NEW GENERALIZATION OF THE FIBONACCI SEQUENCE IN CASE OF THIRD ORDER RECURRENCE EQUATIONS

1.1 INTRODUCTION

In this chapter we define new generalization of the Fibonacci sequence in case of third order recurrence equations. We generate pair of integer sequences using third order recurrence equations:

$$\alpha_{n+3} = \beta_{n+2} + \beta_{n+1} + \beta_n \qquad n \geq 0$$

$$\beta_{n+3} = \alpha_{n+2} + \alpha_{n+1} + \alpha_n \qquad n \geq 0$$

This process of constructing two sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ is called 2-Fibonacci sequences [5,7].

1.2 NEW GENERALIZATION

The process of construction of the Fibonacci numbers is a sequential process [1,2,6]. Atanassov, K. [3,4] consider two infinite sequence $\{a_n\}$ and $\{b_n\}$ which have given initial values a_1, a_2 and b_1, b_2. Sequences $\{a_n\}$ and $\{b_n\}$ are generated for every natural number $n \geq 2$ by the coupled equations,

$$a_{n+2} = b_{n+1} + b_n$$

$$b_{n+2} = a_{n+1} + a_n$$

In this chapter we consider two infinite sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ which have given three initial values a, c, e and b, d, f (which are real numbers). Sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ are generated for every natural number $n \geq 3$ by the coupled equations.

$$\alpha_{n+3} = \beta_{n+2} + \beta_{n+1} + \beta_n \qquad n \geq 0$$

$$\beta_{n+3} = \alpha_{n+2} + \alpha_{n+1} + \alpha_n \qquad n \geq 0$$

If we set a = b, c = d, e = f then the sequence $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ will coincide with each other and with the sequence $\{F_i\}_{i=0}^{\infty}$, which is a generalized Fibonacci sequence.

where, $F_o(a, c, e) = a$, $F_1(a, c, e) = c$, $F_2(a, c, e) = e$,

$$F_{n+3}(a, c, e) = F_{n+2}(a, c, e) + F_{n+1}(a, c, e) + F_n(a, c, e)$$

There are eight different ways to construct sequences $\{\alpha_i\}$ and $\{\beta_i\}$:

First way : $\alpha_{n+3} = \alpha_{n+2} + \alpha_{n+1} + \alpha_n$

$\beta_{n+3} = \beta_{n+2} + \beta_{n+1} + \beta_n$

Second way : $\alpha_{n+3} = \alpha_{n+2} + \alpha_{n+1} + \beta_n$

$\beta_{n+3} = \beta_{n+2} + \beta_{n+1} + \alpha_n$

Third way : $\alpha_{n+3} = \alpha_{n+2} + \beta_{n+1} + \alpha_n$

$\beta_{n+3} = \beta_{n+2} + \alpha_{n+1} + \beta_n$

Fourth way : $\alpha_{n+3} = \alpha_{n+2} + \beta_{n+1} + \beta_n$

$\beta_{n+3} = \beta_{n+2} + \alpha_{n+1} + \alpha_n$

Fifth way : $\alpha_{n+3} = \beta_{n+2} + \alpha_{n+1} + \alpha_n$

$\beta_{n+3} = \alpha_{n+2} + \beta_{n+1} + \beta_n$

Sixth way : $\alpha_{n+3} = \beta_{n+2} + \alpha_{n+1} + \beta_n$

$\beta_{n+3} = \alpha_{n+2} + \beta_{n+1} + \alpha_n$

Seventh way: $\alpha_{n+3} = \beta_{n+2} + \beta_{n+1} + \alpha_n$

$\beta_{n+3} = \alpha_{n+2} + \alpha_{n+1} + \beta_n$

Eighth way : $\alpha_{n+3} = \beta_{n+2} + \beta_{n+1} + \beta_n$

$\beta_{n+3} = \alpha_{n+2} + \alpha_{n+1} + \alpha_n$

Graphically we can show the above generalization as under :

