STUDY OF STRUCTURE AND OPERATORS ON ALMOST KAEHLERIAN MANIFOLDS

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Abstract:

Kodaira and Spencer (1957) have studied on the variation of almost complex structure. Hsiung (1966) has defined and studied structures and operators on almost Hermition manifolds. Also, Ogawa (1970) has studied operators on almost Hermition manifolds. In this paper, we have defined and studied structure and operators on almost Kaehlerian spaces and several theorems have been derived. We have also been demonstrated within nearly Kaehlerian spaces that for the structure to be integrable, it is both necessary and sufficient that the square of the difference between Γ and γ, i. e., $(\Gamma - \gamma)^2 = 0$. Additionally, when the operator Γ^2 vanishes across the entire space, then the space can be classified as Kaehlerian.

Keywords: Almost complex structure, almost Hermition spaces, almost Kaehlerian spaces, Kaehlerian spaces.

MSC: 53C55, 53B35.

1 Introduction:

Consider M^n as a Riemannian space, where its fundamental metric tensor is denoted as g_{ij}, and $g = det|g_{ij}|$. In this context, Greek indices $i, j, k,$ and so on, range from 1 to n, which is the dimension of the space. Let $\varepsilon_{i_1 \dots\dots i_p}^{j_1 \dots\dots j_p}$ represent the generalized Kronecker's delta, and $\varepsilon_{i_1 \dots\dots i_p}$ signify $\varepsilon_{i_1 \dots\dots i_p}^{1 \dots\dots p}$. We define F^p as the algebra of differential p-forms on M^n. Consequently, the operators of exterior differentiation $\boldsymbol{d}: \boldsymbol{F^p} \rightarrow \boldsymbol{F^{p+1}}$, and the adjoint operator $\boldsymbol{d'}: \boldsymbol{F^p} \rightarrow \boldsymbol{F^{n-p}}$ can be expressed for a p-form $u = \left(u_{i_1 \ldots\ldots i_p} \right)$ as follows:

(1.1) $$(du)_{i_0 \ldots\ldots i_p} = \frac{1}{p!} \varepsilon_{i_0 \ldots\ldots i_p}^{\rho j_1 \ldots\ldots j_p} \nabla_\rho u_{j_1 \ldots\ldots j_p}$$

(1.2) $$(d'u)_{i_1 \ldots\ldots i_{n-p}} = \frac{1}{p!} \sqrt{g}\, g^{\rho_1 j_1 \ldots\ldots} g^{\rho_p j_p} u_{\rho_1 \ldots\ldots \rho_p} \varepsilon_{j_1 \ldots\ldots j_p i_1 \ldots\ldots i_{n-p}}$$

where ∇_j Represents the covariant differentiation concerning the Riemannian connection, the exterior co-differentiation $\boldsymbol{\delta}: \boldsymbol{F^p} \rightarrow \boldsymbol{F^{p-1}}$ is specified by

(1.3) $$\delta = (-1)^{np+n+1} \ d'd\, d'$$

can be expressed locally as

(1.4) $$(\delta u)_{i_2 \ldots\ldots i_p} \nabla^\rho u_{\rho i_2 \ldots\ldots i_p}$$

Let Δ be the Laplace-Beltrami operator defined by

$$\Delta = d\delta + \delta d$$

Subsequently, utilizing equations (1.1) and (1.3), it is straightforward to confirm that in the case of a $p -$ degree form u,

(1.5) $$(\Delta u)_{i_1 \ldots\ldots i_p} = -\nabla^\rho \nabla_\rho u_{i_1 \ldots\ldots i_p} + + \sum_{i=1}^{p} R_{i_\lambda}^{\rho}\, u_{i_1 \ldots \overset{\lambda}{\widehat{a}} \ldots i_p}$$

$$+ \sum_{\lambda < \mu} R_{i_\lambda i_\mu}{}^{\rho a}\, u_{i_1 \ldots \overset{\lambda}{\widehat{\rho}} \ldots \overset{\mu}{\widehat{a}} \ldots i_p}$$

holds, where R_{ijkl} $(or\ R_{ij})$ represents the curvature (or Ricci) tensor linked to the Riemann connection. In the notation $u_{i_1 \ldots \overset{\lambda}{\widehat{a}} \ldots i_p}$, the index ρ replaces the index i_λ, while in $u_{i_1 \ldots \hat{a} \ldots\ldots i_p}$ indicates that the subscript i_a is deleted.

