CONFORMAL CURVATURE TENSOR COMPILED WITH A METRIC SEMI-SYMMETRIC CONNECTION OF ALMOST HYPERBOLIC TACHIBANA MANIFOLDS

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Abstract: Negi, et. al. (2019) has established an analytic HP-transformation in almost Kaehlerian spaces. Also, study on Projective recurrent and Symmetric tensor in Almost Kaehlerian Spaces. After that, Negi and Preeti Chauhan (2021), have accomplished Kaehlerian Manifolds with H-Projective and Bochner Recurrent Curvature Tensor of first order. In this chapter, we have calculated Conformal curvature tensor compiled with a metric semi-symmetric connection of Almost Hyperbolic Tachibana Manifolds and some theorems established.

Keywords: Conformal curvature tensor, Recurrent curvature tensor, Riemannian manifolds, Kaehlerian Manifolds and Almost Hyperbolic Tachibana Manifolds.

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1. Introduction:

Let ***n***-dimensional differential manifold (M^n, g) (n >2), with the structure $F_i^{\ h}$ is a tensor field of type (1, 1) and $F_{i,j}^h$ is a covariant derivative with respect to Riemannian curvature tensor. A metric semi-symmetric connection ∇ and Riemannian curvature tensor with coefficients $\Gamma_{i,j}^h$ and $\begin{Bmatrix} h \\ i\ j \end{Bmatrix}$ [Yano and Imai (1982)] and if the torsion tensor Γ of the connection ∇ on (M^n, g) (n >2) satisfies. Then the manifold is called hyperbolic Tachibana manifold which satisfies equations (1.1), (1.2) and (1.3), (1.4), (1.5) and (1.6) respectively:

$$F_j^{\ i} F_i^{\ h} = \delta_j^{\ h}, \tag{1.1}$$

$$F_{ij} = -F_{ji}, (F_{ij} = g_{jk} F_i^{\ k}), \tag{1.2}$$

$$F_{i,j}{}^{h} = 0, \tag{1.3}$$

$$\Gamma_{jk}{}^{i} = p_j A_k{}^{i} - p_k A_j{}^{i}, \tag{1.4}$$

$$\Gamma_{jk}{}^{i} = \begin{Bmatrix} i \\ jk \end{Bmatrix} - p_k U_j{}^{i} + p_j V_k{}^{i} - p^i V_{jk}, \tag{1.5}$$

$$U_{ij} = \tfrac{1}{2}(A_{ij} - A_{ji}),\ V_{ij} = \tfrac{1}{2}(A_{ij} + A_{ji}), \tag{1.6}$$

Also, $A_j{}^{i}$ denotes the components of the tensor of the type (1, 1) as well as $\nabla g = 0$ and p_i are the components of a 1- form. Then equation (1.6) written as:

$$A_{ij} = U_{ij} + V_{ij}. \tag{1.7}$$

Taking $V_{ij} = g_{ij}$ and $U_{ij} = F_{ij}$ in (1.5) [Nevena Pusic (2003)], then we obtain:

$$\Gamma_{jk}{}^{i} = \begin{Bmatrix} i \\ jk \end{Bmatrix} - p_k F_j{}^{i} + p_j \delta_k{}^{i} - p^i g_{jk}. \tag{1.8}$$

The relation between Riemannian curvature tensor (r-4) relating to a metric semi-symmetric connection is given by [Nevena Pusic (2003)]:

$$\begin{aligned}\bar{R}_{ijkh} = {} & R_{ijkh} - g_{ih}p_{kj} + g_{ik}p_{hj} - g_{jk}p_{hi} + g_{hj}p_{ki} + p_j p_h F_{ik} + p_i p_k F_{jh} \\ & - p_j p_k F_{ih} - p_i p_h F_{jk},\end{aligned} \tag{1.9}$$

Where

$$p_{jk} = \nabla_j p_k - p_j p_k + p_k q_j + \tfrac{1}{2} p_s p^s g_{jk}. \tag{1.10}$$

