

# A STUDY ON GENERALIZED RICCI SOLITONS ON LP-SASAKIAN MANIFOLDS

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**Abstract:** In the present Chapter we studied LP-Sasakian manifold. At first we introduced historical background of the concern manifold. Next some rudimentary facts and related properties of LP-Sasakian manifold are discussed. After that LP-Sasakian manifold concerning generalized Ricci soliton is studied and investigate main result in the form of theorem that is LP-Sasakian manifold of odd dimension satisfying the generalized Ricci soliton equation is an Einstein manifolds.

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**Key Words:** LP-Sasakian manifold, Lorentzian para-Sasakian Manifold, Lorentzian Metric, Riemannian manifold, Ricci Soliton, Einstein Manifold.

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## 1. INTRODUCTION

An developing area of contemporary mathematics is the geometry of contact manifolds. The mathematical formalisation of classical mechanics has given way to the concept of contact geometry [7]. K- contact manifolds and sasakian manifolds are two significant kinds of contact manifolds [1], [20]. There are various researchers that have analyzed K-contact and Sasakian manifolds ( [21], [3], [4], [11], [19], [23]) and many others.

The concept of the LP-Sasakian manifold was initially introduced by Matsumoto [13]. Mihai and Rosca defined the same notion independently in [16]. This type of manifold is also discussed in ([14, [22]). A complete regular contact metric manifold  $M^{2n+1}$  carries a K-contact structure  $(\varphi, \xi, \eta, g)$ , which is described in terms of almost kaehler structure  $(J, G)$  of the base manifold's  $M^{2n+1}$ . If the base manifold  $(M^{2n+1}, J, G)$  in this case is Kaehlerian, the K-contact structure  $(\varphi, \xi, \eta, g)$  is Sasakian. If  $(M^{2n+1}, J, G)$  is only almost Kaehler then  $(\varphi, \xi, \eta, g)$  is only K-contact [1]. Recent research in [12] has demonstrated the existence of K-contact manifolds that are not Sasakian. Even yet, Sasakian and contact structures are intermediated by K-contact structures. Numerous writers, including [3, [4], [9], [19], [21], [23], have researched K-contact manifolds.

Let us consider a smooth function  $f$  on  $M$ , the gradient of  $f$  is defined by

$$(1.1) \quad g(\text{grad } f, X) = Xf,$$

The Hessian of  $f$  defined by

$$(1.2) \quad (\text{Hess } f)(X, Y) = g(\nabla_X \text{grad } f, Y),$$

for all smooth vector fields  $X, Y$ . For a smooth vector field  $X$ , we have ([15],[18])

$$(1.3) \quad X^b(Y) = g(X, Y).$$

The generalized Ricci soliton equation in a Riemannian manifold  $(M, g)$  is described by [18]

$$(1.4) \quad \ell_X g = -2c_1 X^b \cdot X^b + 2c_2 S + 2\lambda g,$$

where  $\ell_X g$  is the lie derivative of  $X$ , defined by

$$(1.5) \quad (\ell_X g)(Y, Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y),$$

for all vector fields  $X, Y, Z$  and  $c_1, c_2, \lambda \in R$ . For different values of and equation (1.4) is a generalization of killing equation ( $c_1 = c_2 = \lambda = 0$ ), for homotheties ( $c_1 = c_2 = 0$ ), Ricci soliton ( $c_1 = 0, c_2 = -1$ ), vaccum near-horizon geometry equation ( $c_1 = 1, c_2 = \frac{1}{2}$ ) etc. We suggest the reader for further information ([2], [5], [6], [10], [18]).

If  $X = \text{grad } f$ , then the equation for the generalized Ricci soliton is [8]

$$(1.6) \quad \text{Hess } f = -c_1 df \cdot df + c_2 S + \lambda g.$$

The work in present Chapter motivated by [8], for the fact that relationship between LP-Sasakian and K-contact manifold, so we studied  $(2n+1)$ -dimensional Lorentzian para- Sasakian manifold over generalized Ricci soliton.

