**Dynamical Behavior of Predator-Prey Interaction in Fuzzy Environment**

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**ABSTRACT**

Nowadays the fuzziness of the parameters of any mathematical model plays a crucial role in the field of ecology. In this research article we apply the technique of Hukuhara derivative for mathematical formulation of predator-prey interaction. We formulate a two diamentional predator-prey model, with uncertain behavior of prey population and predator population. We formulate the FDE when both the populations i.e. prey population (x(t)) and predator population (y(t)) are i-gH differentiable, x(t) is i-gH and y(t) is ii-gH differentiable, x(t) is ii-gH and y(t) is i-gH differentiable and finally we present the model when both the population are in ii-gH differentiable. Stability criteria of the equilibrium points are described analytically. Here we observed that the dynamical behavior of the predator-prey interaction depends on the fuzziness of the parameter. From our study, we can conclude that the dynamical behavior of the mathematical model depends on the imprecise nature of the parameters, which is more realistic scenario in the ecology. This study opened a new dimension in ecological study.

Keywords: Predator; Prey; Harvesting; Fuzzy; Hukuhara derivative.

1. **INTRODUCTION:**

In the field of prey-predator system, Harvesting has immense importance as a biological phenomenon to study the management of renewable resources. Fishery and Forestry uses harvesting of prey population [1, 2, 3]. In the recent years, researcher are more interested to study the dynamical behavior of predator prey system with Harvesting in the management of renewable resources [4-10]. Fisheries and floristries has been exploiting renewable resources since the past few decades and has became a major concern in bio economic analysis [1, 11]. The dynamics of multy species harvesting models has demanded the time of many researchers [12, 13, 14, 15, 16]. Non-selective harvesting model is studied in [17, 18], though there is a problem on non-selective harvesting policy with logistic growth [1]. This type of model has maximum sustainable yield (MSY). The population will extinct if the harvesting of a population exceeds its MSY, and the decreased harvested species can be recovered if the harvesting rate is smaller than its MSY [19].

Harvesting Policy are of two types – one is constant quota harvesting and the other one is constant-effort harvesting. A constant number of individual in the population are harvested per unit time and in the constant-effort harvesting, fixed effort is applied per unit of time to catch the animals in a population [20]. Although a particular rule is not followed by harvesting. For example, more number of fishes are harvested in warmer seasons than in cold seasons [21].

Nowadays fuzzy differential equation (FDE) has been broadly explored in various fields like engineering, economics, biology, and physics. In real life situations fuzzy set theory plays an important role for modeling different types of problems. Individuals of a system usually show variations in their characteristic behavior. These differences remain bounded in a relatively small set that represents the characteristic behavior of the group. If we want to quantify the subjective quality that is being studied,we should attribute values or degrees to represent this quality satisfactorily. It is something that cannot always be obtained through this objective menstruation or statistics. Therefore, when the state variables of a given demographic system are uncertain in nature we should include fuzziness in to the system, which are then represented by fuzzy variables. In short, when we formulate models withreal-life situations, some data are not precisely known to us all the time. In such situations if we consider precise real-life data whichmay create an error. So, we should take the parameter as an imprecise parameter. In this situation a question may arise, whetherthe behavior of the model is same as that of the crisp model or not. Obviously, the behavior is different. We need to investigate the behavior in that environment. FDE is a powerful tool for describing the dynamics and characteristics of Harvesting model in predator prey systems.

FL-generalization is one of the remarkable contributions of fuzzy theory, which creates a relationship between crisp sets and the sets which depend on fuzzy sets. One type of FL-generalization is Fuzzy differential equations (FDEs), which is known as fuzzy mathematics. If all the coefficients or parameters of boundary conditions of a differential equations are fuzzy, the differential equation is said to be a fuzzy differential equation [22]. Fuzzy set theory is proposed by Zadeh. Fuzzy set theory describes the term membership function and uncertainty [23]. In the recent years fuzzy set theory is one of the popular branches of mathematics due to its application in different fields of science. Various dynamical systems can be modeled with the help of Fuzzy differential equations due to the uncertain behavior of the parameters [24]. Harvesting in prey-predator system is described elaborately in [25] with the help of fuzziness.

