**SIMILARITY MEASURES OF FERMATEAN NEUTROSOPHIC SETS BASED ON THE COSINE FUNCTION AND THEIR APPLICATIONS**

1 Radha R, 2  Princy R , 3Gayathri P ,4Gomathi S , 5Kavitha A, 6Kalamani A

**ABSTRACT:**

In this study, we analyse the degree of hesitation, non-membership, and membership in Fermatean Neutrosophic sets (FNSs) and give similarity metrics between them based on the cosine function. Next, we utilise these similarity measures along with weighted measures amongst FNSs for diagnosing medical conditions and identifying patterns. Lastly, two instances are provided to show how effective similarity measures are at identifying patterns and making medical diagnoses.

1. **SOME SIMILARITY MEASURE BASED ON THE COSINE FUNCTION FOR FERMATEAN NEUTROSOPHIC SETS**

**3.1 Cosine Similarity Measure for FNSs**

Let $A$ be a FNS in an universe of discourse $X=\{x\}$, the FNS is characterized by the degree of membership $μ\_{A}(x)$, the degree of non-membership $ϑ\_{A}(x)$, and the degree of hesitation $π\_{A}(x)$, $π\_{A}\left(x\right)=\sqrt{1-\left[μ\_{A}^{3}\left(x\right)+ϑ\_{A}^{3}\left(x\right)\right]}$, which can be considered as a vector representation with the three elements. Therefore, a cosine similarity measure and a weighted cosine similarity measure for FNSs are proposed in an analogous manner to the cosine similarity measure based on Bhattacharya’s distance [21, 6]and cosine similarity measure for IFS [28]**.**

 Suppose that there are two FNSs $A=\left\{<x\_{j}, μ\_{A}\left(x\_{j}\right), ϑ\_{A}\left(x\_{j}\right), π\_{A}\left(x\_{j}\right)>\right|x\_{j}\in X\}$ and $B=\left\{<x\_{j}, μ\_{B}\left(x\_{j}\right), ϑ\_{B}\left(x\_{j}\right), π\_{B}\left(x\_{j}\right)>\right|x\_{j}\in X\}$ in the universe of discourse $X=\{x\_{1}, x\_{2}, …,x\_{n}\}$, we further propose the cosine similarity measures between FNSs as follows:

$FNC\left(A, B\right)=\frac{1}{n}\sum\_{j=1}^{n}\frac{μ\_{A}^{3}\left(x\_{j}\right)μ\_{B}^{3}\left(x\_{j}\right)+ ϑ\_{A}^{3}\left(x\_{j}\right)ϑ\_{B}^{3}\left(x\_{j}\right)+π\_{A}^{3}\left(x\_{j}\right)π\_{B}^{3}\left(x\_{j}\right)}{\sqrt{μ\_{A}^{6}\left(x\_{j}\right)+ϑ\_{A}^{6}\left(x\_{j}\right)+π\_{A}^{6}\left(x\_{j}\right)} \sqrt{μ\_{B}^{6}\left(x\_{j}\right)+ϑ\_{B}^{6}\left(x\_{j}\right)+π\_{B}^{6}\left(x\_{j}\right)} }$ ------------(17)

If we take n=1, then the cosine similarity measure between FNSs $A$ and $B$ becomes the correlation coefficient between FNSs $A$ and $B$. Therefore, the cosine similarity measure between $A$ and $B$ also satisfies the following properties:

1. $0\leq FNC(A, B)\leq 1$
2. $FNC\left(A, B\right)=FNC\left(B,A\right)$.
3. $FNC\left(A, B\right)=1,$ if $A=B, i=1,2,…,n$

Proof:

1. It is obvious that the proposition is true according to the cosine value.
2. It is obvious that the proposition is true.
3. When $A=B, $there are $μ\_{A}\left(x\_{j}\right)=μ\_{B}\left(x\_{j}\right)$, $ϑ\_{A}\left(x\_{j}\right)=ϑ\_{B}\left(x\_{j}\right)$ and $π\_{A}\left(x\_{j}\right)=π\_{B}\left(x\_{j}\right) for j=1,2,…,n.$ So there is $C\_{FNS}\left(A, B\right)=1$.

Therefore, we have finished the proofs.

If we consider the weights of $x\_{j}$, a weighted cosine similarity measure between FNSs $A$ and $B$ is proposed as follows:

$$WFNC\left(A, B\right)=\sum\_{j=1}^{n}ω\_{j}\frac{μ\_{A}^{3}\left(x\_{j}\right)μ\_{B}^{3}\left(x\_{j}\right)+ ϑ\_{A}^{3}\left(x\_{j}\right)ϑ\_{B}^{3}\left(x\_{j}\right)+π\_{A}^{3}\left(x\_{j}\right)π\_{B}^{3}\left(x\_{j}\right)}{\sqrt{μ\_{A}^{6}\left(x\_{j}\right)+ϑ\_{A}^{6}\left(x\_{j}\right)+π\_{A}^{6}\left(x\_{j}\right)} \sqrt{μ\_{B}^{6}\left(x\_{j}\right)+ϑ\_{B}^{6}\left(x\_{j}\right)+π\_{B}^{6}\left(x\_{j}\right)} }$$

 --------------(18)

 Where $ω=\left(ω\_{1}, ω\_{2}, …,ω\_{n}\right)^{T}$ is the weight vector of $x\_{j}(j=1,2,…,n)$, with $ω\_{j}\in \left[0,1\right], j=1,2,…,n$, $\sum\_{j=1}^{n}ω\_{j}=1.$ In particular, if $ω=\left({1}/{n}, {1}/{n},…,{1}/{n}\right)^{T}$, then the weighted cosine similarity measure reduces to cosine similarity measure. That is to say, if we take $ω\_{j}=\frac{1}{n}, j=1,2,…,n,$ then there is $WFNC\left(A, B\right)=FNC(A, B)$. Obviously, the weighted cosine similarity measure of two FNSs $A$ and $B$ also satisfies the following properties:

1. $0\leq WFNC(A, B)\leq 1$
2. $WFNC\left(A, B\right)=WFNC\left(B,A\right)$.
3. $WFNC\left(A, B\right)=1,$ if $A=B, i=1,2,…,n$.

