Ecosystem Resilience: A Stochastic Study of Predator-Prey Interactions with Distributed Delay

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Abstract: In this research article, our primary focus is on a stochastic prey - predator system incorporating distributed delay. To analyze this model, we employ the stochastic Lyapunov function approach to prove the presence of a stationary distribution for the non negative solutions. Subsequently, we formulate the necessary criteria for the predator population to go extinct, indicating the survival of the prey population while the predator population diminishes completely.

Keywords: Stationary distribution, Stochastic prey - predator model, Markov Process, Lyapunov function.

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1 Introduction

Mathematical modeling is a potent and essential instrument in ecology, offering a methodical and quantitative approach to comprehend the intricacies of natural ecosystems [1], [2]. The study of ecology involves investigating the interactions between living organisms and their environment, which poses multiple challenges due to the complex interconnections among species and the dynamic nature of ecological systems [3]. By employing mathematical models, researchers can effectively address these intricacies and obtain valuable insights into the operations and conduct of ecological communities[4], [5]. These models have undergone thorough investigation and research following the influential theoretical contributions by Volterra [7] and [6]. However, to accurately capture the system's dynamic behavior, it is crucial to consider the influence of past history on the model's dynamics. This necessitates the incorporation of time delays into the models, leading to a more realistic depiction of predator-prey interactions using delayed differential equations.

Lately, the notion of infinite delay has garnered significant attention within mathematical biology equations, serving as a method to integrate the cumulative impact of a system's historical dynamics, a concept originally championed by Volterra. Numerous scholars (refer to [[8], [9]] as examples) have extensively investigated the stability and bifurcation characteristics of prey-predator systems. This research endeavor aims to capture the intricate interplay between species by considering not only their current interactions but also the enduring influence of their past states, allowing for a more comprehensive understanding of the system's behavior over time.

The predator-prey system with distributed delay was formulated by Chen et al. [9],

$$\begin{aligned} \frac{dm}{dt} &= a_1 m \left(1 - \frac{m}{\mathbb{K}}\right) - \gamma mn, \\ \frac{dn}{dt} &= a_2 n \left(1 - \frac{n}{\mathbb{K}}\right) + \delta \int_{-\infty}^{t} \mathbb{K}\,(t-s)\, m\,(s)\, n\,(s)\, ds - dn. \end{aligned} \tag{1.1}$$

Here, m represent the density of and n is the predator density. a_1 and a_2 are intrinsic growth rate of the prey and predator; $\mathbb{K}$ denotes the carrying capacity. The parameter d denotes the death rate of the predator. γ is the rate of predation by the predator, and δ represents the combined effect of the rate of predation and the rate of converting prey into predators. It is assumed that all of the parameters are positive constants. The kernel $\mathbb{K} : [0, \infty) \to [0, \infty)$ is a L^1 function, normalized such as

$$\int_0^\infty \mathbb{K}\,(s)\, ds = 1.$$

In natural ecosystems, population systems are inescapably subjected to the pervasive impact of environmental variability. This unpredictability often manifests as environmental "white noise," encompassing a wide range of stochastic fluctuations that can significantly affect population dynamics. Recognizing the imperative to mirror this intricate interplay between biological entities and their unpredictable surroundings, researchers have increasingly turned to stochastic differential equation (SDE) models. These models serve as indispensable tools in the study of population dynamics due to their ability to encompass the inherent randomness and complexity present in ecological systems. Unlike their deterministic equivalents, SDE models not only acknowledge the deterministic forces governing populations but also incorporate the essential role of chance events and environmental variability in shaping population behaviors. Consequently, they offer a more comprehensive and nuanced perspective, facilitating a deeper understanding of the intricate ecological processes that unfold over time. Many researchers have explored the impacts of environmental random fluctuations on population dynamics by introducing random perturbations into deterministic models (see, for example, [[10], [11]]). In this paper, we are inspired by the works of Imhof and Walcher [[12]] and adopt their approach, assuming that the environmental white noise is proportionally related to the variables m and n. This assumption leads us to derive the following stochastic model, corresponding to system (1.1),

$$\begin{aligned} dm &= \left[a_1 m\left(1-\frac{m}{\mathbb{K}}\right)-\gamma mn\right]dt+\beta_1 m dQ_1(t),\\ dn &= \left[a_2 n\left(1-\frac{n}{\mathbb{K}}\right)+\delta\int_{-\infty}^{t}\mathbb{K}\left(t-s\right)m(s)n(s)ds-dn\right]dt+\beta_2 n dQ_2(t). \end{aligned} \tag{1.2}$$

where $Q_i(t)$ are standard Brownian motions that are mutually independent, Q stand for the white noise intensities, and $i = 1, 2$.