In the dynamical system the concept of fuzziness plays a crucial role. Hukuhara derivative is one of the pioneering concepts to deal with the fuzzy differential equations [26]. Fuzzy differential equation established in 1978 [22]. So many researchers devoted their time to develop the concept of fuzzy differential equations and its application [22]. The solution of crisp model and the solution of fuzzy differential equation are quite different. To generalize this, a lot of researchers devoted their time [24,27,28–40]. The solution of first order fuzzy differential equation raises many problems since the solution is different for different interpretation [41–43]. There are several methods such as generalized differentiability concepts, operator method etc. for solving first order and second order fuzzy differential equations [44–48]. To solve fuzzy differential equation numerically, generalized Euler approximation method is used [49,50].

In this research article our main aim is to model the predator-prey system with harvesting with the help of fuzzy differential equations and analyze the model when both the population are uncertain in nature which is more realistic scenario. In the section 2 we describe the preliminaries of fuzzy differential equation, In the next section we introduce the mathematical model of predator-prey interaction with harvesting (Section 3). Section 4 is devoted for the fuzzy differential equation, when both the populations i.e. prey population (x(𝑡)) and predator population (y(𝑡)) are i-gH differentiable, 𝑥(𝑡) is i-gH and 𝑦(𝑡) is ii-gH differentiable, 𝑥(𝑡) is ii-gH and 𝑦(𝑡) is i-gH differentiable and finally we present the model when both the population are in ii-gH differentiable and give the corresponding results respectively. Finally we end the paper with discussion and conclusion.

**2. PRELIMINARIES:**

**2.1. Triangular fuzzy number [51]:**

 The membership function of triangular fuzzy number M = ($m\_{1}, m\_{2},m\_{3}$) are given by

n(x) = $\frac{x-m\_{1}}{m\_{2}-m\_{1}}$ , $m\_{1}\leq x\leq m\_{2}$

 = $\frac{m\_{3}-x}{m\_{3}-m\_{2}}$ , $m\_{2}\leq x\leq m\_{3}$

 = 0, $m\_{1}\geq x, m\_{3}\leq x$

**2.2.** **α-cut of triangular fuzzy number [51]:**

The α-cut of triangular fuzzy number M = ($m\_{1}, m\_{2},m\_{3}$), $∀$α$\in $ [0,1] is given by

 $M\_{α} = [m\_{1}+α(m\_{2}-m\_{1}), m\_{3}-α(m\_{3}-m\_{2})].$

**2.3. Generalized Hukuhara difference [52]:**

Let us define the generalized difference of A and B as the set C $\in M(X)$ , where $A, B\in M(X)$; such that

A$Θ\_{g}$B = C $⟺\left\{\left(i\right)A=B+C\right.$

 $\left\{or (ii)B=A+(-1)C\right.$

**2.4. Generalized Hukuhara derivative for fuzzy valued function [53]:**

Let g:(a,b) $\rightarrow R\_{F}$ and p0$\in $ (a,b), then the function g is said to be strongly generalized Hukuhara differentiable or gH-differentiable at p0 if there exist an element g’(p0) $\in R\_{F},$ such that

1. $∀ϵ$> 0 sufficiently small, $∃ $g(p0+$ϵ$)$Θ\_{g}$g(p0), g(p0)$Θ\_{g}$ g(p0-$ϵ)$ and the limits

g’(p0) = $\lim\_{ϵ\to 0}\frac{g(p\_{0}+ ϵ)Θ\_{g}g(p\_{0})}{ϵ}$ = $\lim\_{ϵ\to 0}\frac{g(p\_{0}) Θ\_{g} g(p\_{0}- ϵ)}{ϵ}$

or (ii) $∀ϵ$> 0 sufficiently small, $∃$g(p0)$Θ\_{g}$g(p0+$ϵ$), g(p0-$ϵ$) $Θ\_{g}$ g(p0$)$ and the limits

g’(p0) = $\lim\_{ϵ\to 0}\frac{g\left(p\_{0}\right)Θ\_{g} g(p\_{0}+ ϵ)}{-ϵ}$ = $\lim\_{ϵ\to 0}\frac{g(p\_{0}- ϵ) Θ\_{g} g(p\_{0})}{-ϵ}$