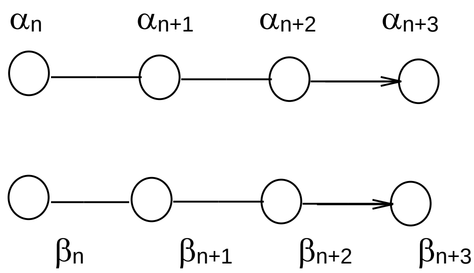

Figure 1

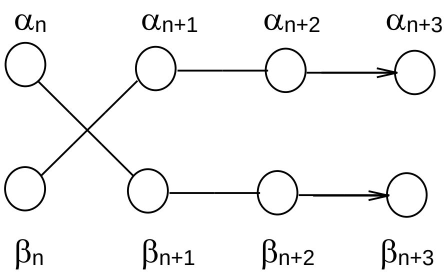

Figure 2

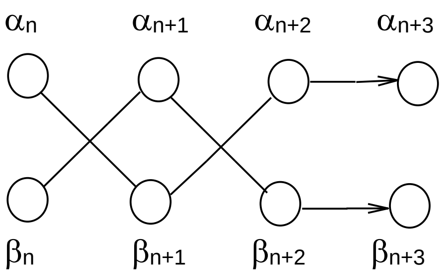

Figure 3

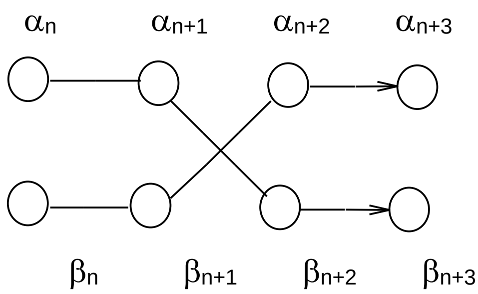

Figure 4

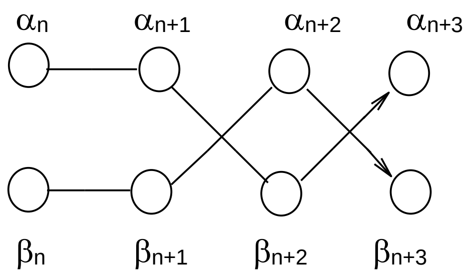

Figure 5

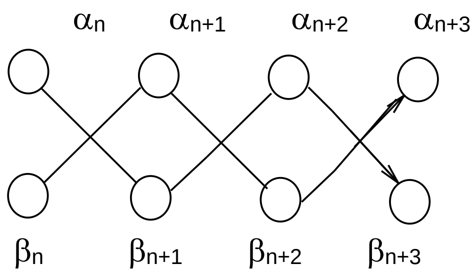

Figure 6

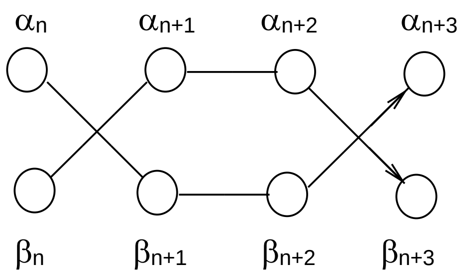

Figure 7

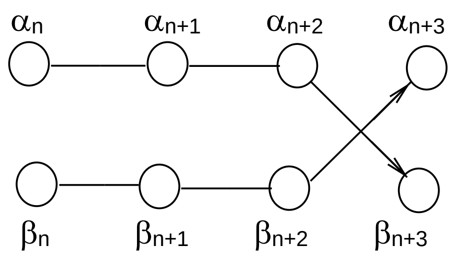

Figure 8

1.3 THE 2F-SEQUENCES

We are constructing two sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ by the following way –

$\alpha_0 = a,\ \alpha_1 = c,\ \alpha_2 = e;\qquad \beta_o = b,\ \beta_1 = d,\ \beta_2 = f$

$$\left.\begin{aligned} \alpha_{n+3} &= \beta_{n+2} + \beta_{n+1} + \beta_n \qquad n \geq 0 \\ \beta_{n+3} &= \alpha_{n+2} + \alpha_{n+1} + \alpha_n \qquad n \geq 0 \end{aligned}\right\} \qquad (1.3.1)$$

where, a, b, c, d, e, f are real numbers.

First we shall study the properties of the sequence $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ defined by equation (1.3.1). The first ten terms of the sequences defined in equation (13.1) are shown in table below :

n	α_n	β_n
0	a	b
1	c	d
2	e	f
3	b + d + f	a + c + e
4	a + c + e + f + d	b + c + d + e + f
5	a + b + 2c + d + 2f + 2e	a + b + c + 2d + 2e + 2f
6	2a + 2b + 3c + 3d + 4e + 3f	2a + 2b + 3c + 3d + 3e + 4f
7	3a + 4b + 5c + 6d + 6e + 7f	4a + 3b + 6c + 5d + 7e + 6f
8	7a + 6b + 10c + 10d + 11e + 12f	6a + 7b + 10c + 10d + 12e + 12f
9	12a + 12b + 18c + 18d + 22e + 22f	12a + 12b + 18c + 19d + 21e + 22f

Theorem – 1 : For every integer $n \geq 0$

(a) $\alpha_{4.n} + \beta_o = \beta_{4.n} + \alpha_o$

(b) $\alpha_{4.n+1} + \beta_1 = \beta_{4.n+1} + \alpha_1$

(c) $\alpha_{4.n+2} + \beta_2 = \beta_{4.n+2} + \alpha_2$

(d) $\alpha_{4.n+3} + \beta_3 = \beta_{4.n+3} + \alpha_3$

We prove the above results by induction hypothesis.