## 2. PRELIMINARIES

A  $(2n+1)$ -dimension differentiable manifold will be LP-Sasakian manifold [13] [16], if it admits the  $(1,1)$  tensor field  $\varphi$ , vector field  $\xi$ ,  $\eta$  is a 1 form on  $M$  and  $g$  is a lorentzian metric, satisfy [14],[17]

$$(2.1) \quad \varphi^2 = I + \eta(X)\xi, \quad \eta(\xi) = -1, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$(2.2) \quad \varphi(\xi) = 0, \quad \eta(\varphi X) = 0, \quad g(X, \xi) = \eta(X), \quad g(\varphi X, Y) = -g(X, \varphi Y),$$

$$(2.3) \quad \nabla_X \xi = \varphi X,$$

$$(2.4) \quad g(R(\xi, X)Y, \xi) = \eta(R(\xi, X)Y) = -g(X, Y) - \eta(X)\eta(Y),$$

$$(2.5) \quad R(\xi, X)\xi = X + \eta(X)\xi,$$

$$(2.6) \quad S(X, \xi) = (n-1)\eta(X),$$

$$(2.7) \quad (\nabla_X \varphi)Y = [g(X, Y) + \eta(X)\eta(Y)]\xi + [X + \eta(X)\xi]\eta(Y),$$

as any vector fields,  $X, Y$  on  $\chi(M)$ .

Additionally, a manifold is considered Einstein if its Ricci tensor has the following form:

$$(2.8) \quad S(X, Y) = ag(X, Y),$$

for vector fields  $X, Y$ .

Substituting  $X = Y = \xi$  in (2.6) and then (2.4) and (2.2), we get

$$(2.9) \quad a = (n-1),$$

Take in account (2.9) we obtain from (2.8)

$$(2.10) \quad S(X, Y) = (n-1)g(X, Y),$$

similarly from (2.10) we infer

$$(2.11) \quad QX = (n-1)X,$$

### 3. GENERALIZED RICCI SOLITON ON LP-SASAKIAN MANIFOLD

**Theorem 3.1.** Let  $(M, \varphi, \xi, \eta, g)$  be a LP-Sasakian manifold then

$$(3.1) \quad (\ell_\xi(\ell_X g))(Y, \xi) = -g(X, Y) + g(\nabla_\xi \nabla_\xi X, Y) + Yg(\nabla_\xi X, \xi),$$

for smooth vector fields  $X, Y$  with  $Y$  orthogonal to  $\xi$ .

**Proof:** It is known that

$$(3.2) \quad (\ell_\xi(\ell_X g))(Y, \xi) = \xi((\ell_X g)(Y, \xi)) - (\ell_X g)(\ell_\xi Y, \xi),$$

using (1.5) in (3.2) yields

$$\begin{aligned}
(3.3) \quad & (\ell_{\xi}(\ell_X g))(Y, \xi) = \xi(g(\nabla_Y X, \xi) + g(\nabla_{\xi} X, Y) - g(\nabla_{[\xi, Y]} X, \xi)) \\
& - g(\nabla_{\xi} X, [\xi, Y]) = g(\nabla_{\xi} \nabla_Y X, \xi) + g(\nabla_Y X, \nabla_{\xi} \xi) + g(\nabla_{\xi} \nabla_{\xi} X, Y) \\
& + g(\nabla_{\xi} X, \nabla_{\xi} Y) - g(\nabla_{[\xi, Y]} X, \xi) - g(\nabla_{\xi} X, \nabla_{\xi} Y) + g(\nabla_{\xi} X, \nabla_Y \xi) \\
& = g(\nabla_{\xi} \nabla_Y X, \xi) + g(\nabla_Y X, \nabla_{\xi} \xi) + g(\nabla_{\xi} \nabla_{\xi} X, Y) \\
& - g(\nabla_{[\xi, Y]} X, \xi) + g(\nabla_{\xi} X, \nabla_Y \xi),
\end{aligned}$$

by definition of Riemannian curvature tensor, from (3.3) it follows that

$$(3.4) \quad (\ell_{\xi}(\ell_X g))(Y, \xi) = g(R(\xi, Y)X, \xi) + g(\nabla_{\xi} \nabla_{\xi} X, Y)(\ell_X g) + Yg(\nabla_{\xi} X, \xi),$$

using (2.4) in (3.4) and with  $Y$  orthogonal to  $\xi$ , we get

$$(3.5) \quad g(R(\xi, Y)X, \xi) = -g(X, Y),$$

so, (3.4) may be expressed as

$$(3.6) \quad (\ell_{\xi}(\ell_X g))(Y, \xi) = -g(X, Y) + g(\nabla_{\xi} \nabla_{\xi} X, Y) + Yg(\nabla_{\xi} X, \xi),$$

**Lemma 3.2:** Let  $M$  be a Riemannian manifold and let  $f$  be a smooth function. Then [15]

$$(3.7) \quad (\ell_{\xi}(df \cdot df))(Y, \xi) = Y(\xi(f))\xi(f) + Y(f)\xi(\xi(f)),$$

for every vector field  $Y$ .