or (iii) $∀ϵ$> 0 sufficiently small, $∃$g(p0+$ϵ$)$Θ\_{g}$g(p0), f(p0-$ϵ$) $Θ\_{g}$ f(p0$)$ and the limits

g’(p0) = $\lim\_{ϵ\to 0}\frac{g\left(p\_{0}+ ϵ\right)Θ\_{g}g(p\_{0})}{ϵ}$ = $\lim\_{ϵ\to 0}\frac{g(p\_{0}- ϵ) Θ\_{g} f(p\_{0})}{-ϵ}$

or (iv) $∀ϵ$> 0 sufficiently small, $∃$g(p0)$Θ\_{g}$g(p0+$ϵ$), g(p0) $Θ\_{g}$ g(p0-$ϵ)$and the limits

g’(p0) = $\lim\_{ϵ\to 0}\frac{g\left(p\_{0}\right)Θ\_{g}g\left(p\_{0}+ϵ\right)}{-ϵ}$ = $\lim\_{ϵ\to 0}\frac{g\left(p\_{0}\right)Θ\_{g}g\left(p\_{0}-ϵ\right)}{ϵ}$

**2.5.** **Strong and weak solution of fuzzy differential equation:**

 Let $\tilde{g}(t)$ be the solution offuzzy differential equation $\frac{d g(t)}{dt} = g\_{1}(t,x(t))$with the initial condition g(t0)=x0. Here the initial condition is fuzzy number. The parametric form of α-cut of the solution $\tilde{g(t)}$ is given by [$g\_{1}(t,α),g\_{2}(t,α)]$.

 The solution $\tilde{g(t)}$is said to be strong fuzzy solution if $g\_{1}(t,α)$ is increasing function and $g\_{2}(t,α) $is decreasing function and $g\_{1}(t,α)\leq g\_{2}(t,α),∀$α$\in $ [0,1]. Otherwise the fuzzy solution is called weak solution.

 The weak fuzzy solution is also a solution of the fuzzy differential equation, but we want to modify the weak fuzzy solution and transformed it to strong fuzzy solutionby using the formula

 $\tilde{g}(t,α) = \left[min \left\{g\_{1}(t,α),g\_{2}(t,α)\right\}, max\left\{g\_{1}(t,α),g\_{2}(t,α)\right\}\right]$.

1. **DEVELOPMENT OF THE MATHEMATICAL MODEL:**

This portion is devoted for mathematical model of predator-prey system with harvesting. Where x(t) is the prey population and y(t) is the predator population.

$$\frac{dx(t)}{dt}=rx(t)\left(1-\frac{x(t)}{k\_{1}}\right)-ax(t)y(t)-ex(t)$$

$\frac{dy\left(t\right)}{dt}=kax\left(t\right)y\left(t\right)-d\_{2}y\left(t\right)$(1)

$$With x\left(0\right)=x0 and y\left(0\right)=y0.$$

The parameters of the above system are described below. The intrinsic growth rate of the prey population is given by r, carrying capacity of the prey population is k1, predation rate of prey population by predator is given by a and k is the conversion rate of prey population in to predator is given by k. The harvesting coefficients of prey population is given by e and d2 is the death rate of the predator population.

1. **PREDATOR-PREY MODEL WITH HARVESTING IN FUZZY ENVIRONMENT:**

 Let us consider the fuzzy solution of the system of equation (1) is $\left(\tilde{x}\left(t\right), \tilde{y}\left(t\right)\right).$ $α$ -cut of $\tilde{x}\left(t\right)$ is given by [$x\_{1}(t,α)$, $x\_{2}(t,α)]$ and the $α$ -cut of $\tilde{y}\left(t\right)$ is given by [$y\_{1}(t,α)$, $y\_{2}(t,α)]$ .