For convenience's sake, we focus on the weak kernel situation in this study, which is $\mathbb{K}(t) = \eta e^{-\eta t}$ with $\eta > 0$ and was first described by MacDonald [[13]]. Analogously, the strong kernel case may be explored. Both the weak kernel and the strong kernel have found extensive application in biological systems, including epidemiology [14] and population systems [15].

Let

$$p(t) = \int_{-\infty}^{t} \eta e^{-\eta(t-s)} m(s)n(s)ds$$

then, (1.2) is changed into the following comparable model using the linear chain approach.

$$\begin{aligned} dm &= \left[a_1 m\left(1-\frac{m}{\mathbb{K}}\right)-\gamma mn\right]dt+\beta_1 m dQ_1(t),\\ dn &= \left[a_2 n\left(1-\frac{n}{\mathbb{K}}\right)+\delta p-dn\right]dt+\beta_2 n dQ_2(t),\\ dp &= \eta\left(mn-p\right)dt. \end{aligned} \tag{1.3}$$

This research paper primarily concentrates on establishing precise and adequate criteria for the existence of a stationary distribution within the context of (1.3). Previous studies have explored the steady distribution of stochastic predator-prey models incorporating time delay, exemplified by the stochastic delay cascade predator-prey model [16] and the stochastic delay two-predator one-prey model [17]. However, these works primarily focused on discrete delay, while our current paper addresses distributed delay, showcasing its novelty and innovative aspect.

The subsequent lemma addresses the presence and uniqueness of global non negative solutions for system (1.3). Due to its conventional nature, we omit the proof here.

Lemma 1. *Given any initial value $(m(0), n(0), p(0)) \in \mathbb{R}^3_+$ for (1.1), $\exists$ a single, unique solution $(m(t), n(t), p(t))$ for the system on the interval $t \geq 0$ and with probability one, the solution $(m(t), n(t), p(t))$ will remain in the real numbers $\mathbb{R}^3_+$ for all values of $t > 0$. In other words, the solution $(m(t), n(t), p(t))$ remains in $\mathbb{R}^3_+$ almost surely for $t \geq 0$.*

2 Main Results

Here we will present the primary outcomes for system (1.3).

Let us examine the integral equation:

$$M(t) = M(t_0) + \int_{t_0}^{t} a(s, M(s))ds + \sum_{j=1}^{k} \int_{t_0}^{t} \eta_j (s, M(s))\, dQ(s). \tag{2.1}$$

Lemma 2. *[18] Assume that the coefficients of (2.1) satisfy the prerequisites for som e constant Q and are t-independent.*

$$|a(s,m) - a(s,n)| + \sum_{j=1}^{k} |\eta_j(s,m) - \eta_j(s,n)| \leq Q|m-n|, |a(s,m)| + \sum_{j=1}^{k} |\eta_j(s,m)| \leq Q(1+|m|) \tag{2.2}$$

in $V_{\mathbb{R}} \subset \mathbb{R}_+^d$ $\forall$ $\mathbb{R} > 0$ and $\exists$ a positive $\mathbb{C}^2$-function $U(x)$ in $\mathbb{R}_+^d$ $\ni$

$$WU(x) \leq -1.$$

outside some compact set, the stationary distribution is the solution to (2.1).

Remark 1. *Remark 5 of Xu [19] reveals that (2.2) mentioned in Lemma 2 is not strictly necessary, it can be substituted with the requirement for the global existence of solutions of (2.1).*

Lemma 3. *[20] The following inequality is true for any $m > 0$*

$$m(1-m) + 2m \leq \sqrt{m}.$$

Theorem 4. *Suppose that $a_1 > \frac{\beta_1^2}{2}$, $a_2 > \frac{\beta_2^2}{2}$ and $\mathbb{R}_0^s > 1$ then solution $(m(t), n(t), p(t))$ of (1.3) is a stationary Markov process where $\mathbb{R}_0^s = \frac{\delta\mathbb{K}\left(1-\frac{\beta_1^2}{(2a_1)}\right)^2}{a_2+\frac{\beta_2^2}{2}}$.*

