**A STUDY ON PARTITION SETS AND GRAPH ENERGIES OF SOME ZERO DIVISOR GRAPHS**

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**Abstract:** This work starts with an idea of finding graph theoretical properties of zero divisor graphs as in [2], characterization of zero divisor graphs is done, and some graph invariants such as chromatic number, and clique number have been tabulated and discussed. We mainly worked with zero divisor graphs of integer Z modulo n ring, in which n can be factored in a particular manner. In this work, we aim to find partition sets of zero divisor graphs and to find other graph invariants which can be associated with them. Also, we worked to find graph energies of such zero divisor graphs. This work converges to find the number of partitions for zero divisor graphs factored into p and q, where p and q are positive integers to predict possible general expressions for the number of partitions.

**Overview:**

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**Introduction:** The concept of zero divisors was first proposed by Beck in 1988 while studying the coloring of commutative rings. The graphs considered here are only simple graphs. Throughout this project, we will delve into various aspects of zero divisor graphs, including their basic definitions and properties. We will study important concepts of zero divisor graph of ring Zn, such as number of partitions of zero divisor graph, cardinality of partition sets, chromatic number, girth, with n = 2p, 3p, p2q,p3 where p and q are prime or composite natural numbers. Also, we try to find generalized expressions for them.

**Motivation:**

* In this project, we aim to discuss the graph-theoretical properties of zero-divisor graphs and to look for the scopes, where these properties can be related to a set of zero-divisors.
* The study of zero divisor graphs has applications in diverse fields such as **coding theory, cryptography, and network analysis.**
* These graphs offer a unique perspective on the algebraic properties of rings and provide a visual representation of the relationships between their zero divisors.
* By undertaking this project, we aim to develop an understanding of zero divisor graphs and the way they establish the relation between graph theory and ring theory.

**Definitions in Group theory:**

**1. Group:** A group (G, ∗) is a non-empty set, G, together with binary operation ∗, satisfying the following axioms:

 **Associative:**(a ∗ b) ∗ c = a ∗ (b ∗ c), ∀a, b, c ∈ G.

 I**dentity:** There exists an element e ∈ G such that a ∗ e = e ∗ a = a, ∀a ∈ G.

 **Inverse:** For each a ∈ G there is an element a −1 ∈ G such that a ∗ a −1 = a−1 ∗ a = e.

**2**. **Abelian Group:** A group (G, ∗) is said to be the commutative or abelian group, for each

 a, b ∈ G, a ∗ b = b ∗ a.

**3.** **Finite Group :** A group (G, ∗) is said to be a finite group if G contains a finite number of

 elements.

**Fundamental definitions in Ring theory:**

**Ring:** A ring is a non-empty set R with two binary compositions denoted by (+) and (.) which

 satisfy the following properties:

 R1 : a + b ∈ R, ∀a, b ∈ R

 R2 : a + b = b + a, ∀a, b ∈ R

 R3 : (a + b) + c = a + (b + c), ∀a, b, c ∈ R

 R4 : ∃ an element 0 ∈ R, such that a + 0 = a, ∀a ∈ R (0 is called additive identity or zero

 element in R)

R5 : For each a ∈ R, there exists an element b ∈ R such that a + b = 0 (b is called additive inverse

 or negative of a)

R6 : a.b ∈ R, ∀a, b ∈ R

R7 : (a.b).c = a.(b.c), ∀a, b, c ∈ R

R8 : a.(b + c) = a.b + a.c, ∀a, b, c ∈ R

R9 : (a + b).c = a.c + b.c, ∀a, b, c ∈ R We denote a ring as (R, +, .) (R, +) is an abelian group,

 and (R, .) is a semi group.

**Commutative Ring:** A ring (R, +, .) is called commutative ring if a.b = b.a, ∀a, b ∈ R

**Finite commutative Ring:** A commutative ring (R, +, .)) is called a finite commutative ring if it

 has a finite number of elements.

**Integral Domain:** It is a commutative ring in which the product of any two non-zero elements is

 non-zero.

 **Zero divisor:** A non-zero element ’a’ of a commutative ring R is called a zero divisor if there

 exists some non-zero element b in R such that a.b = 0.