Where $α$is the parameter which satisfy the condition 0 ≤ $α$≤ 1

**4.1.** **Case 1: when 𝑥(𝑡) and 𝑦(𝑡) both are i-gH differentiable**

The system of equation (1) transformed to the model

$$\frac{dx\_{1}(t,α)}{dt}=rx\_{2}(t,α)\left(1-\frac{x\_{2}(t,α)}{k\_{1}}\right)-ax\_{2}(t,α)y\_{1}(t,α)-ex\_{2}(t,α)$$

$\frac{dx\_{2}(t,α)}{dt}=rx\_{1}(t,α)\left(1-\frac{x\_{1}(t,α)}{k\_{1}}\right)-ax\_{1}(t,α)y\_{2}(t,α)-ex\_{1}(t,α)$(2)

$\frac{dy\_{1}\left(t,α\right)}{dt}=kax\_{1}\left(t,α\right)y\_{2}\left(t,α\right)-d\_{2}y\_{2}\left(t,α\right)$

$$\frac{dy\_{2}(t,α)}{dt}=kax\_{2}(t,α)y\_{1}(t,α)-d\_{2}y\_{1}(t,α)$$

With$ x\_{1}\left(0,α\right)=x\_{01}\left(α\right),x\_{2}\left(0,α\right)=x\_{02}\left(α\right), y\_{1}\left(0,α\right)=y\_{01}\left(α\right) and y\_{2}\left(0,α\right)=y\_{02}\left(α\right)$

**4.1.1.** **Stability analysis of system**

 The system of equation (2) has interior point equilibrium as $E\_{1}^{\*}(x\_{1}^{1\*},x\_{2}^{1\*},y\_{1}^{1\*},y\_{2}^{1\*})$

 Where,

 $x\_{1}^{1\*}=\frac{d\_{2}}{ka}$ , $x\_{2}^{1\*}=\frac{d\_{2}}{ka} $,

 $y\_{1}^{1\*}= \frac{kk\_{1}a(r-e)-d\_{2}r}{kk\_{1}a^{2}},$ $y\_{2}^{1\*}= \frac{kk\_{1}a(r-e)-d\_{2}r}{kk\_{1}a^{2}}$

It is clear that the equilibrium point$ E\_{1}^{\*}$ exists if $kk\_{1}a(r-e)>d\_{2}r$. Which is the feasible condition of the equilibrium point $E\_{1}^{\*}.$

 Therefore the jacobian matrix is denoted by $V\_{1}^{\*}$, and it is defined by,

 $V\_{1}^{\*}= \left(\begin{matrix}\begin{matrix}0\\a\_{21}\\\begin{matrix}a\_{31}\\0\end{matrix}\end{matrix}&\begin{matrix}\begin{matrix}a\_{12}\\0\\\begin{matrix}0\\a\_{42}\end{matrix}\end{matrix}&\begin{matrix}\begin{matrix}a\_{13}\\0\\\begin{matrix}0\\0\end{matrix}\end{matrix}&\begin{matrix}0\\a\_{24}\\\begin{matrix}0\\0\end{matrix}\end{matrix}\end{matrix}\end{matrix}\end{matrix}\right)$

Where,

 $a\_{12}=r-\frac{2rx\_{2}^{1\*}}{k\_{1}}-ay\_{1}^{1\*}-e$, $a\_{13}=-ax\_{2}^{1\*},$

 $a\_{21}=r-\frac{2rx\_{1}^{1\*}}{k\_{1}}-ay\_{2}^{1\*}-e,$ $a\_{24}= -ax\_{1}^{1\*},$

 $a\_{31}= kay\_{2}^{1\*},$

 $a\_{42}= kay\_{1}^{1\*}$

The characteristic equation is given by

 $ξ\_{1}^{4}+A\_{1}ξ\_{1}^{3}+A\_{2}ξ\_{1}^{2}+A\_{3}ξ\_{1}+A\_{4}=0$

Where,

$$A\_{1}=0.$$

$A\_{2}=a\_{12}a\_{21}+a\_{24}a\_{42}+a\_{13}a\_{31}$ . where, $a\_{12}=r-\frac{2rx\_{2}^{1\*}}{k\_{1}}-ay\_{1}^{1\*}-e$ ,$ a\_{13}=-ax\_{2}^{1\*},a\_{21}=r-\frac{2rx\_{1}^{1\*}}{k\_{1}}-ay\_{2}^{1\*}-e,a\_{24}= -ax\_{1}^{1\*},a\_{24}= -ax\_{1}^{1\*},a\_{31}= kay\_{2}^{1\*},a\_{42}= kay\_{1}^{1\*}$.

$A\_{3}=$ 0.

$A\_{4}=a\_{13}a\_{24}a\_{31}a\_{42}$.

Here$ A\_{1}=$ 0$, A\_{3}=$ 0,

So by Routh-Hurwitz criteria, the above system is unstable.