Proof. According to Lemma 2, it suffices to establish the existence of a non negative C-function $U(m,n,p)$ and a closed set $V \subset \mathbb{R}_+^3$ such that,

$$WU(m,n,p) \leq -1 \text{ for any } (m,n,p) \in \mathbb{R}_+^3 \backslash V.$$

Define

$$U_1(m,n,p) = -x_1 ln\ n - x_2 ln\ p + \frac{\sqrt{x_1 x_2 \delta \eta \mathbb{K}}}{a_1}\left(\frac{m}{\mathbb{K}} - 2ln\ m\right) + \frac{2\gamma\sqrt{\delta\eta^2\mathbb{K}\left(d+\frac{\beta_2^2}{2}\right)}}{a_1 d} n + \frac{2\gamma\delta\sqrt{\delta\eta^2\mathbb{K}\left(d+\frac{\beta_2^2}{2}\right)}}{a_1 d\eta} p.$$

where x_1,x_2 are non negative constants. Ito's formula [21] applied to U_1 results in,

$$\begin{aligned}
WU_1 &= -\frac{x_1\delta p}{n} - \frac{x_2\eta mn}{p} + c_1\left(d+\frac{\beta_2^2}{2}\right) + x_2\eta + \\
&\sqrt{x_1x_2\delta\eta\mathbb{K}}\left[\frac{m}{\mathbb{K}}\left(1-\frac{m}{\mathbb{K}}\right) - \frac{\gamma}{a_1\mathbb{K}}mn - 2\left(1-\frac{m}{\mathbb{K}}\right) + \frac{2\gamma}{a_1}n + \frac{\beta_1^2}{a_1}\right] \\
&- \frac{2\gamma\sqrt{x_1x_2\delta\eta\mathbb{K}}}{a_1}n + \frac{2\gamma\delta\sqrt{\delta\eta^2\mathbb{K}\left(d+\frac{\beta_2^2}{2}\right)}}{a_1 d}mn \\
&\leq -2\sqrt{x_1x_2\delta\eta}\sqrt{m} + x_1\left(d+\frac{\beta_2^2}{2}\right) + x_2\eta + \\
&\sqrt{x_1x_2\delta\eta\mathbb{K}}\left[\frac{m}{\mathbb{K}}\left(1-\frac{m}{\mathbb{K}}\right) + \frac{2m}{\mathbb{K}} - 2\left(1-\frac{\beta_1^2}{a_1}\right) + \frac{2\gamma}{a_1}n\right] - \frac{2\gamma\sqrt{x_1x_2\delta\eta\mathbb{K}}}{a_1}n + \\
&\frac{2\gamma\delta\sqrt{\delta\eta^2\mathbb{K}\left(d+\frac{\beta_2^2}{2}\right)}}{a_1 d}mn \\
&\leq -2\sqrt{x_1x_2\delta\eta}\sqrt{m} + x_1\left(d+\frac{\beta_2^2}{2}\right) + x_2 n + \sqrt{x_1x_2\delta\eta\mathbb{K}}\frac{2\sqrt{m}}{\mathbb{K}} \\
&- 2\sqrt{x_1x_2\delta\eta\mathbb{K}}\left(1-\frac{\beta_1^2}{2a_1}\right) + \frac{2\gamma\delta\sqrt{\delta\eta^2\mathbb{K}\left(d+\frac{\beta_2^2}{2}\right)}}{a_1 d}mn \\
&= x_1\left(d+\frac{\beta_2^2}{2}\right) + x_2\eta - 2\sqrt{x_1x_2\delta\eta\mathbb{K}}\left(1-\frac{\beta_1^2}{2a_1}\right) + \frac{2\gamma\delta\sqrt{\delta\eta^2\mathbb{K}\left(d+\frac{\beta_2^2}{2}\right)}}{a_1 d}mn.
\end{aligned} \tag{2.3}$$

Consider $x_1 = \eta$, $x_2 = \left(d+\frac{\beta_2^2}{2}\right)$, then from (2.3), one may observe that,

$$WU_1 \leq 2\left[\eta\left(d+\frac{\beta_2^2}{2}\right) - \sqrt{\delta\eta^2\mathbb{K}\left(d+\frac{\beta_2^2}{2}\right)\left(1-\frac{\beta_1^2}{2a_1}\right)}\right] + \frac{2\gamma\delta\sqrt{\delta\eta^2\mathbb{K}\left(d+\frac{\beta_2^2}{2}\right)}}{a_1 d}mn. \tag{2.4}$$

$$\mathbb{R}_0^s = \frac{\delta\mathbb{K}\left(1-\frac{\beta_1^2}{2a_1}\right)}{d+\frac{\beta_2^2}{2}}, \Lambda = 2\eta\left(d+\frac{\beta_2^2}{2}\right)\sqrt{\mathbb{R}_0^s - 1} > 0.$$