Proof of (a) : If n = 0 the result is true because –

$$\alpha_o + \beta_o = \beta_0 + \alpha_o$$

Assume that the result is true for some integer $n \geq 1$.

Now by equation (1.3.1) we can write –

$$\alpha_{4.n+4} + \beta_0 = \beta_{4.n+3} + \beta_{4.n+2} + \beta_{4.n+1} + \beta_o$$

$$= \alpha_{4.n+2} + \alpha_{4.n+1} + \alpha_{4.n} + \beta_{4.n+2} + \beta_{4.n+1} + \beta_o$$

$$= \alpha_{4.n+2} + \alpha_{4.n+1} + \beta_{4.n+2} + \beta_{4.n+1} + \alpha_{4.n} + \beta_o$$

$$= \alpha_{4.n+2} + \alpha_{4.n+1} + \beta_{4.n+2} + \beta_{4.n+1} + \beta_{4.n} + \alpha_o \text{ (by ind. hyp.)}$$

$$= \alpha_{4.n+2} + \alpha_{4.n+1} + \alpha_{4.n+3} + \alpha_o \text{ (By eq. 1.3.1)}$$

$$= \alpha_{4.n+3} + \alpha_{4.n+2} + \alpha_{4.n+1} + \alpha_o$$

$$= \beta_{n+4} + \alpha_o \text{ (By eq. 1.3.1)}$$

Hence the result is true for all integers $n \geq 0$.

(b) : If n = 0 the result is true because $\alpha_1 + \beta_1 = \beta_1 + \alpha_1$

Assume that the result is true for some integer $n \geq 1$.

Now by eqn. (1.3.1) we can write –

$$\alpha_{4.n+5} + \beta_1 = \beta_{4.n+4} + \beta_{4.n+3} + \beta_{4.n+2} + \beta_1$$

$= \alpha_{4.n+3} + \alpha_{4.n+2} + \alpha_{4.n+1} + \beta_{4.n+3} + \beta_{4.n+2} + \beta_1$ (By eq.1.3.1)

$= \alpha_{4.n+3} + \alpha_{4.n+2} + \beta_{4.n+3} + \beta_{4.n+2} + \alpha_{4.n+1} + \beta_1$

$= \alpha_{4.n+3} + \alpha_{4.n+2} + \beta_{4.n+3} + \beta_{4.n+2} + \beta_{4.n+1} + \alpha_1$ (By ind. hyp.)

$= \alpha_{4.n+3} + \alpha_{4.n+2} + \alpha_{4.n+4} + \alpha_1$ (By eq. 1.3.1)

$= \alpha_{4.n+4} + \alpha_{4.n+3} + \alpha_{4.n+2} + \alpha_1$

$= \beta_{4.n+5} + \alpha_1$ (By eq. 1.3.1)

Hence the result is true for all integer $n \geq 0$.

(c): If n = 0 the result is true because $\alpha_6 + \beta_2 = \beta_6 + \alpha_2$

Now from eqn.(2.3.1) we can write –

$\alpha_{4.n+6} + \beta_2 = \beta_{4.n+5} + \beta_{4.n+4} + \beta_{4.n+3} + \beta_2$

$= \alpha_{4.n+4} + \alpha_{4.n+3} + \alpha_{4.n+2} + \beta_{4.n+4} + \beta_{4.n+3} + \beta_2$ (By eq. 1.3.1)

$= \alpha_{4.n+4} + \alpha_{4.n+3} + \beta_{4.n+4} + \beta_{4.n+3} + \alpha_{4.n+2} + \beta_2$

$= \alpha_{4.n+4} + \alpha_{4.n+3} + \beta_{4.n+4} + \beta_{4.n+3} + \beta_{4.n+2} + \alpha_2$ (By ind. hyp.)

$= \alpha_{4.n+4} + \alpha_{4.n+3} + \alpha_{4.n+5} + \alpha_2$ (By eq. 1.3.1)

$= \alpha_{4.n+3} + \alpha_{4.n+4} + \alpha_{4.n+5} + \alpha_2$

$= \beta_{4.n+6} + \alpha_2$ (By eq. 1.3.1)

Hence the result is true for $n \geq 0$.

(d) : If n = 0 the result is true because $\alpha_7 + \beta_3 = \beta_7 + \alpha_3$

Now from eqn.(2.3.1) we can write

$\alpha_{4.n+7} + \beta_3 = \beta_{4.n+6} + \beta_{4.n+5} + \beta_{4.n+4} + \beta_3$

$= \alpha_{4.n+5} + \alpha_{4.n+4} + \alpha_{4.n+3} + \beta_{4.n+5} + \beta_{4.n+4} + \beta_3$ (By eq. 1.3.1)

$= \alpha_{4.n+5} + \alpha_{4.n+4} + \beta_{4.n+5} + \beta_{4.n+4} + \alpha_{4.n+3} + \beta_3$

$= \alpha_{4.n+5} + \alpha_{4.n+4} + \beta_{4.n+5} + \beta_{4.n+4} + \beta_{4.n+3} + \beta_3$ (By Ind. hyp.)