Therefore when 𝑥(𝑡) and𝑦(𝑡) both are i-gH differentiable, the system is unstable.

**4.2.** **Case 2: when 𝑥(𝑡) is i-gH differentiable and 𝑦(𝑡) is ii-gH differentiable**

The system of equation (1) transformed to the model

$$\frac{dx\_{1}(t,α)}{dt}=rx\_{2}(t,α)\left(1-\frac{x\_{2}(t,α)}{k\_{1}}\right)- ax\_{2}(t,α)y\_{1}(t,α)-ex\_{2}(t,α)$$

$$\frac{dx\_{2}(t,α)}{dt}=rx\_{1}(t,α)\left(1-\frac{x\_{1}(t,α)}{k\_{1}}\right)- ax\_{1}(t,α)y\_{2}(t,α)-ex\_{1}(t,α)$$

$\frac{dy\_{1}\left(t,α\right)}{dt}=kax\_{2}\left(t,α\right)y\_{1}\left(t,α\right)-d\_{2}y\_{1}\left(t,α\right)$(3)

$$\frac{dy\_{2}(t,α)}{dt}=kax\_{1}(t,α)y\_{2}(t,α)-d\_{2}y\_{2}(t,α)$$

With $x\_{1}\left(0,α\right)=x\_{01}\left(α\right),x\_{2}\left(0,α\right)=x\_{02}\left(α\right), y\_{1}\left(0,α\right)=y\_{01}\left(α\right) and y\_{2}\left(0,α\right)=y\_{02}\left(α\right)$

**4.2.1. Stability analysis of system**

 The system of equation (6) has interior point equilibrium as $E\_{2}^{\*}(x\_{1}^{2\*},x\_{2}^{2\*},y\_{1}^{2\*},y\_{2}^{2\*})$

 Where,

 $x\_{1}^{2\*}=\frac{d\_{2}}{ka }$, $x\_{2}^{2\*}=\frac{d\_{2}}{ka} $,

 $y\_{1}^{2\*}= \frac{kk\_{1}a(r-e)-d\_{2}r}{kk\_{1}a^{2}},$ $y\_{2}^{2\*}= \frac{kk\_{1}a(r-e)-d\_{2}r}{kk\_{1}a^{2}}$

It is clear that the equilibrium point $E\_{2}^{\*}$ exists if $kk\_{1}a(r-e)>d\_{2}r$. Which is the feasible condition of the equilibrium point $E\_{2}^{\*}.$

 Therefore the jacobian matrix is denoted by $V\_{2}^{\*}$, and it is defined by,

$$V\_{2}^{\*}= \left(\begin{matrix}\begin{matrix}0\\a\_{21}\\\begin{matrix}0\\a\_{41}\end{matrix}\end{matrix}&\begin{matrix}\begin{matrix}a\_{12}\\0\\\begin{matrix}a\_{32}\\0\end{matrix}\end{matrix}&\begin{matrix}\begin{matrix}a\_{13}\\0\\\begin{matrix}0\\0\end{matrix}\end{matrix}&\begin{matrix}0\\a\_{24}\\\begin{matrix}0\\0\end{matrix}\end{matrix}\end{matrix}\end{matrix}\end{matrix}\right)$$

Where,

 $a\_{12}=r-\frac{2rx\_{2}^{2\*}}{k\_{1}}-ay\_{1}^{2\*}-e$, $a\_{13}=-ax\_{2}^{2\*},$

$a\_{21}=r-\frac{2rx\_{1}^{2\*}}{k\_{1}}-ay\_{2}^{2\*}-e,$ $a\_{24}= -ax\_{1}^{2\*},$

 $a\_{32}= kay\_{1}^{2\*},$

 $a\_{41}= kay\_{2}^{2\*}$.

The characteristic equation corresponding to the equilibrium point $E\_{2}^{\*}$ is given by

 $ξ\_{2}^{4}+B\_{1}ξ\_{2}^{3}+B\_{2}ξ\_{2}^{2}+B\_{3}ξ\_{2}+B\_{4}=0$

Where,

$$B\_{1}=0,$$

$B\_{2}=a\_{12}a\_{21}.$ where, $a\_{12}=r-\frac{2rx\_{2}^{2\*}}{k\_{1}}-ay\_{1}^{2\*}-e,a\_{21}=r-\frac{2rx\_{1}^{2\*}}{k\_{1}}-ay\_{2}^{2\*}-e,$

$B\_{3}=a\_{12 }a\_{24}a\_{41}+a\_{13}a\_{21}a\_{32}$,

$B\_{4}=a\_{13}a\_{24}a\_{32}a\_{41}$.