Define

$$U_2(m,n,p) = \frac{1}{\nu+2}\left(m + \frac{\gamma}{2\delta}n + \frac{\gamma}{n}p\right)^{\nu+2}.$$

where $0 < \nu < \frac{d-\frac{\beta_2^2}{2}}{d+\frac{\beta_2^2}{2}}$ is sufficiently small number. Applying Ito's formula to U_2 results in

$$\begin{aligned}
WU_2 =& \left(m + \frac{\gamma}{2\delta}n + \frac{\gamma}{\eta}p\right)^{\nu+1} \left[a_1 m\left(1 - \frac{m}{\mathbb{K}}\right) - \frac{\gamma d}{2\delta}n - \frac{\gamma}{2}p\right] \\
&+ \frac{\nu+1}{2}\left(m + \frac{\gamma}{2\delta}n + \frac{\gamma}{n}p\right)^{\nu}\left(\beta_1^2 m^2 + \frac{\gamma^2\beta_2^2}{4\delta^2}n^2\right) \\
&\leq a_1 m\left(m + \frac{\gamma}{2\delta}n + \frac{\gamma}{n}p\right)^{\nu+1} - \frac{a_1}{\mathbb{K}}m^{\nu+3} - d\left(\frac{\gamma}{2\delta}\right)^{\nu+2} n^{nu+2} - \frac{\gamma^{\nu+2}}{2\eta^{\nu+1}}p^{\nu+2} \quad (2.5) \\
&+ \frac{\gamma+1}{2}\left(m + \frac{\gamma}{2\delta}n + \frac{\gamma}{n}p\right)^{\nu}\left(\beta_1^2 m^2 + + \frac{\gamma^2\beta_2^2}{4\delta^2}n^2\right) \\
&\leq -\frac{a_1}{2\mathbb{K}}m^{\nu+3} - d\nu\left(\frac{\gamma}{2\delta}\right)^{\nu+2} - \frac{\gamma^{\nu+2}}{4\eta^{\nu+1}}p^{\nu+2} + A,
\end{aligned}$$

where

$$Q = sup_{(m,n,p)\in R_+^3}\left\{\begin{array}{l} -\frac{a_1}{2\mathbb{K}}m^{\nu+3} - d\left(1-\nu\right)\left(\frac{\gamma}{2\delta}\right)^{\nu+2}n^{\nu+2} - \frac{\gamma^{\nu+2}}{4\eta^{\nu+1}}p^{\nu+2} + a_1 m\left(m + \frac{\gamma}{2\delta}n + \frac{\gamma}{n}p\right)^{\nu+1} \\ +\frac{\gamma+1}{2}\left(m + \frac{\gamma}{2\delta}n + \frac{\gamma}{n}p\right)^{\nu}\left(\beta_1^2 m^2 + \frac{\gamma^2\beta_2^2}{4\delta^2}n^2\right) \end{array}\right\}.$$

Let us define Lyapunov function as follows,

$$\tilde{U}(m,n,p) = \zeta U_1\left(m,n,p\right) + U_2\left(m,n,p\right) - ln\ p.$$

where $\zeta > 0$ is a constant that satisfies $-\zeta\lambda + g_1^v + g_2^v + g_3^v \leq -2$ and the functions $g_i, i = 1,2,3$ will be found later. Furthermore, $\tilde{U}(m,n,p)$ tends to $+\infty$ as (m,n,p) approaches the boundary of $\mathbb{R}_+^3$ and as $||\left(m,n,p\right)|| \to \infty$ where $||.||$ represent the euclidean norm of a point in $\mathbb{R}_+^3$. As a result, it must be lower bounded and reach this lower bound at a point (m_0, n_0, p_0) in R_+^3 interior. Let us denote a positive C^2-function $U : \mathbb{R}_+^3 \to \mathbb{R}_+ \cup \{0\}$ by

$$U\left(m,n,p\right) = \tilde{U}(m,n,p) - \tilde{U}(m_0,n_0,p_0) = \zeta U_1 + U_2 + U_3$$

where $U_3 = -ln\ p - \tilde{U}(m_0,n_0,p_0)$. Applying Itô's formula to U_3, we obtain

