A hybrid numerical technique for solving fractional Riccati equations

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Abstract.

In this article, we extends the application of hybrid numerical technique known as 'Laplace adomian decomposition mehtod' to the fractional Riccati equations. The hybrid numerical technique is the combination of Laplace transform and Adomian decomposition method. It increase the performance of the numerical technique and reduces the computing time. Finally, the efficiency of the proposed technique is shown by solving two examples and comparing the results with existing methods.

Keywords: Fractional reaction-diffusion system (FRDS); Caputo derivative; Laplace transform; Adomian decomposition technique.

1. Introduction

The hybrid numerical technique, Laplace adomian decomposition technique (LADM), is presented for solving the Riccati differential equation as follows:

$$u^{\alpha}(x) = p(x) + q(x)u(x) + r(x)u^2(x), \quad 0 \le x \le X, \tag{1}$$

with $u(x,0) = g(x)$.

Where $p(x), q(x), r(x)$ are continuous, which plays a key role in many disciplines of applied mathematics as well as sciences, [1]. For example, it is well known that static Schrödinger equation is closely related to a Riccati differential equation. Also, solitary wave solution of a nonlinear partial differential equation (NPDE) is expressed as a polynomial in two basic elementary functions, which satisfy a projective Riccati equation, [2]. These problems are also useful in analyzing optimal control strategies. Optimal control analysis is a very useful tool in the application of mathematical modeling of infectious disease dynamics. Several researchers are interested in the analysis and applications of these problems. Also, only up to some extent the analytical solutions are obtained because it is really difficult to get an explicit form except only in very few cases. For example, analytical method is obtained in solving the Riccati differential equation with constant coefficients, [3].

This is the situation where numeric techniques or approximate approaches to get the solution nearly similar to the analytical solution. Once the numerical solutions are obtained, they get verified and compare with the other numerical methods. In the recent investigations, solution of Riccati differential equation has been obtained by Adomian's decomposition method in [4]. Moreover, a special Riccati differential equation i.e., the quadratic Riccati differential equation is solved by Abbasbandy using He's VIM, the homotopy perturbation method (HPM) and the iterated He's HPM, [5–7]. In this investigation, the accuracy of solution of Riccati Differential equation has been compared with the Adomian's decomposition method. Furthermore, piecewise VIM has been introduced by Geng for solving Riccati differential equation, [8]. Originally, He has proposed the VIM [9-14]. Also, several mathematicians have suggested the VIM to be an important tool to solve different linear and nonlinear problems, [15-24].

In this paper, we present the LADM to solve Riccati Differential equation. Our results will show the method LADM is more reliable and accurate in comparison to VIM. The method LADM will reduce the burden of computation work for sure.

2. LADM solution process

We assume the following general non-linear FDE

$$ {}_0^cD_t^\alpha u(x,t) = L[u(x,t)] + N[u(x,t)] + h(x,t), \qquad (2.1) $$

with $u(x,0) = g(x)$.

Here ${}_0^cD_t^\alpha$ is the Caputo derivative, $h(x,t)$ is the source term, L and N is the linear and non-linear terms.

Taking LT of (2.1), we get

$$ LT[{}_0^cD_t^\alpha w(x,y,t)] = LT[L[w(x,y,t)] + N[w(x,y,t)] + g(x,y,t)]. $$

$$ LT[w(x,y,t)] = \frac{1}{s}h(x,y) + \frac{1}{s^\alpha}LT[L[w(x,y,t)] + N[w(x,y,t)] + g(x,y,t)], $$

Applying the inverse LT and LADM, we get

$$ \sum_{n=1}^{\infty} p^n u_n(x,t) = g(x) \ + LT^{-1}\left\{\frac{1}{s^\alpha}LT\left[L\left[\sum_{n=1}^{\infty} p^n u_n(x,t)\right] + \sum_{n=1}^{\infty} H_n[u(x,t)] + h(x,t)\right]\right\} $$

then for equal powers of p, we get

$u_0(x,t) = g(x),$

$u_1(x,t) = LT^{-1}\left\{\frac{1}{s^\alpha}LT[L[u_0(x,t)] + H_0[u(x,t)] + h(x,t)]\right\},$

and so on..

The LADM solution is

$u(x,t) = \ u_0(x,t) + u_1(x,t) + u_2(x,t) \dots \dots$

3. Test Examples

In this segment, we will apply the proposed method to some test problems.

Example1. Consider an inhomogeneous nonlinear system of time-fractional equations as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = 1 + 2u(x) - u^2(x)$$

Subject to the initial values $u(x, 0) = 0$ and the exact solution is $1 + \sqrt{2}\tanh(\sqrt{2}xt + \frac{\log(\frac{-1+\sqrt{2}}{1+\sqrt{2}})}{2})$.

Solution.

By using the LADM, the solution is given by-

$$u_0(x, t) = 0$$

$$u_1(x, t) = \frac{t^\alpha}{\Gamma(1+\alpha)}$$

$$u_2(x, t) = \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{2\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}$$

and so on......

The LADM solution is

$u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) \dots ....$

Table1. It shows approximate and exact solution $u(x, t)$ numeration using LADM , MVIM and VIM for example 1

x	***Exact Solution*** $\boldsymbol{u(x)}$	***FHPTM***	***MVIM***	***VIM***
0.4	0.567812	0.567812	0.513543	0.538667
1.2	1.95136	1.95136	1.90195	2.064
2.0	2.35777	2.35777	2.41229	3.33333
2.8	2.40823	2.40823	2.30603	3.32267
3.6	2.41359	2.41359	2.40026	1.008
4.0	2.41401	2.41401	2.50735	-1.33333

Example 2. Consider an inhomogeneous nonlinear system of time-fractional equations as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = 1 + x^2 - u^2(x)$$

Subject to the initial values $u(x, 0) = 1$ and the exact solution is $x + \frac{e^{-x^2}}{1+\int_0^x e^{-t^2}} dt$

Solution.

By using the LADM, the solution is given by-

$$u_0\,(x,t) = 1$$

$$u_1(x,t) = \frac{x^2 t^\alpha}{\Gamma(1+\alpha)}$$

$$u_2(x,t) = \frac{t^2\,(1+x^2)\Gamma(1+\alpha)}{\Gamma(1+\alpha)^2} - \frac{t^{3\alpha}x^4\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)\Gamma(1+\alpha)^2}$$

and so on......

The LADM solution is

$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t)$ … ….

Table 2. It shows approximate and exact solution $u(x,t)$ numeration using LADM, MVIM and VIM for example 2

X	***Exact Solution*** $\boldsymbol{u(x)}$	***FHPTM***	***MVIM***	***VIM***
0.4	1.01765	1.01765	1.0153	1.01704
1.2	1.11809	1.11809	1.10893	1.09907
2.0	1.33114	1.33114	1.32233	1.17352
2.8	2.00973	2.00973	2.04175	-1.03075
3.6	2.80021	2.80021	2.76833	-23.3443
4.0	4.00000	4.00000	3.56075	-135.829

4. Conclusion

In this paper, LADM is employed successfully to solve very famous Riccati differential equation. Numerical results suggest that solution of Riccati equation has a rapid convergence to approximate numerical solutions. These numerical results obtained are compared with those approximations, which are obtained from MVIM and VIM. Further, LADM has the ability to reduce the computation cost significantly. Thus, it can be easily used for big and small parameters in differential equations. In wide class of differential equations involving highly nonlinear terms, LADM can also be applied which will give solution exactly similar to the solution obtained by analytical methods. Hence LADM is more convenient and accurate than other existing methods.

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