Here $B\_{1}=$ 0 ,

 So by Routh-Hurwitz criteria, the above system is unstable.

 Therefore, when 𝑥(𝑡) is i-gH differentiable and 𝑦(𝑡) is ii-gH differentiable, the system is unstable.

**4.3. Case 3:** **When 𝑥(𝑡) is ii-gH differentiable and 𝑦(𝑡) is i-gH differentiable**

The system of equation (1) transformed into the model

$$\frac{dx\_{1}(t,α)}{dt}=rx\_{1}(t,α)\left(1-\frac{x\_{1}(t,α)}{k\_{1}}\right)-ax\_{1}(t,α)y\_{2}(t,α)-ex\_{1}(t,α)$$

$$\frac{dx\_{2}(t,α)}{dt}=rx\_{2}(t,α)\left(1-\frac{x\_{2}(t,α)}{k\_{1}}\right)-ax\_{2}(t,α)y\_{1}(t,α)-ex\_{2}(t,α)$$

$\frac{dy\_{1}\left(t,α\right)}{dt}=kax\_{1}\left(t,α\right)y\_{2}\left(t,α\right)-d\_{2}y\_{2}\left(t,α\right)$(4)

$$\frac{dy\_{2}(t,α)}{dt}=kax\_{2}(t,α)y\_{1}(t,α)-d\_{2}y\_{1}(t,α)$$

With$ x\_{1}\left(0,α\right)=x\_{01}\left(α\right),x\_{2}\left(0,α\right)=x\_{02}\left(α\right), y\_{1}\left(0,α\right)=y\_{01}\left(α\right) and y\_{2}\left(0,α\right)=y\_{02}\left(α\right)$

**4.3.1. Stability analysis of system**

 The system of equation (4) has interior point equilibrium as $E\_{3}^{\*}(x\_{1}^{3\*},x\_{2}^{3\*},y\_{1}^{3\*},y\_{2}^{3\*})$

 Where,

 $x\_{1}^{3\*}=\frac{d\_{2}}{ka}$ , $x\_{2}^{3\*}=\frac{d\_{2}}{ka}$ ,

 $y\_{1}^{3\*}= \frac{kk\_{1}a(r-e)-d\_{2}r}{kk\_{1}a^{2}}$ $, y\_{2}^{3\*}= \frac{kk\_{1}a(r-e)-d\_{2}r}{kk\_{1}a^{2}}$

It is clear that the equilibrium point$E\_{3}^{\*}$ exists if $kk\_{1}a(r-e)>d\_{2}r$. Which is the feasible condition of the equilibrium point $E\_{3}^{\*}.$

 Therefore the jacobian matrix is denoted by $V\_{3}^{\*}$, and it is defined by,

 $V\_{3}^{\*}=\left[\begin{matrix}\begin{matrix}a\_{11}\\0\\\begin{matrix}a\_{31}\\0\end{matrix}\end{matrix}&\begin{matrix}\begin{matrix}0\\a\_{22}\\\begin{matrix}0\\a\_{42}\end{matrix}\end{matrix}&\begin{matrix}\begin{matrix}0\\a\_{23}\\\begin{matrix}0\\0\end{matrix}\end{matrix}&\begin{matrix}a\_{14}\\0\\\begin{matrix}0\\0\end{matrix}\end{matrix}\end{matrix}\end{matrix}\end{matrix}\right]$

Where,

 $a\_{11}= r-\frac{2rx\_{1}^{3\*}}{k\_{1}}-ay\_{2}^{3\*}-e$, $a\_{14}=-ax\_{1}^{3\*},$

 $a\_{22}=r-\frac{2rx\_{2}^{3\*}}{k\_{1}}-ay\_{1}^{3\*}-e,$ $a\_{23}= -ax\_{2}^{3\*},$