Similar to the previous proof method, we can prove the above three properties.

In the following, we shall investigate the distance measure of the angle as

$d\left(A, B\right)=arccos\left(C\_{FNS}\left(A, B\right)\right).$ It satisfies the following properties:

1. $d\left(A, B\right)\geq 0,$ if $0\leq C\_{FNS}\left(A, B\right)\leq 1$;
2. $d\left(A, B\right)=$ $\arccos(\left(1\right))=0$, if $C\_{FNS}\left(A, A\right)=1.$
3. $d\left(A, B\right)=d\left(B,A\right),$ if $C\_{FNS}\left(A, B\right)=C\_{FNS}\left(B,A\right)$
4. $d\left(A, C\right)\leq d\left(A, B\right)+d\left(B, C\right),$ if $A⊆B⊆C$ for any FNS $C$.

Proof:

Obviously, $d\left(A, B\right)$ satisfies the property (1) - (3). In the following, $d\left(A, B\right)$ will be proved to satisfy the property (4).

 For any $C=\left\{<x\_{j}, μ\_{C}\left(x\_{j}\right), ϑ\_{C}\left(x\_{j}\right), π\_{C}\left(x\_{j}\right)>\right|x\_{j}\in X\}$, $A⊆B⊆C$, Since Equation (16) is the sum of terms, let us investigate the distance measures of the angle between the vectors: $d\_{j}\left(A\left(x\_{j}\right), B\left(x\_{j}\right)\right)=\arccos(\left(FNC\left(A\left(x\_{j}\right), B\left(x\_{j}\right)\right)\right)),$

 $d\_{j}\left(B\left(x\_{j}\right), C\left(x\_{j}\right)\right)=\arccos(\left(FNC\left(B\left(x\_{j}\right), C\left(x\_{j}\right)\right)\right))$ and

$d\_{j}\left(A\left(x\_{j}\right), C\left(x\_{j}\right)\right)=\arccos(\left(FNC\left(A\left(x\_{j}\right), C\left(x\_{j}\right)\right)\right)), j=1,2,…,n$ where

$$FNC\left(A\left(x\_{j}\right), B\left(x\_{j}\right)\right)=\sum\_{j=1}^{n}\frac{μ\_{A}^{3}\left(x\_{j}\right)μ\_{B}^{3}\left(x\_{j}\right)+ ϑ\_{A}^{3}\left(x\_{j}\right)ϑ\_{B}^{3}\left(x\_{j}\right)+π\_{A}^{3}\left(x\_{j}\right)π\_{B}^{3}\left(x\_{j}\right)}{\sqrt{μ\_{A}^{6}\left(x\_{j}\right)+ϑ\_{A}^{6}\left(x\_{j}\right)+π\_{A}^{6}\left(x\_{j}\right)} \sqrt{μ\_{B}^{6}\left(x\_{j}\right)+ϑ\_{B}^{6}\left(x\_{j}\right)+π\_{B}^{6}\left(x\_{j}\right)} }$$

$$FNC\left(B\left(x\_{j}\right), C\left(x\_{j}\right)\right)=\sum\_{j=1}^{n}\frac{μ\_{B}^{3}\left(x\_{j}\right)μ\_{C}^{3}\left(x\_{j}\right)+ ϑ\_{B}^{3}\left(x\_{j}\right)ϑ\_{C}^{3}\left(x\_{j}\right)+π\_{B}^{3}\left(x\_{j}\right)π\_{C}^{3}\left(x\_{j}\right)}{\sqrt{μ\_{B}^{6}\left(x\_{j}\right)+ϑ\_{B}^{6}\left(x\_{j}\right)+π\_{B}^{6}\left(x\_{j}\right)} \sqrt{μ\_{C}^{6}\left(x\_{j}\right)+ϑ\_{C}^{6}\left(x\_{j}\right)+π\_{C}^{6}\left(x\_{j}\right)} }$$

$$FNC\left(A\left(x\_{j}\right), C\left(x\_{j}\right)\right)=\sum\_{j=1}^{n}\frac{μ\_{A}^{3}\left(x\_{j}\right)μ\_{C}^{3}\left(x\_{j}\right)+ ϑ\_{A}^{3}\left(x\_{j}\right)ϑ\_{C}^{3}\left(x\_{j}\right)+π\_{A}^{3}\left(x\_{j}\right)π\_{C}^{3}\left(x\_{j}\right)}{\sqrt{μ\_{A}^{6}\left(x\_{j}\right)+ϑ\_{A}^{6}\left(x\_{j}\right)+π\_{A}^{6}\left(x\_{j}\right)} \sqrt{μ\_{C}^{6}\left(x\_{j}\right)+ϑ\_{C}^{6}\left(x\_{j}\right)+π\_{C}^{6}\left(x\_{j}\right)} }$$

 For three vectors $A\left(x\_{j}\right)=<μ\_{A}\left(x\_{j}\right), ϑ\_{A}\left(x\_{j}\right), π\_{A}\left(x\_{j}\right)>$, $B\left(x\_{j}\right)=<μ\_{B}\left(x\_{j}\right), ϑ\_{B}\left(x\_{j}\right), π\_{B}\left(x\_{j}\right)>,$

$C\left(x\_{j}\right)=<μ\_{C}\left(x\_{j}\right), ϑ\_{C}\left(x\_{j}\right), π\_{C}\left(x\_{j}\right)>$ in one plane, if $A\left(x\_{j}\right)⊆B\left(x\_{j}\right)⊆C\left(x\_{j}\right), j=1,2,…,n.$ Then, it is obvious that $d\_{j}\left(A\left(x\_{j}\right), C\left(x\_{j}\right)\right)\leq d\_{j}\left(A\left(x\_{j}\right), B\left(x\_{j}\right)\right)+d\_{j}\left(B\left(x\_{j}\right), C\left(x\_{j}\right)\right)$ according to the triangle inequality. Combining the inequality with Equation (16), we can obtain $d\left(A, C\right)\leq d\left(A, B\right)+d\left(B, C\right).$ Thus $d\left(A, B\right)$ satisfies the property (4). So we finished the proof.