If a Riemannian space M^n admits an almost complex structure A_i^j satisfying

(1.6) $$g_{kh}\, A_i^k\, A_j^h = g_{ij}$$

then it is called an almost Hermitian space.

And If in an almost Kaehler space, the Nijenhuis tensor satisfies the condition

$$N_{jih} + N_{jhi} = 0,$$

then we deduce from it $G_{jih} = 0$, i.e.

$$F_{i,j}^h + F_{j,i}^h = 0$$

and the space is an almost Tachibana space. Thus, we have

$$3\, F_{ih,j} = F_{ji,h} = 0.$$

Consequently, the space is a Kaehler space i.e., an almost Kaehler space is a Kaehler space, if and only if the Nijenhuis tensor equation is satisfied.

Let $T^c(M)$ represent complexified tangent space of the manifold M^n. consider F_c^p as the space of complexified differential p-forms, which are essentially complex-valued functions defined on $T^c(M) \wedge \ldots . \wedge T^c(M)$. For non-negative integers r, s we introduce the projection mapping denoted by $\prod_{r,s} : F_c^p \rightarrow F_c^p$, where

p = r + s as follows. At first

(1.7) $$\prod_{1,0}{}_i^j = \left(\tfrac{1}{2}\right)\left(\delta_i^j - \sqrt{-1}\, A_i^j\right)$$

and its conjugate

(1.8) $$\prod_{0,1}{}_i^j = \overline{\prod_{1,0}}{}_i^j = \left(\tfrac{1}{2}\right)\left(\delta_i^j + \sqrt{-1}\, A_i^j\right)$$

which will be abbreviated to $\prod$ *and* $\overline{\prod}$ respectively. Then for a p-form **u** of F_c^p, we define

(1.9) $$\Big(\prod_{r,s} u\Big)_{i_2 \ldots\ldots i_p} = \left(\frac{1}{p!}\right) \prod_{r,s}{}_{i_2 \ldots\ldots i_p}^{j_1 \ldots\ldots j_p} u_{j_1 \ldots\ldots j_p}$$

$$= \left[\frac{1}{(r!s!)}\right] \varepsilon_{i_1 \ldots\ldots\ldots\ldots i_p}^{t_1 \ldots t_r h_1 \ldots h_s} \prod{}_{t_1}^{j_1} \ldots\ldots \prod{}_{t_r}^{j_r} \overline{\prod}{}_{h_1}^{k_1} \ldots\ldots \overline{\prod}{}_{h_s}^{k_s} u_{j_1 \ldots j_r k_1 \ldots k_s}.$$

A p-form u of F_c^p is called of type (r, s) if it satisfies ($\prod_{r,s} u$) = u.

Now, here following two Lemmas given by [Kodaira and Spencer (1957)], Ogawa (1970),

Lemma (1.1): In an almost complex space, for any set of functions $u_{i_1 \ldots\ldots i_p}$, we have

(1.10) $$\sum_{v=0}^{p} \Big(\prod_{(p-v,\, v)} u\Big)_{i_1 \ldots\ldots i_p} == u_{i_1 \ldots\ldots i_p}$$

and

(1.11) $$\sum_{v=0}^{p} {}_p C_v\, \varepsilon_{j_1 \ldots\ldots j_p}^{\rho_1 \ldots\ldots \rho_p} \prod{}_{\rho_1}^{j_1} \ldots\ldots \prod{}_{\rho_v}^{j_v} \overline{\prod}{}_{\rho_{v+1}}^{j_{v+1}} \ldots\ldots \overline{\prod}{}_{\rho_p}^{j_p} u_{j_1 \ldots\ldots j_p}$$

$$= \varepsilon_{i_1 \ldots\ldots i_p}^{j_1 \ldots\ldots j_p} u_{j_1 \ldots\ldots j_p}$$

holds for any p-form $u_{j_1 \ldots\ldots j_p}$, $1 \le p \le n$.

Now we define the operators $d_1 : F_c^p \to F_c^{p+1}$ of type (1, 0) and $d_2 : F_c^p \to F_c^{p+1}$ of type $(2, -1)$ in accordance with [Kodaira and Spencer (1957)] given by

(1.12) $$d_1 = \sum_{r+s=p} \prod_{r+1,s} d \prod_{r,s},$$

(1.13) $$d_2 = \sum_{r+s=p} \prod_{r+2,s-1} d \prod_{r,s}.$$

Here we denote the conjugate operator of d_1 $(or\ d_2)$ by $\bar{d}_1$ $(or\ \bar{d}_2)$.