$= \alpha_{4.n+5} + \alpha_{4.n+4} + \alpha_{4.n+6} + \beta_3$ (By eq. 1.3.1)

$= \alpha_{4.n+6} + \alpha_{4.n+5} + \alpha_{4.n+4} + \beta_3$

$= \beta_{4.n+7} + \beta_3$ (By eq. 1.3.1)

Hence result is true for $n \geq 0$.

Some results for particular value of sequences $\{\alpha_i\}$ and $\{\beta_i\}$ defined in equation (1.3.1).

2.4 RESULTS

Result I:

(1) For K = 0, $\alpha_{4.K+3} = \sum_{i=0}^{4K+2} \beta_i + \beta_1 + \beta_2$

(2) For K = 1, $\alpha_{4.K+3} = \sum_{i=0}^{4K+2} \beta_i + \beta_1 + \beta_2 + \alpha_5$

Result II:

(1) For K = 0, $\sum_{i=0}^{4K} \alpha_i - \beta_i = \alpha_0 - \beta_0$

(2) For K = 1, $\sum_{i=0}^{4K} \alpha_i - \beta_i = \alpha_0 - \beta_0$

(3) For K = 2, $\sum_{i=0}^{4K} \alpha_i - \beta_i = \alpha_0 - \beta_0 - \alpha_2$

Result III: Relationship between sequence defined in (1.3.1) and Fibonacci numbers :

(1) $\alpha_{n+3} + \beta_{n+3} = F_{n+1} (\alpha_0 + \beta_0) + F_{n+2} (\alpha_1 + \beta_1) + F_{n+3} (\alpha_2 + \beta_2) - \alpha_0 - \beta_0$

above result is true for n = 0.

(2) $\alpha_{n+3} + \beta_{n+3} = F_{n+1} (\alpha_0 + \beta_0) + F_{n+2} (\alpha_1 + \beta_1) + F_{n+3} (\alpha_2 + \beta_2)$

above result is true for n = 1, n = 2.

(3) $\alpha_{n+3} + \beta_{n+3} = F_{n+1} (\alpha_0 + \beta_0) + F_{n+2} (\alpha_1 + \beta_1) + F_{n+3} (\alpha_2 + \beta_2) - \alpha_0 - \beta_0 + \alpha_1 + \alpha_1 - \alpha_2 - \beta_2$

above result is true for n = 3.

2.5 PARTICULAR CASES

(1) If we take $\alpha_0 = 1$, $\alpha_1 = 2$, $\alpha_2 = 3$ and $\beta_0 = 3$, $\beta_1 = 2$, $\beta_2 = 1$ then with the help of equation (1.3.1) we get the sequences $\{\alpha_i\}$ and $\{\beta_i\}$ in the following form :

Table - 2

n	α_n	β_n
0	1	3
1	2	2
2	3	1
3	6	6
4	9	11
5	18	18
6	35	33
7	62	62
8	113	115
9	210	210

By induction we can show the following results from the Table above:

(a) $\alpha_{4.n} + \beta_0 = \beta_{4.n} + \alpha_0$

(b) $\alpha_{4.n+1} + \beta_1 = \beta_{4.n+1} + \alpha_1$

(c) $\alpha_{4.n+2} + \beta_2 = \beta_{4.n+2} + \alpha_2$

(d) $\alpha_{4.n+3} + \beta_3 = \beta_{4.n+3} + \alpha_3$

(2) If we take $\alpha_0 = 1$, $\alpha_1 = 2$, $\alpha_2 = 3$

and $\beta_0 = 1$, $\beta_1 = 2$, $\beta_2 = 3$

Then with the help of equation (1.3.1) we get the sequences $\{\alpha_i\}$ and $\{\beta_i\}$ in the following form :

Table - 3

n	α_n	β_n
0	1	1
1	2	2
2	3	3
3	6	6
4	11	11
5	20	20
6	37	37
7	68	68
8	125	125
9	230	230

In Table 3 sequence $\{\alpha_i\}$ and $\{\beta_i\}$ coincide with each other because of the reason that we set initial values –

$\alpha_o = \beta_o$, $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$

In this case sequences $\{\alpha_i\}$ and $\{\beta_i\}$ also coincide with the sequence $\{F_i\}_{i=0}^{\infty}$, which is generalized Fibonacci sequence which is defined by the recurrence relation.

$F_{n+3} = F_{n+2} + F_{n+1} + F_n$

where, $F_o = \alpha_0$, $F_1 = \alpha_1$, $F_2 = \alpha_2$.

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