* 1. **Similarity measures of FNSs based on cosine function:**

Based on the cosine function, in this section, we shall propose two cosine similarity measures between FNSs and analyse their properties.

**Definition 3.2.1:**

Suppose that there are two FNSs $A=\left\{<x\_{j}, μ\_{A}\left(x\_{j}\right), ϑ\_{A}\left(x\_{j}\right), π\_{A}\left(x\_{j}\right)>\right|x\_{j}\in X\}$ and $B=\left\{<x\_{j}, μ\_{B}\left(x\_{j}\right), ϑ\_{B}\left(x\_{j}\right), π\_{B}\left(x\_{j}\right)>\right|x\_{j}\in X\}$ in the universe of discourse $X=\{x\_{1}, x\_{2}, …,x\_{n}\}$, we further propose the cosine similarity measures between FNSs as follows:

$FNCS^{1}\left(A, B\right)=\frac{1}{n}\sum\_{j=1}^{n}cos\left[\frac{π}{2}\left(\begin{array}{c}\left|μ\_{A}^{2}\left(x\_{j}\right)-μ\_{B}^{2}\left(x\_{j}\right)\right|\\∨\left|ϑ\_{A}^{2}\left(x\_{j}\right)-ϑ\_{B}^{2}\left(x\_{j}\right)\right|\\∨\left|π\_{A}^{2}\left(x\_{j}\right)-π\_{B}^{2}\left(x\_{j}\right)\right|\end{array}\right)\right]$ --------------(19)

$FNCS^{2}\left(A, B\right)=\frac{1}{n}\sum\_{j=1}^{n}cos\left[\frac{π}{4}\left(\begin{array}{c}\left|μ\_{A}^{2}\left(x\_{j}\right)-μ\_{B}^{2}\left(x\_{j}\right)\right|\\+\left|ϑ\_{A}^{2}\left(x\_{j}\right)-ϑ\_{B}^{2}\left(x\_{j}\right)\right|\\+\left|π\_{A}^{2}\left(x\_{j}\right)-π\_{B}^{2}\left(x\_{j}\right)\right|\end{array}\right)\right]$ --------------(20)

Where the symbol $"∨"$ is the maximum operator.

**Proposition 3.2.2:**

For any two FNSs $A$ and $B$ in $X=\left\{x\_{1}, x\_{2},…,x\_{n}\right\},$ the cosine similarity measures $FNCS^{k}\left(A, B\right) (k=1,2)$ should satisfy the following properties (1) – (4):

1. $0\leq FNCS^{k}\left(A, B\right)\leq 1$
2. $FNCS^{k}\left(A, B\right)=1 if and only if A=B$
3. $FNCS^{k}\left(A, B\right)=FNCS^{k}\left(B,A\right)$
4. If $C$ is a FNS in $X$ and $A⊆B⊆C$, then $FNCS^{k}\left(A, C\right)\leq FNCS^{k}\left(A, B\right)$ and $FNCS^{k}\left(A, C\right)\leq FNCS^{k}\left(B, C\right)$.

Proof:

1. Since the value of the cosine function is within [0,1], the similarity measure based on the cosine function is also within [0,1]. Thus, there is $0\leq FNCS^{k}\left(A, B\right)\leq 1$.
2. For any two FNSs $A$ and $B$ in $X=\left\{x\_{1}, x\_{2},…,x\_{n}\right\},$ if $A=B$, then $μ\_{A}^{3}\left(x\_{j}\right)=μ\_{B}^{3}\left(x\_{j}\right),$ $ϑ\_{A}^{3}\left(x\_{j}\right)=ϑ\_{B}^{3}\left(x\_{j}\right)$ and $π\_{A}^{3}\left(x\_{j}\right)=π\_{B}^{3}\left(x\_{j}\right)$ for $j=1,2,…,n.$ Thus,

$\left|μ\_{A}^{3}\left(x\_{j}\right)-μ\_{B}^{3}\left(x\_{j}\right)\right|=0,$ $\left|ϑ\_{A}^{3}\left(x\_{j}\right)-ϑ\_{B}^{3}\left(x\_{j}\right)\right|=0,$ $\left|π\_{A}^{3}\left(x\_{j}\right)-π\_{B}^{3}\left(x\_{j}\right)\right|=0.$ So, $FNCS^{k}\left(A, B\right)=1, \left(k=1,2\right).$

If $FNCS^{k}\left(A, B\right)=1, \left(k=1,2\right),$ this implies $\left|μ\_{A}^{3}\left(x\_{j}\right)-μ\_{B}^{3}\left(x\_{j}\right)\right|=0,$ $\left|ϑ\_{A}^{3}\left(x\_{j}\right)-ϑ\_{B}^{3}\left(x\_{j}\right)\right|=0,$ $\left|π\_{A}^{3}\left(x\_{j}\right)-π\_{B}^{3}\left(x\_{j}\right)\right|=0,$ for $j=1,2,…,n.$ Since cos(0) =1. Then, there are

$μ\_{A}^{3}\left(x\_{j}\right)=μ\_{B}^{3}\left(x\_{j}\right),$ $ϑ\_{A}^{3}\left(x\_{j}\right)=ϑ\_{B}^{3}\left(x\_{j}\right)$ and $π\_{A}^{3}\left(x\_{j}\right)=π\_{B}^{3}\left(x\_{j}\right)$ for $j=1,2,…,n.$ Hence

$A=B$.