Lemma (1.2): In an almost complex space, on F_c^p , we have

(1.14) $$\prod_{r+3,s-2} d \prod_{r,s} = 0,$$

where, r + s = p.

From Lemmas (1.1) and (1.2), we have [2] [Kodaira and Spencer (1957)] given by

(1.15) $$d = d_1 + d_2 + \bar{d}_1 + \bar{d}_2$$

The definitions of complex counterparts of the real operators $d\ and\ \delta$, as per the framework established by Kodaira-Spencer in their (1957) work [2], can be stated as follows:

(1.16) $\partial = 2d_2 + d_1 - \bar{d}_2$

(1.17) $\mathfrak{D} = -*\partial*$

On the other hand, Hsiung (1966) defined them by the following operators

(1.18) $$(\boldsymbol{\partial} u)_{i_0 \ldots\ldots i_p} = \left(\frac{1}{p!}\right) \sum_{r+s=p} \prod^{tj_1\ldots\ldots j_p}_{i_0\ldots\ldots i_p\ r+1,s} \prod^{h}_{t} \nabla_h u_{j_1\ldots\ldots j_p} ,$$

(1.19) $$(\boldsymbol{\mathfrak{D}} u)_{i_0 \ldots\ldots i_p} = -\sum_{r+s=p} \prod^{j_1\ldots\ldots j_p}_{ti_2\ldots\ldots i_p\ r,s} \prod^{t}_{h} \nabla^h u_{j_1\ldots\ldots j_p} ,$$

for a p-form $u = (u_{i_1\ldots\ldots i_p})$. After then we shall show that the relation

(1.20) $\boldsymbol{\mathfrak{D}} = -*\boldsymbol{\partial}*$

is valid.

2. Operators on Almost Kaehlerian Manifolds:

We have studied the following properties of the operators

Lemma (2.1): In an almost Kaehlerian space, the operator Γ is a skew-derivation and satisfies

(2.1) $*\Gamma* = -D$

Proof: Ogawa (1967) gives that Γ is a skew-derivation and that for any *p*-form *u* $= u_{i_1\ldots\ldots i_p}$,

$(*\Gamma*u)_{i_2\ldots\ldots i_p} = (-1)^{np+n+1} (Du)_{i_2\ldots\ldots i_p}$

holds, where ***n*** is the dimension of the space. Since ***n*** is even, therefore (2.1) is proof.

Lemma (2.2): In an almost Kaehlerian space, the operator $\emptyset$ is a derivation and satisfies for any *p*-form u_p,

(2.2) $*\emptyset*u_p = (-1)^p \emptyset\, u_p$,

(2.3) $d\emptyset - \emptyset d = -\Gamma + \Upsilon$

Proof: From directive calculation with respect to an orthonormal local coordinate system for any *p*-form $u = u_{i_1\ldots\ldots i_p}$, we have

$$(*\emptyset*u)_{i_1\ldots\ldots i_p} = \left(\frac{1}{(n-p)!\ p!}\right) g\, g^{j_1 j_1 \ldots\ldots} g^{j_{n-p} j_{n-p}} g^{k_1 r_1 \ldots\ldots} g^{k_p r_p} u_{k_1\ldots\ldots k_p}$$

$$= (-1)^{p(n-p)} (\emptyset\, u)_{k_1\ldots\ldots k_p}.$$

Since n is even, we have $(-1)^{p(n-p)} = (-1)^p$, and thus (2.2) is proved.

Now, we have

$$(d\emptyset u)_{i_0 \ldots\ldots i_p} = \nabla_{i_0} A^t_{i_r} u_{i_1 \ldots\ldots \overset{r}{\overset{\frown}{t}} \ldots i_p} - \nabla_{i_r} A^t_{i_0} u_{i_1 \ldots\ldots \overset{r}{\overset{\frown}{t}} \ldots i_p}$$

$$- \textstyle\sum_{r \neq s} \nabla_{i_r} A^t_{i_s} \; u_{i_1 \ldots \overset{r}{\overset{\frown}{i_0}} \ldots \overset{r}{\overset{\frown}{t}} \ldots i_p} + A^t_{i_r} \nabla_{i_0} u_{i_1 \ldots\ldots \overset{r}{\overset{\frown}{t}} \ldots i_p}$$

$$- A^t_{i_0} \nabla_{i_r} u_{i_1 \ldots\ldots \overset{r}{\overset{\frown}{t}} \ldots i_p} - \textstyle\sum_{r \neq s} A^t_{i_s} \; \nabla_{i_r} u_{i_1 \ldots \overset{r}{\overset{\frown}{i_0}} \ldots \overset{r}{\overset{\frown}{t}} \ldots i_p}$$