 $a\_{31}= kay\_{2}^{3\*},$

 $a\_{42}= kay\_{1}^{3\*}$

The characteristic equation is given by

 $ξ\_{3}^{4}+C\_{1}ξ\_{3}^{3}+C\_{2}ξ\_{3}^{2}+C\_{3}ξ\_{3}+C\_{4}=0$

Where,

$C\_{1}=a\_{11}+ a\_{22} .$ where, $a\_{11}= r-\frac{2rx\_{1}^{3\*}}{k\_{1}}-ay\_{2}^{3\*}-e$ , $a\_{22}=r-\frac{2rx\_{2}^{3\*}}{k\_{1}}-ay\_{1}^{3\*}-e$.

$C\_{2}=a\_{11}a\_{22}$ . where, $a\_{11}= r-\frac{2rx\_{1}^{3\*}}{k\_{1}}-ay\_{2}^{3\*}-e$ , $a\_{22}=r-\frac{2rx\_{2}^{3\*}}{k\_{1}}-ay\_{1}^{3\*}-e$.

$C\_{3}=$ 0.

$C\_{4}=a\_{14}a\_{23}a\_{31}a\_{42}$. Where $a\_{14}=-ax\_{1}^{3\*},a\_{23}= -ax\_{2}^{3\*},a\_{31}= kay\_{2}^{3\*},a\_{42}= kay\_{1}^{3\*}$.

Here$C\_{3}=$ 0,

So by Routh-Hurwitz criteria, the above system is unstable.

Therefore when 𝑥(𝑡) is ii-gH differentiable and 𝑦(𝑡) is i-gH differentiable, the system is unstable.

**4.4. Case 4:** **when 𝑥(𝑡) and 𝑦(𝑡) both are ii-gH differentiable**

The system of equation (1) transformed to the model

$$\frac{dx\_{1}(t,α)}{dt}=rx\_{1}(t,α)\left(1-\frac{x\_{1}(t,α)}{k\_{1}}\right)- ax\_{1}(t,α)y\_{2}(t,α)-ex\_{1}(t,α)$$

$\frac{dx\_{2}(t,α)}{dt}=rx\_{2}(t,α)\left(1-\frac{x\_{2}(t,α)}{k\_{1}}\right)- ax\_{2}(t,α)y\_{1}(t,α)-ex\_{2}(t,α)$ (5)

$\frac{dy\_{1}\left(t,α\right)}{dt}=kax\_{2}\left(t,α\right)y\_{1}\left(t,α\right)-d\_{2}y\_{1}\left(t,α\right)$

$$\frac{dy\_{2}(t,α)}{dt}=kax\_{1}(t,α)y\_{2}(t,α)-d\_{2}y\_{2}(t,α)$$

With $x\_{1}\left(0,α\right)=x\_{01}\left(α\right),x\_{2}\left(0,α\right)=x\_{02}\left(α\right), y\_{1}\left(0,α\right)=y\_{01}\left(α\right) and y\_{2}\left(0,α\right)=y\_{02}\left(α\right)$

**4.4.1. Stability analysis of system**

The system of equation (5) has interior point equilibrium as $E\_{4}^{\*}(x\_{1}^{4\*},x\_{2}^{4\*},y\_{1}^{4\*},y\_{2}^{4\*})$

Where,

 $x\_{1}^{4\*}=\frac{d\_{2}}{ka}$, $x\_{2}^{4\*}=\frac{d\_{2}}{ka}$,

 $y\_{1}^{4\*}= \frac{kk\_{1}a(r-e)-d\_{2}r}{kk\_{1}a^{2}},$ $y\_{2}^{4\*}= \frac{kk\_{1}a(r-e)-d\_{2}r}{kk\_{1}a^{2}}$

It is clear that the equilibrium point $E\_{4}^{\*}$ exists if $kk\_{1}a(r-e)>d\_{2}r$. Which is the feasible condition of the equilibrium point $E\_{4}^{\*}.$