1. Proof is straightforward.
2. If $A⊆B⊆C,$ then there are $μ\_{A}\left(x\_{j}\right)\leq μ\_{B}\left(x\_{j}\right)\leq μ\_{C}\left(x\_{j}\right)$, $ϑ\_{A}\left(x\_{j}\right)\geq ϑ\_{B}\left(x\_{j}\right)\geq ϑ\_{C}\left(x\_{j}\right)$ and $π\_{A}\left(x\_{j}\right)\geq π\_{B}\left(x\_{j}\right)\geq π\_{C}\left(x\_{j}\right),$ for $j=1,2,…n.$ Then, $μ\_{A}^{3}\left(x\_{j}\right)\leq μ\_{B}^{3}\left(x\_{j}\right)\leq μ\_{C}^{3}\left(x\_{j}\right)$, $ϑ\_{A}^{3}\left(x\_{j}\right)\geq ϑ\_{B}^{3}\left(x\_{j}\right)\geq ϑ\_{C}^{3}\left(x\_{j}\right)$ and $π\_{A}^{3}\left(x\_{j}\right)\geq π\_{B}^{3}\left(x\_{j}\right)\geq π\_{C}^{3}\left(x\_{j}\right).$

Thus, we have

$\left|μ\_{A}^{3}\left(x\_{j}\right)-μ\_{B}^{3}\left(x\_{j}\right)\right|\leq \left|μ\_{A}^{3}\left(x\_{j}\right)-μ\_{C}^{3}\left(x\_{j}\right)\right|$,

$\left|μ\_{B}^{3}\left(x\_{j}\right)-μ\_{C}^{3}\left(x\_{j}\right)\right|\leq \left|μ\_{A}^{3}\left(x\_{j}\right)-μ\_{C}^{3}\left(x\_{j}\right)\right|$,

$\left|ϑ\_{A}^{3}\left(x\_{j}\right)-ϑ\_{B}^{3}\left(x\_{j}\right)\right|\leq \left|ϑ\_{A}^{3}\left(x\_{j}\right)-ϑ\_{C}^{3}\left(x\_{j}\right)\right|$,

$\left|ϑ\_{B}^{3}\left(x\_{j}\right)-ϑ\_{C}^{3}\left(x\_{j}\right)\right|\leq \left|ϑ\_{A}^{3}\left(x\_{j}\right)-ϑ\_{C}^{3}\left(x\_{j}\right)\right|$,

$\left|π\_{A}^{3}\left(x\_{j}\right)-π\_{B}^{3}\left(x\_{j}\right)\right|\leq \left|π\_{A}^{3}\left(x\_{j}\right)-π\_{C}^{3}\left(x\_{j}\right)\right|$ and

$\left|π\_{B}^{3}\left(x\_{j}\right)-π\_{C}^{3}\left(x\_{j}\right)\right|\leq \left|π\_{A}^{3}\left(x\_{j}\right)-π\_{C}^{3}\left(x\_{j}\right)\right|$.

Hence, $FNCS^{k}\left(A, C\right)\leq FNCS^{k}\left(A, B\right)$ and$ FNCS^{k}\left(A, C\right)\leq FNCS^{k}\left(B, C\right)$ for $k=1,2,$ as the cosine function is a decreasing function with the interval $\left[0, {π}/{2}\right].$

Thus, the proofs of these properties are completed.

 In many situations, the weight of the elements $x\_{j}\in X$ should be taken into account. For example, in Multiple Attribute Decision Making (MADM), the considered attributes usually have different importance, and thus need to be assigned different weights. As a result, two weighted cosine similarity measure between FNSs $A$ and $B$ is proposed as follows:

$WFNCS^{1}\left(A, B\right)=\sum\_{j=1}^{n}ω\_{j} cos\left[\frac{π}{2}\left(\begin{array}{c}\left|μ\_{A}^{3}\left(x\_{j}\right)-μ\_{B}^{3}\left(x\_{j}\right)\right|\\∨\left|ϑ\_{A}^{3}\left(x\_{j}\right)-ϑ\_{B}^{3}\left(x\_{j}\right)\right|\\∨\left|π\_{A}^{3}\left(x\_{j}\right)-π\_{B}^{3}\left(x\_{j}\right)\right|\end{array}\right)\right]$ --------------(21)

$WFNCS^{2}\left(A, B\right)=\sum\_{j=1}^{n}ω\_{j} cos\left[\frac{π}{4}\left(\begin{array}{c}\left|μ\_{A}^{3}\left(x\_{j}\right)-μ\_{B}^{3}\left(x\_{j}\right)\right|\\+\left|ϑ\_{A}^{3}\left(x\_{j}\right)-ϑ\_{B}^{3}\left(x\_{j}\right)\right|\\+\left|π\_{A}^{3}\left(x\_{j}\right)-π\_{B}^{3}\left(x\_{j}\right)\right|\end{array}\right)\right]$ --------------(22)

Where $ω=\left(ω\_{1}, ω\_{2}, …,ω\_{n}\right)^{T}$ is the weight vector of $x\_{j}(j=1,2,…,n)$, with $ω\_{j}\in \left[0,1\right], j=1,2,…,n$, $\sum\_{j=1}^{n}ω\_{j}=1$ and the symbol $"∨" $is the maximum operator. In particular, if $ω=\left({1}/{n}, {1}/{n},…,{1}/{n}\right)^{T}$, then the weighted cosine similarity measure reduces to cosine similarity measure. That is to say, if we take $ω\_{j}=\frac{1}{n}, j=1,2,…,n,$ then there is $WFNCS^{k}\left(A, B\right)=WFNCS^{1}\left(A, B\right) (k=1,2)$.