Again, the Ricci tensor and the scalar curvature are given by [Nevena Pusic (2003)]:

$$\bar{R}_{jk} = R_{jk} - (n-2)p_{kj} - g_j p_m{}^{m} - p_j q_k + p_k g_j - p^s p_s F_{kj}, \tag{1.11}$$

$$\bar{R} = R - 2(n-1)p_m{}^{m}. \tag{1.12}$$

$$p_j = p^h g_{jh};\ q_i = F_{ti}p^t;\ p^r = g^{ir}p_i. \tag{1.13}$$

2. Conformal Curvature Tensor compiled with a Metric semi-Symmetric Connection of Almost Hyperbolic Tachibana Manifolds.

We have Conformal curvature tensor (rank-4) in a Riemannian manifolds is defined as following:

$$\begin{aligned}C_{ijkh} = {} & R_{ijkh} - \tfrac{1}{n-2}(R_{jk}g_{ih} - R_{ik}g_{jh} + R_{ih}g_{jk} - R_{jh}g_{ki}) \\ & + \tfrac{R}{(n-1)(n-2)}(g_{ih}g_{jk} - g_{jh}g_{ki})\end{aligned} \tag{3.1}$$

The Conformal curvature tensor (rank-4) relating to a metric semi-symmetric connection is given by:

$$\bar{C}_{ijkh} = \bar{R}_{ijkh} - \tfrac{1}{n-2}(\bar{R}_{jk}g_{ih} - \bar{R}_{ik}g_{jh} + \bar{R}_{ih}g_{jk} - \bar{R}_{jh}g_{ki})$$

$$+\frac{\bar{R}}{(n-1)(n-2)}(g_{ih}g_{jk} - g_{jh}g_{ki}) \quad (3.2)$$

Theorem 3.1 The Conformal curvature tensor (r-4) relating to a metric semi-symmetric connection will be equal to the Conformal curvature tensor (r-4) with respect to a Riemannian curvature tensor (r-4) in almost hyperbolic Tachibana manifold, if the following conditions satisfies:

$$p_h F_{ik} = p_k F_{ih}. \quad (3.3)$$

Proof. Wed have from (1.9), (1.11) and (1.12) in (3.2), we obtain:

$$\begin{aligned}\bar{C}_{ijkh} = {} & R_{ijkh} - g_{ih}p_{kj} + g_{ik}p_{hj} - g_{jk}p_{hi} + g_{hj}p_{ki} + p_j p_h F_{ik} + p_i p_k F_{jh} - p_j p_k F_{ih} \\ & - p_i p_h F_{jk} - \frac{1}{n-2}[g_{ih}(R_{jk} - (\text{n-2})p_{kj} - g_{jk}p_m{}^m - p_j q_k + p_k q_j - p^s p_s F_{kj}) \\ & - g_{jh}(R_{ik} - (n-2)p_{ki} - g_{ik}p_m{}^m - p_i q_k + p_k q_i - p^s p_s F_{ki}) \\ & + g_{kj}(R_{ih} - (n-2)p_{hi} - g_{ih}p_m{}^m - p_i q_h + p_h q_i - p^s p_s F_{hi}) \\ & - g_{ik}(R_{jh} - (n-2)p_{hj} - g_{jh}p_m{}^m - p_j q_h + p_h q_j - p^s p_s F_{hj})] \\ & + \frac{R - 2(n-1)p_m{}^m}{(n-1)(n-2)}(g_{ih}g_{jk} - g_{jh}g_{ik}) \end{aligned} \quad (3.4)$$

Again, from (1.13 and (3.4), we get:

$$\begin{aligned}\bar{C}_{ijkh} = {} & R_{ijkh} - \frac{1}{(n-2)}(R_{jk}g_{ih} - R_{ik}g_{jh} + R_{ih}g_{jk} - R_{jh}g_{ki}) \\ & + \frac{R}{(n-1)(n-2)}(g_{ih}g_{jk} - g_{jh}g_{ki}) + \frac{n+1}{n-2}(p_j p_h F_{ik} - p_j p_k F_{ih}) \\ & + \frac{n+1}{n-2}(p_i p_k F_{jh} - p_i p_h F_{jk}) \end{aligned} \quad (3.5)$$