**Fundamental definitions in Graph Theory:**

**Graph:** A graph G is defined by G = (V, E), consisting of a finite non-empty set V = V (G) of p vertices (or points) together with a prescribed set E of q unordered pair of distinct vertices of V .

 Each pair e = (u, v) of vertices E is called an edge (or line) and e is said to join u and v.

 We write e = uv and say that u and v are adjacent vertices and we say that u and v are

 Incident with an edge e.

**Loop:** An edge is drawn from a vertex to itself.

**Walk:** A walk in graph is an alternate sequence of vertices and edges that starts and ends at a

 vertex.

**Path:** A path is a walk in which, no vertices are repeated (hence no edge).

**Trail:** It is a walk in a path, in which no edge is repeated (vertex can be repeated).

 **Cycle:** A cycle in a graph is a non-empty trail, in which only the first and last vertices are the

 same.

**Types of Graph Null graph:** A graph with no edges. Trivial graph: A graph having only one

 vertex.

 **Non-trivial graph:** A graph with more than one vertex.

**Directed graph:** A graph, in which all the edges have some directions.

**Non-directed graph:** A graph, where the edges do not contain any direction.

**Connected graph:** A graph, in which there exists at least one path between every pair of

 vertices.

**Disconnected graph:** A graph, in which there does not exist any path between at least one pair

 of vertices.

**Simple graph:** A graph, that does not contain any loop and multi-edges (or parallel edges).

**Multi graph:** A graph that has no self-loops but has parallel edges.

**Labeled graph:** The graph, whose n-vertices are distinguished from one another by labels such

 As v1, v2, ..., vn.

**Complete graph:** A simple graph, in which every pair of vertex are adjacent.

**Cyclic graph:** A graph containing at least one cycle in it. Acyclic graph: A graph that does not

 contain any cycle( or is not closed).

**Finite graph:** A graph that contains a finite number of vertices and edges.

**Infinite graph:** A graph consists of an infinite number of vertices.

**Bipartite graph:** A graph G = (V, E) is bipartite if the set of vertices V can be partitioned into

 two disjoint sets V1 and V2 such that every edge of G joins one vertex of V1 to

 one vertex of V2.

**Complete bipartite graph:** A bipartite graph, where every vertex of V1 is joined to every vertex

 of V2.

**n-partite graph:** A graph G = (V, E) is bipartite if the set of vertices V can be partitioned into n

 disjoint sets V1, V2, V3, ..., Vn. 3-partite, 4-partite and 5-partite are also called

 tripartite, tetrapartite and pentapartite respectively.

**Subgraph:** Given a graph G = (V, E),its subgraph G′ = (V ′ , E′ ) of a given graph G with a set of

 vertices V and a set of edges E is a graph, whose vertex set V ′ and edge set E′ is

 such that V ′ ⊆ V and E′ ⊆ E.

**Star graph:** A graph with one center vertex and n pendent (vertices that have degree 1) vertices.

**Zerodivisor graph:** The zero divisor graph of a ring R, is a simple graph whose set of vertices

 consists of all zero divisors with an edge defined between two zero divisors a and

 b if and only if a.b = 0,a ̸= 0,b ̸= 0.

**Graph Invariants Length:** The number of edges contained in the graph.

**Eccentricity of graph:** The maximum distance from one vertex to any other vertex.

**Distance of graph:** The number of edges in the shortest path between vertices u and v.

**Diameter:** The maximum eccentricity over all the vertices and denoted as diam G.

**Girth:** The girth of an undirected graph is the length of the shortest cycle contained in the graph

 And denoted as Girth G.

**Radius of graph:** The minimum eccentricity over all the vertices.

**Incidency, adjacency and deletion of a vertex from a graph:**

**Incidency:** An edge is said to be incident to a vertex, if it ends at the vertex.

**Adjacency:** Two vertices are adjacent if they share a common edge or two edges are adjacent if

 they share a common vertex.

**Deletion of vertex from graph**: If v ∈ V for a graph G = (V, E), then graph G − v is a subgraph

 of G with vertex V − v and also all edges incident to v are removed.