Therefore the jacobian matrix is denoted by $V\_{4}^{\*}$, and it is defined by,

$$V\_{4}^{\*}= \left[\begin{matrix}\begin{matrix}a\_{11}\\0\\\begin{matrix}0\\a\_{41}\end{matrix}\end{matrix}&\begin{matrix}\begin{matrix}0\\a\_{22}\\\begin{matrix}a\_{32}\\0\end{matrix}\end{matrix}&\begin{matrix}\begin{matrix}0\\a\_{23}\\\begin{matrix}a\_{33}\\0\end{matrix}\end{matrix}&\begin{matrix}a\_{14}\\0\\\begin{matrix}0\\a\_{44}\end{matrix}\end{matrix}\end{matrix}\end{matrix}\end{matrix}\right]$$

Where,

 $a\_{11}= r-\frac{2rx\_{1}^{4\*}}{k\_{1}}-ay\_{2}^{4\*}-e$, $a\_{14}=-ax\_{1}^{4\*},$

 $a\_{22}=r-\frac{2rx\_{2}^{4\*}}{k\_{1}}-ay\_{1}^{4\*}-e,$ $a\_{23}= -ax\_{2}^{4\*},$

 $a\_{32}= kay\_{1}^{4\*},$ $a\_{33} = 0,$

 $a\_{41}= kay\_{2}^{4\*}$, $a\_{44} = 0.$

The characteristic equation is given by

 $ξ^{\*4}+P\_{1}ξ^{\*3}+P\_{2}ξ^{\*2}+P\_{3}ξ^{\*}+P\_{4}=0$

Where,

$P\_{1}=a\_{11}+ a\_{22} .$ where, $a\_{11}= r-\frac{2rx\_{1}^{\*4}}{k\_{1}}-ay\_{2}^{\*4}-e$,$a\_{22}=r-\frac{2rx\_{2}^{\*4}}{k\_{1}}-ay\_{1}^{\*4}-e,$

$P\_{2}=a\_{23}a\_{32}-a\_{11}a\_{22}+a\_{14}a\_{41}$ . where, $a\_{14}=-ax\_{1}^{\*4},a\_{23}= -ax\_{2}^{\*4}, a\_{32}= kay\_{1}^{\*4},a\_{41}= kay\_{2}^{\*4}$.

$P\_{3}=a\_{11}a\_{23}a\_{32}+a\_{14}a\_{22}a\_{41}$,

$P\_{4}=a\_{14}a\_{23}a\_{32}a\_{41}$.

Now,

$P\_{1}>$0

$P\_{2}>$0

$P\_{3}>$0

$P\_{4}>$0

Therefore by Routh-Hurwitz criteria the system is stable if $P\_{1},P\_{2},P\_{3},P\_{4}$ all ae positive and $P\_{1}P\_{2}-P\_{3}>0$ and $P\_{1}P\_{2}P\_{3}-P\_{3}^{2}-P\_{1}^{2}P\_{4}>0.$

Therefore when 𝑥(𝑡) and 𝑦(𝑡) both are ii-gH differentiable, the system is stable.

1. **DISCUSSION AND CONCLUSION**

In this research article we study the dynamical behavior of Predator-Prey interaction in fuzzy environment. In any mathematical representation of ecological problem it is not necessary that all the parameters are crisp or precise. For different behavioral aspects, some parameters may be imprecise. Due to presence of the impreciseness among the parameters the dynamical behavior of the system changes. The concepts of fuzziness plays crucial role in the dynamical system.

In this research article at first we consider a two diamentional Predator-Prey system then we establish four set of fuzzy differential equation model of Predator-Prey system with fuzzy initial value. In the fuzzy differential equation at first we consider both the population i.e. 𝑥(𝑡, 𝛼), 𝑦(𝑡, 𝛼) are i-gH differentiable and observe the dynamical behavior of the model analytically, then we consider that 𝑥(𝑡, 𝛼) is i-gH differentiable and 𝑦(𝑡, 𝛼) is ii-gH differentiable and observe the dynamical behavior of the model, then we consider that 𝑥(𝑡, 𝛼) is ii-gH differentiable and 𝑦(𝑡, 𝛼) is i-gH differentiable and observe the dynamical behavior of the model analytically. At last we consider both the population 𝑥(𝑡, 𝛼), 𝑦(𝑡, 𝛼) are ii-gH differentiable and observe the dynamical behavior of the model analytically. Here we observe that the equilibrium point corresponding to the first three fuzzy system of equation are unstable and the equilibrium point corresponding to the last fuzzy system of equation is stable i.e when both the population is ii-gH differentiable then the solution is stable.

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