Also, from (3.1) and (3.5), we get:

$$\bar{C}_{ijkh} = C_{ijkh} + \frac{n+1}{n-2}(p_j p_h F_{ik} - p_j p_k F_{ih}) + \frac{n+1}{n-2}(p_i p_k F_{jh} - p_i p_h F_{jk}) \quad (3.6)$$

If we take $p_h F_{ik} = p_k F_{ih}$ then (3.6) reduces to the form:

$$\bar{C}_{ijkh} = C_{ijkh}. \quad (3.7)$$

Hence completes the proof.

Theorem 3.2 The Conformal curvature tensor (r-4) relating to a metric semi-symmetric connection in almost hyperbolic Tachibana manifold satisfies Bianchi identity if:

$$p_j F_{ik} + p_i F_{kj} + p_k F_{ji} = 0. \quad (3.8)$$

Proof. We have Interchanging i, j *and* k in a cyclic order in (3.5), we get:

$$\bar{C}_{ijkh} = C_{ijkh} + \frac{n+1}{n-2}(p_j p_h F_{ik} - p_j p_k F_{ih}) + \frac{n+1}{n-2}(p_i p_k F_{jh} - p_i p_h F_{jk}), \quad (3.9)$$

$$\bar{C}_{jkih} = C_{jkih} + \frac{n+1}{n-2}(p_k p_h F_{ji} - p_k p_i F_{jh}) + \frac{n+1}{n-2}(p_j p_i F_{kh} - p_j p_h F_{ki}) \quad (3.10)$$

And

$$\bar{C}_{kijh} = C_{kijh} + \frac{n+1}{n-2}(p_i p_h F_{kj} - p_i p_j F_{kh}) + \frac{n+1}{n-2}(p_k p_j F_{ih} - p_k p_h F_{ij}) \quad (3.11)$$

Adding (3.9), (3.10) and (3.11), we obtain:

$$\bar{C}_{ijkh} + \bar{C}_{jkih} + \bar{C}_{kijh} = C_{ijkh} + C_{jkih} + C_{kijh} + 2\left(\frac{n+1}{n-2}\right) p_h (p_j F_{ik} + p_i F_{kj} + p_k F_{ji}). \quad (3.12)$$

Since, the Conformal curvature tensor (r-4) in a Riemannian manifold satisfies the condition:

$$C_{ijkh} + C_{jkih} + C_{kijh} = 0, \quad (3.13)$$

By using (3.13) in (3.12), we find

$$\bar{C}_{ijkh} + \bar{C}_{jkih} + \bar{C}_{kijh} = 2\left(\frac{n+1}{n-2}\right) p_h (p_j F_{ik} + p_i F_{kj} + p_k F_{ji}). \quad (3.14)$$

If we take $p_j F_{ik} + p_i F_{kj} + p_k F_{ji} = 0$ then from (3.14), we get

$$\bar{C}_{ijkh} + \bar{C}_{jkih} + \bar{C}_{kijh} = 0. \quad (3.15)$$

Hence completes the proof.

Theorem 3.3 The Conformal curvature tensor (r-4) relating to a metric semi-symmetric connection in almost hyperbolic Tachibana manifold satisfies the following properties:
$\bar{C}_{ijkh} = -\bar{C}_{jikh}, \quad \bar{C}_{ijkh} = -\bar{C}_{ijhk}.$

Proof. We have Interchanging i and j in (3.5), we get:

$$\bar{C}_{jikh} = C_{jikh} + \left(\frac{n+1}{n-2}\right)(p_i p_h F_{jk} - p_i p_k F_{jh}) + \left(\frac{n+1}{n-2}\right)(p_j p_k F_{ih} - p_j p_h F_{ik}). \quad (3.16)$$

Adding (3.5) and (3.16), we obtain

$$\bar{C}_{ijkh} + \bar{C}_{jikh} = C_{ijkh} + C_{jikh}. \quad (3.17)$$

Since the conformal curvature tensor (r-4) in a Riemannian manifold satisfies:

$$C_{ijkh} + C_{jikh} = 0, \quad (3.18)$$

Now using (3.18) and (3.17), we get the result.