**Domination and Strong Split domination sets:**

**Dominating set:** A subset D of the set of vertices V of a graph G is a dominating set of G if

 every vertex in (V − D) is adjacent to at least one vertex in D.

**Domination number:** The cardinality of a minimum dominating set of G and denoted as γ(G).

**Split dominating set:**  A dominating set D of a graph G is a split dominating set if the induced ]

 sub-graph (V − D) is disconnected.

 **Split domination number:** The split domination number of G is the minimum cardinality

 taken over all split dominating sets and is denoted as γs(G).

 **Strong split dominating set:** A dominating set D of graph G is a strong split dominating set if
 theinduced sub-graph (V − D) is totally disconnected with at least two vertices. **Strong split domination number:** The strong split dominating number of G is the minimum

 cardinality of a strong split dominating set and is denoted as γss(G).

**Coloring of graph :**

**Coloring of a graph:** Coloring of a graph refers to the assignment of colors to the vertices or

 edges of a graph.

**Proper coloring or vertex coloring:** A proper vertex coloring of a graph G = (V, E) is an

 assignment of colors to each vertex in V such that if (u, v) is an edge in E, then the colors

 assigned to u and v are different. In other words, no two adjacent vertices are assigned the

 same color in a proper coloring.

**Chromatic number of a graph:** The chromatic number of a graph is the minimum number of

 colors required to properly color its vertices.

**Matrix representation of a graph:**

**Adjacency matrix:** Adjacency matrix of a graph G with ′n ′ vertices is an n ∗ n matrix whose rows andcolumns both correspond to vertices and matrix M(G) = [mij ] is such that

 mij = ( 1, if i th vertex is adjacent to j th vertex. if 0 otherwise).

**Incidence matrix:** Incidence matrix of a graph G with vertex set V and edge set E and no self-l

 oops is given by matrix M(G) = [mij ] in which the rows correspond to the vertices

 and columns correspond to the edges is such that mij = ( 1, if j th edge is incident with i th

 vertex .if 0 otherwise) .

**Partition set, Order, Size and Energy of a graph**:

**Partition of a graph:** A partition of a graph refers to the division of the vertices of the graph

 into disjoint exhaustive subsets or groups such that each vertex belongs to exactly one

 subset and each edge connects vertices of different partitions.

**Order of a graph :** The cardinality of the vertex set of a graph G is called the order of G and is

 denoted by p.

**Size of a graph:** The cardinality of the edge set of graph G is called the size of G and is denoted

 by q.

**Energy of a graph:** The energy E(G) of a graph G is the sum of the absolute values of the eigen

 values of G.

**History** :

 The idea of a zero divisor graph was first given by Beck and he was mainly interested in

 labeling. Beck considered all ring elements to be in a vertex set of zero divisor graphs and

 joined every element to the ’zero’ by an edge along with connecting all the zero divisors to

 each other whose product is zero.

**Review of some standard results on zero divisor graphs:**

 In this section, we discuss some properties of Γ(Zn) discussed in [2].

**Theorem:** Γ(Zn) is finite only when Zn is itself finite.

**Theorem:** Γ(Zn) is empty graph if and only if Zn is an integral domain.

**Theorem:** If Γ(Zn) is a star graph then |Γ(Zn)| = p n for some prime p and integer n ≥ 0.

 Conversely, each star graph of order p n can be realized as Γ(Zn).

**Theorem:**[7] If p is a prime number, then Γ(Zp 2 ) is complete.

**Theorem:** The chromatic number, χ((Γ(Zn)) = Number of partitions.

**Lemma:**[5] For any ring R, there always exists at least one cycle of length 3 in Γ(R), whenever

 there are more than one nonzero zero-divisors. Theorem:[5] For the ring R, there exists

 a cycle of length 3 in Γ(R) if and only if |Z(R) ∗ | > 1.

**Theorem:**[6] Let Zn be a commutative ring. Then Γ(Zn) is connected and dia(Γ(Zn)) ≤ 3.

**Theorem:** If Γ(Zn) is a star graph then |Γ(Zn)| = p n for some prime p and integer n ≥ 0.

 Conversely, each star graph of order p n can be realized as Γ(Zn).