Obviously, the weighted cosine similarity measures also satisfy the axiomatic requirements of similarity measures in Proposition 2.

**Proposition 3.2.3:**

For any two FNSs $A$ and $B$ in $X=\left\{x\_{1}, x\_{2},…,x\_{n}\right\},$ the cosine similarity measures $WFNCS^{k}\left(A, B\right) (k=1,2,3,4)$ should satisfy the following properties (1) – (4):

1. $0\leq WFNCS^{k}\left(A, B\right)\leq 1$
2. $WFNCS^{k}\left(A, B\right)=1 if and only if A=B$
3. $WFNCS^{k}\left(A, B\right)=WFNCS^{k}\left(B,A\right)$
4. If $C$ is a FNS in $X$ and $A⊆B⊆C$, then W$FNCS^{k}\left(A, C\right)\leq WFNCS^{k}\left(A, B\right)$ and $WFNCS^{k}\left(A, C\right)\leq WFNCS^{k}\left(B, C\right)$.

By using similar proof in Proposition 1, we can give the proofs of these properties (1) – (4).

 **3.3 Similarity Measures of FNSs based on the Cotangent Function:**

In this section, we shall propose two cotangent similarity measures between FNSs.

**Definition 3.3.1**:

Suppose that there are two FNSs $A=\left\{<x\_{j}, μ\_{A}\left(x\_{j}\right), ϑ\_{A}\left(x\_{j}\right), π\_{A}\left(x\_{j}\right)>\right|x\_{j}\in X\}$ and $B=\left\{<x\_{j}, μ\_{B}\left(x\_{j}\right), ϑ\_{B}\left(x\_{j}\right), π\_{B}\left(x\_{j}\right)>\right|x\_{j}\in X\}$ in the universe of discourse $X=\{x\_{1}, x\_{2}, …,x\_{n}\}$, we further propose the cotangent similarity measures between FNSs as follows:

$FNCT^{1}\left(A, B\right)=\frac{1}{n}\sum\_{j=1}^{n}cot\left[\frac{π}{4}+\frac{π}{4}\left(\begin{array}{c} \left|μ\_{A}^{3}\left(x\_{j}\right)-μ\_{B}^{3}\left(x\_{j}\right)\right|\\∨\left|ϑ\_{A}^{3}\left(x\_{j}\right)-ϑ\_{B}^{3}\left(x\_{j}\right)\right|\\∨\left|π\_{A}^{3}\left(x\_{j}\right)-π\_{B}^{3}\left(x\_{j}\right)\right|\end{array}\right)\right]$ --------------(23)

$FNCT^{2}\left(A, B\right)=\frac{1}{n}\sum\_{j=1}^{n}cot\left[\frac{π}{4}+\frac{π}{12}\left(\begin{array}{c}\left|μ\_{A}^{3}\left(x\_{j}\right)-μ\_{B}^{3}\left(x\_{j}\right)\right|\\+\left|ϑ\_{A}^{3}\left(x\_{j}\right)-ϑ\_{B}^{3}\left(x\_{j}\right)\right|\\+\left|π\_{A}^{3}\left(x\_{j}\right)-π\_{B}^{3}\left(x\_{j}\right)\right|\end{array}\right)\right]$ --------------(24)

Where the symbol $"∨"$ is the maximum operator.

**Proposition 3.3.2:**

For any two FNSs $A$ and $B$ in $X=\left\{x\_{1}, x\_{2},…,x\_{n}\right\},$ the cotangent similarity measures $FNCT^{k}\left(A, B\right) (k=1,2)$ should satisfy the following properties (1) – (4):

1. $0\leq FNCT^{k}\left(A, B\right)\leq 1$
2. $FNCT^{k}\left(A, B\right)=1 if and only if A=B$
3. $FNCT^{k}\left(A, B\right)=FNCT^{k}\left(B,A\right)$
4. If $C$ is a FNS in $X$ and $A⊆B⊆C$, then $FNCT^{k}\left(A, C\right)\leq FNCT^{k}\left(A, B\right)$ and $FNCT^{k}\left(A, C\right)\leq FNCT^{k}\left(B, C\right)$.

**Proof:**

1. Since,

 $\frac{π}{4}\leq \left(\frac{π}{12}\left(3+\left|μ\_{A}^{3}\left(x\_{j}\right)-μ\_{B}^{3}\left(x\_{j}\right)\right|+\left|ϑ\_{A}^{3}\left(x\_{j}\right)-ϑ\_{B}^{3}\left(x\_{j}\right)\right|+\left|π\_{A}^{3}\left(x\_{j}\right)-π\_{B}^{3}\left(x\_{j}\right)\right|\right)\right)\leq \frac{π}{2}$,

It is obvious that the cotangent function $FNCT^{k}\left(A, B\right)$ are within 0 and 1.