$$WU_3 = \frac{-\eta mn}{p} + \eta. \quad (2.6)$$

From (2.4), (2.5) and (2.6), we get

$$\begin{aligned}
WU \leq & -\zeta\Lambda + \frac{2\zeta\Lambda\gamma\delta\sqrt{\delta\eta^2\mathbb{K}\left(d + \frac{\beta_2^2}{2}\right)}}{a_1 d}mn - \frac{a_1}{2\mathbb{K}}m^{\nu+3} \\
&- \frac{d}{2}\left(\frac{\gamma}{2\delta}\right)^{\nu+2} n^{\nu+2} - \frac{\gamma^{\nu+2}}{4\eta^{\nu+1}}p^{\nu+2} - \frac{\eta}{p}mn + Q + \eta \quad (2.7) \\
=& g_1(m) + g_2(n) + g_3(p) - \zeta\Lambda + \frac{2\zeta\Lambda\gamma\delta\sqrt{\delta\eta^2\mathbb{K}\left(d + \frac{\beta_2^2}{2}\right)}}{a_1 d}mn - \frac{\eta}{p}mn,
\end{aligned}$$

where $g_1(m) = -\frac{a_1}{2k}m^{\nu+3}$, $g_2(n) = -\frac{d}{2}\left(\frac{\gamma}{2\delta}\right)^{\nu+2} n^{\nu+2}$, $g_3(p) = -\frac{\gamma^{\nu+2}}{4\eta^{\nu+1}}p^{\nu+2} + Q + \eta$.

Represent,

$$F(m,n,p) = g_1(m) + g_2(n) + g_3(p) - X\Lambda + \frac{2X\Lambda\gamma\delta\sqrt{\delta\eta^2\mathbb{K}\left(d+\frac{\beta_2^2}{2}\right)}}{a_1 d}mn - \frac{\eta}{p}mn.$$

Then $F(m,n,p) \leq \begin{cases} F(+\infty,n,p) \to -\infty \text{ as } m \to +\infty, \\ F(m,+\infty,p) \to -\infty \text{ as } n \to +\infty, \\ F(m,n,+\infty) \to -\infty \text{ as } p \to +\infty, \\ g_1^u + g_2^u + g_3^u - \zeta\Lambda \leq -2, \text{ as } m \to 0^+ \ or\ n \to 0^+, \\ F(m,n,0) \to -\infty \text{ as } p \to 0^+. \end{cases}$

Therefore, we can choose a sufficiently small positive value for $\epsilon > 0$,

$$WU(m,n,p) \leq -1 \text{ for any } (m,n,p) \in \mathbb{R}_+^3/V,$$

where $V = \left[\epsilon, \frac{1}{\epsilon}\right] \times \left[\epsilon, \frac{1}{\epsilon}\right] \times \left[\epsilon^3, \frac{1}{\epsilon^3}\right]$. Based on the findings in Lemma 2, it can be concluded that the system (1.3) possesses a solution that exhibits the characteristics of a stationary Markov process. This concludes the proof. □

Theorem 5. *Consider a solution $(m(t), n(t), p(t))$ of (1.3) with any initial conditions $(m(0), n(0), p(0)) \in \mathbb{R}$. Given $a_1 > \frac{\beta_1^2}{2}$, it follows that for almost $\upsilon \in \Upsilon$, the following holds:*

$$\lim_{t\to\infty} sup\frac{1}{t}ln\left(\frac{\mathbb{K}}{d}n(t) + \frac{\sqrt{\mathbb{R}_0}}{\eta}p(t)\right) \leq \varkappa.$$

where $\varkappa = min\{d,\eta\}\left(\sqrt{\mathbb{R}_0} - 1\right)I_{\sqrt{\mathbb{R}_0}\leq 1} + max\{d,\eta\}\left(\sqrt{\mathbb{R}_0} - 1\right)I_{\sqrt{\mathbb{R}_0}>1} + \beta_1 d\left(\frac{\mathbb{R}_0}{2a_1}\right)^{\frac{1}{2}}$ and $\mathbb{R}_0 = \frac{k\delta}{d}$.

When $\varkappa < 0$, the predator population n is expected to undergo exponential decay with a probability of one, implying that the population will inevitably diminish.

$$\lim_{t\to\infty} n(t) = 0.$$

Furthermore, the weak convergence of the distribution of $m(t)$ occurs, leading to the emergence of a measure characterized by the density:

$$\varpi(\upsilon) = \mathbb{Z}\beta_1^{-2}\upsilon^{-2+\frac{2a_1}{\beta_1^2}} w^{-2+\frac{2a_1}{\mathbb{K}\beta_1^2}\upsilon}, \upsilon \in (0,\infty),$$

where $\mathbb{Z} = \left[\beta_1^{-2}\left(\frac{\mathbb{K}\beta_1^2}{2a_1}\right)^{\frac{2a_1}{\beta_1^2}-1}\Gamma\left(\frac{2a_1}{\beta_1^2}\right)\right]^{-1}$ is a constant that satisfies $\int_0^\infty \varpi(\upsilon)\,d\upsilon = 1$.