**Theorem:**[7] If p is a prime number, then Γ(Zp 2 ) is complete.

**Graph energies of Γ(Zn) for varied values of n**:

 **Theorem:**[4] For any graph Γ(Z2p) , where p is any prime number, then the Eigen values are √

 p − 1, − √ p − 1 and E(Γ(Z2p)) = 2√ p − 1.

**Theorem:**[4] For any graph Γ(Z3p), where p is any prime number, then the Eigen values are p

 2(p − 1), − p 2(p − 1) and E(Γ(Z3p)) = 2p 2(p − 1).

**Structures and partition sets of some Zero divisor graphs** :

 In this section, we try to find partition sets of zero divisor graphs for different Zn with n = 2p, 3p, p2 q and p 3 . Also, we observe different graph theoretic constants related to Γ(Zn). We construct the Γ(Zn) for different values of n and observe their partition number, and their diameter, girth, energy etc.

 Let (Zn, +, .) be a commutative ring and Γ(Zn) be a zero divisor graph of Zn with vertex set V and edge set E, where (n ∈ Z +) V (Γ(Zn)) = {0, 1, 2, ...,(n − 1)} E(Γ(Zn)) = {(a, b) : a, b ∈ V, a.b = 0, a ̸= 0, b ̸= 0} and let U1, U2, ..., Uk be the k partition sets of V (Γ(Zn)).

**A result on Graph energy of Γ(Zp 2q):**

**Energy of a graph:**The energy E(G) of a graph G is the sum of the absolute values of the eigenvalues of

 G.

**Result 5:**For any graph Γ(Zp 2q ), where p = 2, 3 and q is prime numbers, then the energy for the given

 graph is 2q − 1.5 ≤ E(Γ(Zp 2q )) ≤ 2q + 1.5.

 We can find the molecular energy of a molecule, which has the same structure as Γ(Zp 2q ) from the

 above result. Also, we can verify it practically by collaborating with the Chemists.

**Conclusion and future scope**:

 In this, we worked on the zero divisor graph of Zn for varied values of n, factored in a particular manner. For Zn, when n = p 2 q, we find that Γ(Zn) is p-partite, where 1 < p < 6 and q is any prime.

 In a similar way, we also observed that when n = 2p the Γ(Zn) is a star graph, and when n = 3p the Γ(Zn) is a complete bipartite graph.

 For Zn, where n = p 2 q if p = 2, 3, and 5, we noted that the number of partitions when q ≤ 17, q ≤ 11 and q ≤ 7 as discussed above. And also for the above p values, we get the associated zero divisor graphs as bipartite, tripartite, and pentapartite respectively.

 We need to find out that up to what values of p and q , Γ(Zn) is p-partite. We want to generalize this result for any value of n.

 Also, we can observe that Γ(Zn), for any values of n, the diameter is ≤ 3 and girth is ≤ 4, where

 2 ≤ p ≤ 5.

 Ultimately we observed that the energy of the zero divisor graph Γ(Zn), n = p 2 q, p = 2, 3, the upper bound and lower bound is 2q ± 1.5.

 The whole discussion in this report needs to be supported by proofs hence it can be further researched.

**References:**

**1.** Beck, ”Colouring of a commutative ring”, J. Algebra, 116(1988), 208-226.

 2. D. F. Anderson and P. F. Livingston, ”The zero-divisor graph of a commutative ring”, J. Algebra,

 217(1999), 437-447.

3. I. Gutman, The energy of a Graph,Ber.Math - Statist.Sekt.Forschungsz.Graz, 103,(1978),1-22. 4.https://www.ijert.org/research/on-domination-and-energy-of-zerodivisorgraphs-

 IJERTCONV3IS22025.pdf

5. https://www.researchgate.net/publication/337912142 On Beck’s zero-divisor graph. 6.https://www.researchgate.net/publication/226466409 7.https://trace.tennessee.edu/cgi/viewcontent.cgi?referer httpsredir=1 and article=3156 and context =

 utk gradthes

8.F. Harary, Graph Theory, Addison Wesley, 1969.

9.Contemporary Abstract Algebra by Joseph A. Gallian, tenth edition.