + G(p,q,u(\tau(p,q)))\},\ p \in C. \tag{3.1}$$

Some functional equations appear naturally during the multistage process (cf. Bellman [4] and Bellman and Lee [6]; see also Bhakta and Mitra [7] and others). In this part, we investigate whether the functional equation (3.1) that arises in dynamic programming has a solution.

Let $B(C)$ denote the set of all bounded real-valued functions on C. For an arbitrary $l \in B(C)$, define $\|l\| = \sup_{p\in C}|l(p)|$. Then $(B(C), \|.\|)$is a Banach space. Imagine that the following circumstances hold:

(DP-1) G and g are bounded.

(DP-2) For every $(p, q) \in C \times D$, $l, n \in B(C)$ and $t \in C$,

$$\frac{1}{2}|l(t) - Jl(t)| \leq |l(t) - n(t)|$$

implies

$$|G(p,q,l(t)) - G(p,q,n(t))| \leq \left(\frac{|l(t)-An(t)| + |n(t)-Al(t)|}{|l(t)-Al(t)|,\ |n(t)-An(t)|+1}\right)|l(t) - n(t)|,$$

where A is defined as follows:

$$Al(p) = \sup_{q\in D}\{g(p,q) + G\left(p,q,l\tau\left((p,q)\right)\right)\},\ p \in C,\ l \in B(C).$$

Theorem 4.1. Assume (DP-1) and (DP-2) are true. Therefore, $B(C)$ contains at least one solution to the functional equation (3.1).

Proof : For any l, $n \in B(C)$, let $\delta(l,n) = \sup\{|l(p) - n(p)|: p \in C\}$. Then $(B(C), \delta)$ is a complete metric space.

Let λ represent any random positive number and $l_1, l_2 \in B(C)$. Pick $p \in C$ and choose $q_1, q_2 \in D$such that

$$Al_j < g(\mathrm{p}, q_j) + G\left(p, q_j, l_j(p_j)\right), +\lambda, \tag{3.2}$$

where $p_i = (p, q_i)$, $i = 1, 2$ and $p_j = \tau(p, q_j)$.

Further,

$$Al_1 \geq g(p, q_2) + G\big(p, q_2, l_1(p_2)\big), \tag{3.3}$$

$$Al_2 \geq g(p, q_1) + G\big(p, q_1, l_2(p_1)\big), \tag{3.4}$$

Consequently, the initial inequality in (DP-2) becomes

$$\frac{1}{2}|l_1(p) - Al_1(p)| \leq |l_1(p) - l_2(p)| \tag{3.5}$$

and more over this with (3.2) and (3.4) gives

$$Al_1 - Al_2 < G(p, q_1, l_1(\mathrm{p})) - G(p, q_1, l_2(\mathrm{p})) + \lambda$$

$$\leq |G(p, q_1, l_1(\mathrm{p})) - G(p, q_1, l_2(\mathrm{p}))| + \lambda$$

$$\leq \left(\frac{|l_1(p) - Al_2(p)| + |l_2(p) - Al_1(p)|}{|l_1(p) - Al_1(p)|,\ |l_2(p) - Al_2(p)| + 1}\right)|l_1(p) - l_2(p)| + \lambda. \tag{3.6}$$

Similarly, (3.2), (3.3) and (3.5) implies

$$Al_2 - Al_1 \leq \left(\frac{|l_1(p) - Al_2(p)| + |l_2(p) - Al_1(p)|}{|l_1(p) - Al_1(p)|,\ |l_2(p) - Al_2(p)| + 1}\right)|l_1(p) - l_2(p)| + \lambda. \tag{3.7}$$

So, from (3.6) and (3.7), we obtain

$$|Al_1(p) - Al_2(p)| \leq \left(\frac{|l_1(p) - Al_2(p)| + |l_2(p) - Al_1(p)|}{|l_1(p) - Al_1(p)|,\ |l_2(p) - Al_2(p)| + 1}\right)|l_1(p) - l_2(p)| + \lambda. \tag{3.8}$$

Since $\lambda > 0$ is arbitrary and this inequality is true for any $p \in C$, and on taking supremum, we conclude from (3.5) and (3.8) that

$$\frac{1}{2}\delta(l_1, \mathrm{A}l_1) \leq \delta(l_1, l_2)$$

implies

$$\delta(Al_1, Al_2) \leq \left(\frac{|l_1(p)-Al_2(p)|+\ |l_2(p)-Al_1(p)|}{|l_1(p)-Al_1(p)|,\ |l_2(p)-Al_2(p)|+1}\right)|l_1(p) - l_2(p)|.$$

Theorem 2.1 is thus applicable, where A corresponds to the map E. So A has atleast one fixed point l^*, that is, $l^*(p)$is solution of the functional equation (3.1).