Proof. Given an $(m(0), n(0), p(0)) \in \mathbb{R}^3_+$, the solution to system (1.3) remains positive. As a result, we obtain:

$$rm \leq a_1 m \left(1 - \frac{m}{\mathbb{K}}\right) dt + \beta_1 mrQ_1(t).$$

Consider the 1-dimensional stochastic differential equation below.

$$rM = a_1 M \left(1 - \frac{M}{\mathbb{K}}\right) dt + \beta_1 MrQ_1(t). \tag{2.8}$$

It can be easily shown that Equation (2.8) possesses a stationary solution denoted as $\tilde{M}(t)$, and this solution's density is outlined in [22].

$$\varpi(v) = \mathbb{Z}\beta_1^{-2} v^{-2+\frac{2a_1}{\beta_1^2}} w^{-2+\frac{2a_1}{K\beta_1^2}v}, v \in (0, \infty),$$

where $\mathbb{Z} = \left[\beta_1^{-2}\left(\frac{K\beta_1^2}{2a_1}\right)^{\frac{2a_1}{\beta_1^2}-1} \Gamma\left(\frac{2a_1}{\beta_1^2}\right)\right]^{-1}$ is a constant that satisfies $\int_0^\infty \varpi(v)\, dv = 1$. Consider $M(t)$ as the solution to the stochastic differential equation denoted by (2.7), where the initial condition is $M(0) = m(0) > 0$. Utilizing the comparison theorem for 1-D SDE [23], it can be deduced that $m(t)$ remains less than or equal to $M(t)$ for all $t \geq 0$, almost surely.

Furthermore, we possess

$$\begin{aligned}
H_1 :=& \int_0^\infty v\varpi(v)dv \\
=& \mathbb{Z}\beta_1^{-2} \int_0^\infty v^{\frac{2a_1}{\beta_1^2}-1} w^{\frac{-2a_1}{\mathbb{K}\beta_1^2}v} dv \\
=& \mathbb{Z}\beta_1^{-2} \int_0^\infty \left(\frac{\mathbb{K}\beta_1^2}{2a_1}\right)^{\frac{2a_1}{\beta_1^2}-1} t^{\frac{2a_1}{\beta_1^2}-1} w^{-t}\frac{\mathbb{K}\beta_1^2}{2a_1} \\
=& \mathbb{Z}\beta_1^{-2}\left(\frac{\mathbb{K}\beta_1^2}{2a_1}\right)^{\frac{2a_1}{\beta_1^2}} \Gamma\left(\frac{2a_1}{\beta_1^2}\right) \\
=& \frac{\mathbb{K}\beta_1^2}{2a_1} \frac{\Gamma\left(\frac{2a_1}{\beta_1^2}\right)}{\Gamma\left(\frac{2a_1}{\beta_1^2} - 1\right)} \\
=& \frac{\mathbb{K}\beta_1^2}{2a_1}\left(\frac{2a_1}{\beta_1^2} - 1\right) \\
=& \frac{\mathbb{K}\left(a_1 - \frac{\beta_1^2}{2}\right)}{a_1}.
\end{aligned}$$

and

$$
\begin{aligned}
H_2 :=& \int_0^\infty v^2 \varpi(v) dv \\
=& \mathbb{Z}\beta_1^{-2} \int_0^\infty v^{\frac{2a_1}{\beta_1^2}-1} w^{\frac{-2a_1}{\mathbb{K}\beta_1^2}v} dv \\
=& \mathbb{Z}\beta_1^{-2} \int_0^\infty \left(\frac{\mathbb{K}\beta_1^2}{2a_1}\right)^{\frac{2a_1}{\beta_1^2}-1} t^{\frac{2a_1}{\beta_1^2}-1} w^{-t} \frac{\mathbb{K}\beta_1^2}{2a_1} \\
=& \mathbb{Z}\beta_1^{-2} \left(\frac{\mathbb{K}\beta_1^2}{2a_1}\right)^{\frac{2a_1}{\beta_1^2}+1} \Gamma\left(\frac{2a_1}{\beta_1^2}+1\right) \\
=& \left(\frac{\mathbb{K}\beta_1^2}{2a_1}\right)^2 \frac{\Gamma\left(\frac{2a_1}{\beta_1^2}+1\right)}{\Gamma\left(\frac{2a_1}{\beta_1^2}-1\right)} \\
=& \left(\frac{\mathbb{K}\beta_1^2}{2a_1}\right)^2 \frac{2a_1}{\beta_1^2} \left(\frac{2a_1}{\beta_1^2}-1\right) \\
=& \frac{\mathbb{K}^2\left(a_1 - \frac{\beta_1^2}{2}\right)}{a_1}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_0^\infty (v-\mathbb{K})^2 \varpi(v) dv =& \int_0^\infty \left(v^2 - 2\mathbb{K}v + \mathbb{K}^2\right) \varpi(v) dv \\
=& H_2 - 2H_1 + \mathbb{K}^2 \\
=& \frac{\mathbb{K}^2\left(a_1 - \frac{\beta_1^2}{2}\right)}{a_1} - \frac{2\mathbb{K}^2\left(a_1 - \frac{\beta_1^2}{2}\right)}{a_1} + \mathbb{K}^2 \\
=& \frac{\mathbb{K}^2\beta_1^2}{2a_1}.
\end{aligned} \tag{2.9}
$$