1. It is obvious that the proposition is true.
2. When $A=B$, then obviously $FNCT^{k}\left(A, B\right)=1$. On the other hand if $FNCT^{k}\left(A, B\right)=1$ then,

$μ\_{A}^{3}\left(x\_{j}\right)=μ\_{B}^{3}\left(x\_{j}\right),$ $ϑ\_{A}^{3}\left(x\_{j}\right)=ϑ\_{B}^{3}\left(x\_{j}\right)$ and $π\_{A}^{3}\left(x\_{j}\right)=π\_{B}^{3}\left(x\_{j}\right)$ for $j=1,2,…,n.$

This implies $A=B.$

1. If $A⊆B⊆C$ then we can write $μ\_{A}\left(x\_{j}\right)\leq μ\_{B}\left(x\_{j}\right)\leq μ\_{C}\left(x\_{j}\right)$, $ϑ\_{A}\left(x\_{j}\right)\geq ϑ\_{B}\left(x\_{j}\right)\geq ϑ\_{C}\left(x\_{j}\right)$ and $π\_{A}\left(x\_{j}\right)\geq π\_{B}\left(x\_{j}\right)\geq π\_{C}\left(x\_{j}\right),$ for $j=1,2,…n.$ Then, $μ\_{A}^{3}\left(x\_{j}\right)\leq μ\_{B}^{3}\left(x\_{j}\right)\leq μ\_{C}^{3}\left(x\_{j}\right)$, $ϑ\_{A}^{3}\left(x\_{j}\right)\geq ϑ\_{B}^{3}\left(x\_{j}\right)\geq ϑ\_{C}^{3}\left(x\_{j}\right)$ and $π\_{A}^{3}\left(x\_{j}\right)\geq π\_{B}^{3}\left(x\_{j}\right)\geq π\_{C}^{3}\left(x\_{j}\right).$

The cotangent function is decreasing function within the interval $\left[\frac{π}{4}, \frac{π}{2}\right]$.

Hence we can write $FNCT^{k}\left(A, C\right)\leq FNCT^{k}\left(A, B\right)$ and $FNCT^{k}\left(A, C\right)\leq FNCT^{k}\left(B, C\right)$.

Thus, the proofs of these properties are completed.

 In many situations, the weight of the elements $x\_{j}\in X$ should be taken into account. For example, in Multiple Attribute Decision Making (MADM), the considered attributes usually have different importance, and thus need to be assigned different weights. As a result, two weighted cotangent similarity measure between FNSs $A$ and $B$ is proposed as follows:

$WFNCT^{1}\left(A, B\right)=\sum\_{j=1}^{n}ω\_{j}cot\left[\frac{π}{4}+\frac{π}{4}\left(\begin{array}{c}\left|μ\_{A}^{3}\left(x\_{j}\right)-μ\_{B}^{3}\left(x\_{j}\right)\right|\\∨\left|ϑ\_{A}^{3}\left(x\_{j}\right)-ϑ\_{B}^{3}\left(x\_{j}\right)\right|\\∨\left|π\_{A}^{3}\left(x\_{j}\right)-π\_{B}^{3}\left(x\_{j}\right)\right|\end{array}\right)\right]$ --------------(25)

$WFNCT^{2}\left(A, B\right)=\sum\_{j=1}^{n}ω\_{j}cot\left[\frac{π}{4}+\frac{π}{12}\left(\begin{array}{c}\left|μ\_{A}^{3}\left(x\_{j}\right)-μ\_{B}^{3}\left(x\_{j}\right)\right|\\+\left|ϑ\_{A}^{3}\left(x\_{j}\right)-ϑ\_{B}^{3}\left(x\_{j}\right)\right|\\+\left|π\_{A}^{3}\left(x\_{j}\right)-π\_{B}^{3}\left(x\_{j}\right)\right|\end{array}\right)\right]$ ----------------(26)

Where $ω=\left(ω\_{1}, ω\_{2}, …,ω\_{n}\right)^{T}$ is the weight vector of $x\_{j}(j=1,2,…,n)$, with $ω\_{j}\in \left[0,1\right], j=1,2,…,n$, $\sum\_{j=1}^{n}ω\_{j}=1$ and the symbol $"∨" $is the maximum operator. In particular, if

$ω=\left({1}/{n}, {1}/{n},…,{1}/{n}\right)^{T}$, then the weighted cotangent similarity measure reduces to cotangent similarity measure. That is to say, if we take $ω\_{j}=\frac{1}{n}, j=1,2,…,n,$ then there is $WFNCT^{k}\left(A, B\right)=WFNCT^{1}\left(A, B\right) (k=1,2)$.

**Proposition 3.3.3:**

For any two FNSs $A$ and $B$ in $X=\left\{x\_{1}, x\_{2},…,x\_{n}\right\},$ the cosine similarity measures $WSFCS^{k}\left(A, B\right) (k=1,2,3,4)$ should satisfy the following properties (1) – (4):

1. $0\leq WFNCT^{k}\left(A, B\right)\leq 1$
2. $WFNCT^{k}\left(A, B\right)=1 if and only if A=B$
3. $WFNCT^{k}\left(A, B\right)=WFNCT^{k}\left(B,A\right)$
4. If $C$ is a FNS in $X$ and $A⊆B⊆C$, then W$FNCT^{k}\left(A, C\right)\leq WFNCT^{k}\left(A, B\right)$ and $WFNCT^{k}\left(A, C\right)\leq WFNCT^{k}\left(B, C\right)$.