**Theorem 3.2:** Let  $(M, \varphi, \xi, \eta, g)$  is a LP-Sasakian manifold which satisfies the generalized Ricci soliton equation. Then

$$(3.8) \quad \nabla_{\xi} \text{grad } f = (\lambda + (n-1)c_2 n)\xi - c_1 \xi(f) \text{grad } f.$$

**Proof:** Using (2.6) we have

$$(3.9) \quad \lambda \eta(Y) + c_2 S(\xi, Y) = [\lambda + (n-1)]\eta(Y).$$

Making use of (1.6) and (3.9) implies

$$(3.10) \quad (\text{Hess } f)(\xi, Y) = -c_1 \xi(f)g(\text{grad } f, Y) + [\lambda + (n-1)]\eta(Y).$$

The lemma thus follows from (3.5) and (1.6), which gives the Hessian definition.

Next, Suppose that is  $Y$  orthogonal to  $\xi$ . From Lemma 3.1, and taking  $X = \text{grad } f$ , we get

$$(3.11) \quad 2(\ell_{\xi}(\text{Hess } f))(Y, \xi) = Y(f) + g(\nabla_{\xi} \nabla_{\xi} \text{grad } f, Y) + Yg(\nabla_{\xi} \text{grad } f, \xi),$$

by Lemma (3.2) and above equation, we obtain

$$(3.12) \quad \begin{aligned} 2(\ell_{\xi}(Hess f)(Y, \xi) &= Y(f) + (\lambda + (n-1)c_2)g(\nabla_{\xi}\xi, Y) - c_1g(\nabla_{\xi}(\xi(f)grad f), Y) \\ &+ (\lambda + (n-1)c_2)Yg(\xi, \xi) - c_1Y(\xi(f)^2), \end{aligned}$$

since and from equation (2.10), we obtain

$$(3.13) \quad 2(\ell_{\xi}(Hess f)(Y, \xi) = Y(f) - c_1\xi(\xi(f))Y(f) - c_1\xi(f)g(\nabla_{\xi}(grad f, Y) - 2c_1(\xi(f))Y(\xi(f)),$$

Note that, from equation (2.3) , we have  $\ell_{\xi}g = 0$  it implies . Using the above fact and taking the Lie derivative to the generalized Ricci soliton equation (1.6) yields

$$(3.14) \quad 2(\ell_{\xi}(Hess f)(Y, \xi) = -2c_1(\ell_{\xi}(df \circ df))(Y, \xi).$$

Using (3.13), (3.14) and Lemma (3.2) we infer that

$$(3.15) \quad Y(f)[1 + c_1\xi\xi(f) + c_1\xi(f)^2] = 0.$$

According to Lemma 3.2 we have

$$(3.16) \quad \begin{aligned} c_1\xi(\xi(f)) &= c_1\xi g(\xi, grad f) \\ &= c_1g(\xi, \nabla_{\xi}grad f) \\ &= c_1(\lambda + (n-1)c_2) - c_1^2\xi(f)^2, \end{aligned}$$

by equation (3.15) and (3.16), we obtain

$$(3.17) \quad Y(f)[1 + c_1(\lambda + (n-1)c_2)] = 0.$$

Which implies  $\Rightarrow Y(f)0$ .

Provided  $1 + c_1(\lambda + (n-1)c_2) \neq 0$ . Therefore  $grad f$  is parallel to  $\xi$ . Hence  $grad f$  as  $d = \ker \eta$  is nowhere integrable, that is,  $f$  is a constant function. Thus the manifold is an Einstein one follows from (1.6), so we concluded that

**Theorem 3.3:** If  $(M, \varphi, \xi, \eta, g)$  is a LP-Sasakian manifold of odd dimension, which satisfies the generalized Ricci soliton equation with  $c_1(\lambda + (n-1)c_2) \neq -1$ . Then  $f$  has a constant value. Additionally, manifold is an Einstein manifold if  $c_2 \neq 0$ . The lemma thus follows from (3.5) and (1.6), which gives the Hessian definition.

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