Moreover, let

$$\sqrt{\mathbb{R}_0}\,(\upsilon_1, \upsilon_2) = (\upsilon_1, \upsilon_2)\, X_0.$$

where $\sqrt{\mathbb{R}_0}\,(\upsilon_1, \upsilon_2) = \left(\mathbb{K}, \sqrt{\mathbb{R}_0}\right)$ and $X_0 = \begin{pmatrix} 0 & \frac{\delta}{d} \\ \mathbb{K} & 0 \end{pmatrix}$. Define a $\mathbb{C}^2$-function $\bar{U} : \mathbb{R}_+^2 \to \mathbb{R}_+$ by

$$\bar{U}\,(n, p) = \mu_1 n + \mu_2 p.$$

where $\mu_1 = \frac{\upsilon_1}{d}$, $\mu_2 = \frac{\upsilon_1}{\eta}$.

Applying Itō's formula for differentiating $ln\bar{U}$ results in:

$$d(ln\bar{U}) = W(ln\bar{U})dt + \frac{\mu_1\beta_2 n}{\bar{U}} dQ_2(t). \tag{2.10}$$

where

$$
\begin{aligned}
W(ln\bar{U}) =& \frac{\mu_1}{\bar{U}}[\delta p - dn] + \frac{\mu_2}{\bar{U}}[\eta mn - \eta p] - \frac{\mu_1^2\beta_2^2 n^2}{2\bar{U}^2} \\
\leq& \frac{\mu_1}{\bar{U}}[\delta p - dn] + \frac{\mu_2}{\bar{U}}[\eta mn - \eta p] \\
=& \frac{\mu_2 n}{\bar{U}}(\eta m - \eta\mathbb{K}) + \frac{1}{\bar{U}}\{\mu_2[\eta\mathbb{K}n - \eta p] + \mu_1[\delta p - dn]\} \\
=& \frac{\mu_2\eta n(m-\mathbb{K})}{\bar{U}} + \frac{1}{U}\left\{\frac{\upsilon_1}{d}[\delta p - dn] + \frac{\upsilon_2}{\eta}[\eta\mathbb{K}n - \eta p]\right\} \\
\leq& \frac{\mu_2\eta n(M-\mathbb{K})}{\bar{U}} + \frac{1}{U}\left\{\frac{\upsilon_1}{d}[\delta p - dn] + \frac{\upsilon_2}{\eta}[\eta\mathbb{K}n - \eta p]\right\} \\
\leq& \frac{\mu_2\eta}{\mu_1}|M-\mathbb{K}| + \frac{1}{\bar{U}}(\upsilon_1,\upsilon_2)\left(M_0(n,p)^T - (n,p)^T\right) \\
=& \frac{\mu_2\eta}{\mu_1}|M-\mathbb{K}| + \frac{1}{\bar{U}}\left(\sqrt{R_0}-1\right)(\upsilon_1 n + \upsilon_2 p) \\
\leq& min\{d,n\}\left(\sqrt{\mathbb{R}_0}-1\right)I_{\sqrt{\mathbb{R}_0}\leq 1} + max\{d,n\}\left(\sqrt{\mathbb{R}_0}-1\right)I_{\sqrt{\mathbb{R}_0}>1} + \frac{\mu_2\eta}{\mu_1}|M-\mathbb{K}|.
\end{aligned}
\tag{2.11}
$$

From (2.10), we obtain

$$
\begin{aligned}
d(ln\bar{U}) \leq& \left[min\{d,n\}\left(\sqrt{\mathbb{R}_0}-1\right)I_{\sqrt{\mathbb{R}_0}\leq 1} + max\{d,n\}\left(\sqrt{\mathbb{R}_0}-1\right)I_{\sqrt{\mathbb{R}_0}>1} + \frac{\mu_2\eta}{\mu_1}|M-\mathbb{K}|\right] \\
&+ \frac{\mu_1\beta_2 n}{\bar{U}}dQ_2(t).
\end{aligned}
\tag{2.12}
$$