By using similar proof in Proposition 3, we can give the proofs of these properties (1) – (4)

4.**APPLICATIONS**

In this section, the cosine and cotangent similarity measures for FNSs are applied to pattern recognition and medical diagnosis to illustrate the feasibility of the proposed methods and deliver a comparative analysis

* 1. **Example 1: Pattern Recognition**

Let us consider, a three known patterns $A\_{i}\left(i=1,2,3\right),$ which are represented by the FNSs: $A\_{i}\left(i=1,2,3\right)$ in the feature space $X=\{x\_{1}, x\_{2}, x\_{3}\}$ as

$$A\_{1}=\{\left(0.8,0.1,0.2\right), \left(0.7,0.2,0.3\right), \left(0.7,0.4,0.3\right)\}$$

$$A\_{2}=\{\left(0.7,0.2,0.2\right), \left(0.8,0.2,0.3\right),\left(0.6,0.5,0.4\right)\}$$

$$A\_{3}=\{\left(0.8,0.3,0.3\right), \left(0.7,0.2,0.2\right), \left(0.8,0.4,0.4\right)\}$$

Consider an unknown pattern $A\in FNSs(X)$ that will be recognized, where

$$A=\{\left(x\_{1},0.5,0.5,0.4\right), \left(x\_{2}, 0.5,0.2,0.3\right), \left(x\_{3}, 0.9,0.1,0.2\right)\}$$

 The purpose of this problem is classify the pattern $A$ in one classes $A\_{1}, A\_{2}$ and $A\_{3}$. For it, the proposed similarities degrees have been computed from $A$ to$ A\_{i}(i=1,2,3)$ and are given in Table 1.

|  |  |  |  |
| --- | --- | --- | --- |
| Similarity Measures | $$(A\_{1}, A)$$ | $$(A\_{2}, A)$$ | $$(A\_{3}, A)$$ |
| $$FNS^{1}(A\_{i},A)$$ | 0.8704 | 0.8320 | **0.8992** |
| $$FNCS^{1}(A\_{i},A)$$ | 0.8747 | 0.8360 | **0.9041** |
| $$FNCS^{2}(A\_{i},A)$$ | 0.9104 | 0.8863 | **0.9328** |
| $$FNCT^{1}(A\_{i},A)$$ | 0.5967 | 0.5533 | **0.654** |
| $$FNCT^{2}(A\_{i},A)$$ | 0.7695 | 0.7327 | **0.7898** |

Table 1: The similarity measures between $A\_{i}(i=1,2,3)$ and $A$

 From the numerical results presented in Table 1, we know that the degree of similarity between $A\_{3}$ and $A$ is the largest one as derived by five similarity measures. That is, all the 5 similarity measures assign the unknown class $A$ to the known class $A\_{3}$according to the principle of maximum degree of similarity between FNSs. Compared with Garg’s correlation coefficient method [10], we can get the same result that all the 5 similarity measures assign the unknown class $A$ to the known class $A\_{3}$according to the principle of maximum degree of similarity between FNSs.

 If we consider the weight of $x\_{i}(i=1,2,3)$ are 0.5, 0.3 and 0.2 respectively. Then we use the proposed weighted similarities measures have been computed from $A$ to $A\_{i}(i=1,2,3)$ and are given in Table 2.

|  |  |  |  |
| --- | --- | --- | --- |
| Similarity Measures | $$(A\_{1}, A)$$ | $$(A\_{2}, A)$$ | $$(A\_{3}, A)$$ |
| $$WFNS^{1}(A\_{i},A)$$ | 0.8244 | 0.8145 | **0.8692** |
| $$WFNCS^{1}(A\_{i},A)$$ | 0.8631 | 0.8622 | **0.8808** |
| $$WFNCS^{2}(A\_{i},A)$$ | 0.8938 | 0.8975 | **0.9221** |
| $$WFNCT^{1}(A\_{i},A)$$ | 0.5818 | 0.5877 | **0.6161** |
| $$WFNCT^{2}(A\_{i},A)$$ | 0.7503 | 0.7426 | **0.7753** |

Table 2: The weighted similarity measures between $A\_{i}(i=1,2,3)$ and $A$

 From the numerical results presented in Table 2, we know that the weighted similarity measures between $A\_{3}$ and $A$ is the largest one as derived by five similarity measures. That is, all the 5 similarity measures assign the unknown class $A$ to the known class $A\_{3}$according to the principle of maximum degree of similarity between FNSs. Compared with Garg’s correlation coefficient method [10], we can get the same result that all the 5 similarity measures assign the unknown class $A$ to the known class $A\_{3}$according to the principle of maximum degree of similarity between FNSs.

* 1. **Example 2: Medical Diagnosis**

Let us consider a set of diagnosis $D=\{D\_{1}\left(Viral fever\right), D\_{2}\left(typhoid\right), $

$D\_{3}\left(Stomach Problem\right), D\_{4}\left(Malaria\right),$ $D\_{5}\left(Chest Problem\right)\}$ and a set of symptoms

$S=\left\{s\_{1}\left(Temperature\right), s\_{2}\left(Head Ache\right), s\_{3}\left(Stomach Pain\right), s\_{4}\left(Cough\right), s\_{5}\left(Chest Pain\right)\right\}.$ Suppose that a patient, with respect to all symptoms, can be depicted by the following FNS:

 $P\left(Patient\right)=\{\left(s\_{1}, 0.8, 0.5, 0.2\right), \left(s\_{2}, 0.6, 0.4, 0.4\right), (s\_{3}, 0.7, 0.5, 0.1)$

 ($s\_{4}, 0.9,0.3,0.2),$ ($s\_{5}, 0.7, 0.6, 0.3)\}$

And then each diagnoses $D\_{i}(i=1,2,3,4,5)$ can be viewed as FNSs with respect to all the symptoms as follows:

 $D\_{1}\left(Viral Fever\right)=\{\left(s\_{1},0.4,0.1,0.9\right), \left(s\_{2}, 0.7, 0.3, 0.2\right), (s\_{3}, 0.5, 0.6, 0.1)$

 $(s\_{4}, 0.2, 0.5, 0.4), (s\_{5}, 0.1, 0.7, 0.5)\}$

$D\_{2}\left(Typhoid\right)=\{(s\_{1}, \left(0.6, 0.2, 0.1\right), \left(s\_{2}, 0.8, 0.3, 0.2\right), \left(s\_{3},0.5, 0.5, 0.3\right), $