References

1. R. P. Agarwal, M. Meehan and D. O's Regan, Fixed point theory and applications, vol. 141, Cambridge University Press, Cambridge, 2001.
2. B. Ali and M. Abbas, Suzuki-type fixed point theorem for fuzzy mappings in ordered metric spaces, Fixed Point Theory Appl. 2013 (2013). 104
3. Altun and A. Erduran, A Suzuki type fixed-point theorem, Internat. J. Math. Math. Sci. 2011.
4. R. Bellman, Methods of Nonlinear Analysis. Vol. II, Academic Press, New York, 1973.
5. V. Berinde, Iterative approximation of fixed points, Efemeride Publishing House, Romania, 2002.
6. R. Bellman and E. S. Lee, Functional equations in dynamic programming, *Aequations Math.* 17(1)(1978), 1–18.
7. P. C. Bhakta and S. Mitra, Some existence theorems for functional equations arising in dynamic programming, *J. Math. Anal. Appl.* 98 (2)(1984), 348–362.
8. D. W. Boyd and J. S. Wong, On nonlinear contractions, Proc. Amer. Math. Soc. 20 (1969), 458–464.
9. Lj. B. Ciri`c, ` Generalized contractions and fixed point theorems, Publ. Inst. Math.(Beograd)(N. S.) 12(26) (1971), 19–26.
10. Lj. B. Ciri`c, A generalization of Banach's contraction principle, Proc. Amer. Math. Soc. 45 (1974), 267–273.
11. D. Dori`c and R. Lazovi`c, Some Suzuki-type fixed point theorems for generalized multivalued mappings and applications, Fixed Point Theory Appl. 2011 (2011), 1–13.
12. K. Goebel and W. A. Kirk, Topics in metric fixed point theory, vol. vol. 28 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, London, UK, 1990.
13. J. Jachymski, Equivalent conditions for generalized contractions on (ordered) metric spaces, Nonlinear Anal. 74 (2011), 768–774.
14. F. Khojasteh, M. Abbas and S. Costache, "Two New Types of Fixed Point Theorems in Complete Metric Spaces," Abstract and Applied Analysis Volume 2014, Article ID 325840, 5 pages.
15. R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc. 60 (1968), 71–76.
16. M. Kikkawa and T. Suzuki, Some similarity between contractions and Kannan mappings, Fixed Point Theory Appl. 2008 (2008), 1–8.
17. M. Kikkawa and T. Suzuki, Three fixed point theorems for generalized contractions with constants in complete metric spaces, Nonlinear Anal. 69(9) (2008), 2942–2949.
18. O. Popescu, Two fixed point theorems for generalized contractions with constants in complete metric space, Cent. Eur. J. Math. 7(3) (2009), 189–196.
19. B. E. Rhoades, A comparision of various definitions of contractive mappings, Trans. Amer. Math. Soc. 226 (1977), 257–290.

20. B. E. Rhoades, S. L. Singh and Chitra Kulshrestha, Coincidence theorems for some multivalued mappings, Internat. J. Math. Math. Sci. 7(3) (1984), 429–434.
21. S.L.Singh and S.N.Mishra, Remarks on recent fixed point theorems, Fixed Point Theory Appl. 2010 (2010), 1–18.
22. S. L. Singh, S. N. Mishra, R. Chugh and R. Kamal, General common fixed point theorems and applications, *J. Appl. Math*. 2012 (2012), Art. ID 902312, 14 pages.
23. T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc. 136(5) (2008), 1861–1869.
24. T. Suzuki, A new type of fixed point theorem in metric spaces, Nonlinear Anal. 71(11) (2009), 5313–5317.