Performing integration from 0 to t and subsequently dividing both sides of equation (2.12) by t results in,

$$
\begin{aligned}
\frac{ln\bar{U}(t)}{t} \leq& \frac{ln\bar{U}(0)}{t} + min\{d,n\}\left(\sqrt{\mathbb{R}_0}-1\right)I_{\sqrt{\mathbb{R}_0}\leq 1} + max\{d,n\}\left(\sqrt{\mathbb{R}_0}-1\right)I_{\sqrt{\mathbb{R}_0}>1} \\
&+ \frac{\mu_2\eta}{\mu_1}\int_0^t |M(s)-\mathbb{K}|\,ds + \frac{1}{t}\int_0^t \frac{\mu_1\beta_2 n(s)}{\bar{U}(s)}dQ_2(s) \\
=& \frac{ln\bar{U}(0)}{t} + min\{d,n\}\left(\sqrt{\mathbb{R}_0}-1\right)I_{\sqrt{\mathbb{R}_0}\leq 1} + max\{d,n\}\left(\sqrt{\mathbb{R}_0}-1\right)I_{\sqrt{\mathbb{R}_0}>1} \\
&+ \frac{\mu_2\eta}{\mu_1}\int_0^t |M(s)-\mathbb{K}|\,ds + \frac{X(t)}{t}.
\end{aligned}
\tag{2.13}
$$

Here, let $X(t) = \frac{\mu_1\beta_2 n(s)}{\bar{U}(s)}dQ_2(s)$ represent a local martingale with a quadratic variation of $< X, X >_t = \beta_2^2\int_0^t\left(\frac{\mu_1 n(s)}{\bar{U}(s)}\right)^2 ds \leq \beta_2^2 t$. Applying the strong law of large numbers to a local martingale [21] results in

$$\lim_{t\to\infty}\frac{X(t)}{t}=0. \tag{2.14}$$

Given the ergodic nature of $M(t)$ and $\int_0^\infty v\varpi(v)dv<\infty$, we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t|M(s)-\mathbb{K}|\,ds=\int_0^\infty|v-\mathbb{K}|\,\varpi(v)dv\leq\left(\int_0^\infty(v-\mathbb{K})^2\,\varpi(v)dv\right)^{\frac{1}{2}}. \tag{2.15}$$

Applying the upper limit to both sides of equation (2.13) and combining it with (2.14) and (2.15) yields

$$\begin{aligned}\lim_{t\to\infty}sup\frac{ln\bar{U}(t)}{t}\leq&min\left\{d,n\right\}\left(\sqrt{\mathbb{R}_0}-1\right)I_{\sqrt{\mathbb{R}_0}\leq1}+max\left\{d,n\right\}\left(\sqrt{\mathbb{R}_0}-1\right)I_{\sqrt{\mathbb{R}_0}>1}\\&+\frac{\mu_2\eta}{\mu_1}\left(\frac{\mathbb{K}^2\beta_1^2}{2a_1}\right)^{\frac{1}{2}}\\=&min\left\{d,n\right\}\left(\sqrt{\mathbb{R}_0}-1\right)I_{\sqrt{\mathbb{R}_0}\leq1}+max\left\{d,n\right\}\left(\sqrt{\mathbb{R}_0}-1\right)I_{\sqrt{\mathbb{R}_0}>1}\\&+\beta_1d\frac{\mathbb{R}_0}{2a_1}^{\frac{1}{2}}\\=&\varkappa.\end{aligned} \tag{2.16}$$

This stands as the necessary assertion. Additionally, in the case where $\varkappa<0$, it can be readily deduced that $\lim_{t\to\infty}sup\frac{lnn(t)}{t}<0$

This implies $\lim_{t\to\infty}y(t)=0$ a.s. In other words, the predator population n exhibits exponential decay with a probability of one. This concludes the proof. □

3 Conclusion

In this paper, we investigated a stochastic model that captures the interactions between predator and prey species, accounting for distributed delays. The study's initial focus was on establishing the existence of a stable pattern, known as a stationary distribution, for positive solutions within this model. This was achieved by employing the stochastic Lyapunov function approach. Additionally, the research moved forward to outline specific conditions that lead to the complete elimination of predator populations. This extinction scenario points to the coexistence of a thriving prey population with the absence of predators. In summation, this study contributes to our comprehension of ecological systems by offering insights into the intricate dynamics between predator and prey populations under the influence of distributed delays.

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