 ($s\_{4}, 0.7, 0.2, 0.4), (s\_{5},0.2, 0.7, 0.4)\}$

$D\_{3}\left(Stomach Problem\right)=\{\left(s\_{1}, 0.5, 0.3, 0.2\right), \left(s\_{2}, 0.7, 0.5, 0.3\right), \left(s\_{3},0.8, 0.2, 0.5\right),$

 $(s\_{4}, 0.7, 0.3, 0.1), (s\_{5}, 0.9, 0.1, 0.3)\}$

 $D\_{4}\left(Malaria\right)=\{\left(s\_{1},0.7, 0.4, 0.3\right), \left(s\_{2},0.6, 0.5, 0.4\right), \left(s\_{3}, 0.4, 0.6, 0.2\right),$

 $(s\_{4},0.8, 0.2, 0.4), (s\_{5},0.3, 0.2, 0.1)\}$

 $D\_{5}\left(Chest Problem\right)=\{\left(s\_{1},0.9, 0.2, 0.1\right), \left(s\_{2},0.7, 0.6, 0.4\right), \left(s\_{3}, 0.3, 0.2, 0.1\right),$

 $(s\_{4}, 0.6, 0.3, 0.1), (s\_{5}, 0.8,0.2, 0.5)\}$

The purpose of this problem is classify the pattern $P$ in one classes $D\_{i}(i=1,2,3,4,5)$. For this, the proposed similarities measures have been computed from $P$ to $D\_{i}(i=1,2,3,4,5)$ and are given in Table 3.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Similarity Measures | $$\left(D\_{1},P\right)$$ | $$\left(D\_{2},P\right)$$ | $$\left(D\_{3},P\right)$$ | $$\left(D\_{4},P\right)$$ | $$\left(D\_{5},P\right)$$ |
| $$FNS^{1}\left(D\_{i},P\right)$$ | 0.5811 | 0.8863 | 0.9288 | 0.9420 | **0.9469** |
| $$FNCS^{1}(D\_{i},P)$$ | 0.7440 | 0.8752 | 0.8911 | **0.9208** | 0.8685 |
| $$FNCS^{2}(D\_{i},P)$$ | 0.7644 | 0.9236 | 0.9200 | **0.9355** | 0.9224 |
| $$FNCT^{1}(D\_{i},P)$$ | 0.5083 | 0.6002 | 0.6328 | **0.7018** | 0.6005 |
| $$FNCT^{2}(D\_{i},P)$$ | 0.6628 | 0.7741 | 0.7700 | **0.8155** | 0.7717 |

Table 3: The similarity measures between $D\_{i}(i=1,2,3,4,5)$ and $P$

From the numerical results presented in Table 3, expect for the $FNS^{1}\left(D\_{i},P\right)(i=1,2,3,4,5)$, we know that the similarity measures between $D\_{4}$ and $P$ is the largest one as derived by five similarity measures. That is, the four similarity measures assign the unknown class $P$ to the known class $D\_{4}$ according to the principle of the maximum degree of similarity between FNSs. Compared with Garg’s correlation coefficients method [10] we can get same result that the four similarity measures assign the unknown class $P$ to the known class $D\_{4}$according to the principle of the maximum degree of similarity between FNSs expect for the $FNS^{1}\left(D\_{i},P\right)\left(i=1,2,3,4,5\right).$

If we consider the weight of $s\_{i}(i=1,2,3,4,5)$ is $0.15, 0.20, 0.25, 0.16, 0.13$ respectively. Then we apply the proposed weighted similarities measures, which have been computed from $P$ to $D\_{i}(i=1,2,3,4,5)$ and are given in Table 4.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Similarity Measures | $$\left(D\_{1},P\right)$$ | $$\left(D\_{2},P\right)$$ | $$\left(D\_{3},P\right)$$ | $$\left(D\_{4},P\right)$$ | $$\left(D\_{5},P\right)$$ |
| $$WFNS^{1}\left(D\_{i},P\right)$$ | 0.5608 | 0.7994 | 0.8251 | 0.8235 | **0.8517** |
| $$WFNCS^{1}(D\_{i},P)$$ | 0.6889 | 0.7881 | 0.8020 | **0.8204** | 0.7699 |
| $$WFNCS^{2}(D\_{i},P)$$ | 0.7083 | 0.8280 | 0.8201 | **0.8382** | 0.8206 |
| $$WFNCT^{1}(D\_{i},P)$$ | 0.4830 | 0.5465 | 0.5780 | **0.6245** | 0.5304 |
| $$WFNCT^{2}(D\_{i},P)$$ | 0.6149 | 0.6986 | 0.6892 | **0.7327** | 0.6865 |

Table 4: The weighted similarity measures between $D\_{i}(i=1,2,3,4,5)$ and $P$

From the numerical results presented in table 4, we get the following results:

1. For similarity measures $WFNS^{1}\left(D\_{i},P\right)(i=1,2,3,4,5)$, the degree of similarity between $D\_{5}$ and P is the largest one, so the pattern P should belong to the class of known diagnoses $D\_{5}$ according to the principle of the maximum degree of similarity between FNSs.
2. For similarity measures $WFNCS^{1}\left(D\_{i},P\right), WFNCS^{2}\left(D\_{i},P\right), WFNCT^{1}\left(D\_{i},P\right), WFNCT^{2}\left(D\_{i},P\right), i=1,2,3,4,5,$ the degree of similarity between $D\_{4}$ and P is the largest one, so the pattern P should belong to the class of known diagnoses $D\_{4}$ according to the principle of the maximum degree of similarity between FNSs. At the same time, for this case compared with Garg’s correlation coeeficients method [10], we can get the same result that the pattern P should belong to the class of the known diagnoses $D\_{4}$ according to the principle of the maximum degree of similarity between FNSs.