Again, interchanging k and h in (3.5), we have

$$\bar{C}_{ijhk} = C_{ijhk} + \frac{n+1}{n-2}(p_j p_k F_{ih} - p_j p_h F_{ik}) + \frac{n+1}{n-2}(p_i p_h F_{jk} - p_i p_k F_{jh}). \quad (3.19)$$

Adding (3.5) and (3.19), we have

$$\bar{C}_{ijkh} + \bar{C}_{ijhk} = C_{ijkh} + C_{ijhk}. \quad (3.20)$$

Since the conformal curvature tensor (r-4) in a Riemannian manifold satisfies:

$$C_{ijkh} + C_{ijhk} = 0, \quad (3.21)$$

Now by using (3.21) and (3.20), we get the result.

Theorem 3.4 The Conformal curvature tensor (r-4) relating to a metric semi-symmetric connection relating to Riemannian recurrent curvature tensor (r-4) in almost hyperbolic Tachibana manifold, if following condition satisfies:

$$C_{rjkh} p_i + C_{irkh} p_j + C_{ijrh} p_k + C_{ijkr} p_h = 0. \quad (3.22)$$

Proof. We have by taking covariant differentiation of the Conformal curvature tensor (r-4) relating to the Riemannian manifolds and metric semi-symmetric connection respectively, we get:

$$D_m C_{ijkh} = \partial_m C_{ijkh} - C_{rjkh}\begin{Bmatrix} r \\ mi \end{Bmatrix} - C_{irkh}\begin{Bmatrix} r \\ mj \end{Bmatrix} - C_{ijrh}\begin{Bmatrix} r \\ mk \end{Bmatrix} - C_{ijkr}\begin{Bmatrix} r \\ mh \end{Bmatrix} \quad (3.23)$$

And

$$\nabla_m C_{ijkh} = \partial_m C_{ijkh} - C_{rjkh}\Gamma_{mi}{}^{r} - C_{irkh}\Gamma_{mj}{}^{r} - C_{ijrh}\Gamma_{mk}{}^{r} - C_{ijkr}\Gamma_{mh}{}^{r} \quad (3.24)$$

Subtracting (3.23) from (3.24), we get:

$$\nabla_m C_{ijkh} - D_m C_{ijkh} = C_{rjkh}(\begin{Bmatrix} r \\ mi \end{Bmatrix} - \Gamma_{mi}{}^{r}) + C_{irkh}(\begin{Bmatrix} r \\ mj \end{Bmatrix} - \Gamma_{mj}{}^{r})$$

$$+ C_{ijrh}(\begin{Bmatrix} r \\ mk \end{Bmatrix} - \Gamma_{mk}{}^{r}) + C_{ijkr}(\begin{Bmatrix} r \\ mh \end{Bmatrix} - \Gamma_{mh}{}^{r}) \quad (3.25)$$

Now using (1.8) in (3.25), we find:

$$\nabla_m C_{ijkh} - D_m C_{ijkh} = C_{rjkh}(p_i F_m{}^{r} - p_m \delta_i{}^{r} + p^r g_{mi}) + C_{irkh}(p_j F_m{}^{r} - p_m \delta_j{}^{r} + p^r g_{mj})$$

$$+ C_{ijrh}(p_k F_m{}^{r} - p_m \delta_k{}^{r} + p^r g_{mk}) + C_{ijkr}(p_h F_m{}^{r} - p_m \delta_h{}^{r} + p^r g_{mh}) \quad (3.26)$$

Again using (1.13) in (3.26), we obtain:

$$\nabla_m C_{ijkh} - D_m C_{ijkh} = (C_{rjkh} p_i + C_{irkh} p_j + C_{ijrh} p_k + C_{ijkr} p_h) F_m{}^{r} \quad (3.27)$$

If (3.22) is satisfied, then we find:

$$\nabla_m C_{ijkh} = D_m C_{ijkh}. \quad (3.28)$$

Hence we get the result.

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