$$(\emptyset du)_{i_0 \ldots\ldots i_p} = A^t_{i_0} \; \nabla_t \; u_{i_1 \ldots\ldots i_p} - A^t_{i_s} \nabla_t \; u_{i_1 \ldots\ldots \overset{s}{\overset{\frown}{i_0}} \ldots\ldots i_p}$$

$$+ A^t_{i_r} \nabla_{i_0} u_{i_1 \ldots\ldots \overset{r}{\overset{\frown}{t}} \ldots\ldots i_p} - A^t_{i_0} \nabla_{i_r} u_{i_1 \ldots\ldots\ldots \overset{r}{\overset{\frown}{t}} \ldots\ldots i_p}$$

$$- \textstyle\sum_{r \neq s} \; A^t_{i_s} \; \nabla_{i_r} \; u_{i_0 \ldots\ldots \overset{s}{\overset{\frown}{t}} \ldots \overset{r}{\overset{\frown}{i_0}} \ldots i_p} .$$

Hence it follows that

$$(d\emptyset u - \emptyset du)_{i_0 \ldots\ldots\ldots i_p} = (\nabla_{i_0} A^t_{i_r} - \nabla_{i_r} A^t_{i_0}) u_{i_1 \ldots\ldots \overset{r}{\overset{\frown}{t}} \ldots i_p}$$

$$- \textstyle\sum_n (-1)^n \; A^t_{i_n} \nabla_t \, u_{i_0 \ldots\ldots \hat{n} \ldots i_p}$$

$$+ \sum_{r<s} (-1)^r (\nabla_{i_r} \; A^t_{i_s} - \nabla_{i_s} A^t_{i_r}) \, u_{i_0 i_1 \ldots \hat{r} \ldots \overset{s}{\overset{\frown}{t}} \ldots\ldots i_p} .$$

$$= \textstyle\sum_{n<m} \; (-1)^n (\nabla_{i_n} \; A^t_{i_m} - \nabla_{i_m} A^t_{i_n}) \, u_{i_0 \ldots \hat{n} \ldots \overset{m}{\overset{\frown}{t}} \ldots\ldots i_p} .$$

$$- \textstyle\sum_n \; (-1)^n \; A^t_{i_n} \nabla_t \; u_{i_0 \ldots\ldots \hat{n} \ldots\ldots i_p}.$$

$$= (\Upsilon \, u - \Gamma \, u)_{i_0 \ldots\ldots\ldots i_p}.$$

Now, we have consider the following relation

$$\textstyle\sum_{s=1}^{p} \; \varepsilon^{j_1 \ldots\ldots \overset{s}{\overset{\frown}{t}} \ldots j_p j_{p+1} \ldots\ldots j_{p+q}}_{i_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots i_{p+q}} \; A^{j_s}_t + \textstyle\sum_{s'=p+1}^{p+q} \; \varepsilon^{j_1 \ldots\ldots j_p j_{p+1} \ldots\ldots \overset{s'}{\overset{\frown}{t}} \ldots j_{p+q}}_{i_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots i_{p+q}} \; A^{j_s}_t$$

$$= \textstyle\sum_{n=1}^{p+q} \; A^t_{i_n} \; \varepsilon^{j_1 \ldots\ldots\ldots\ldots\ldots\ldots j_{p+q}}_{i_1 \ldots\ldots\ldots \overset{n}{\overset{\frown}{t}} \ldots\ldots i_{p+q}} ,$$

Then, we have

$$(\emptyset u \wedge v)_{i_1 \ldots\ldots\ldots i_{p+q}} + (u \wedge \emptyset v)_{i_1 \ldots\ldots\ldots\ldots i_{p+q}}$$

$$= \left(\frac{1}{(p!q!)}\right) [\textstyle\sum_{r=1}^{p} \varepsilon^{j_1 \ldots\ldots\ldots j_{p+q}}_{i_1 \ldots\ldots\ldots i_{p+q}} \; A^t_{j_r} \, u_{j_1 \ldots\ldots \overset{s}{\overset{\frown}{t}} \ldots\ldots j_p} \; v_{j_{p+1} \ldots\ldots j_{p+q}}$$

$$+ \textstyle\sum_{s'=p+1}^{p+q} \; \varepsilon^{j_1 \ldots\ldots j_{p+q}}_{i_1 \ldots\ldots\ldots i_{p+q}} \; A^t_{j_{s'}} \, u_{j_1 \ldots\ldots j_p} \; v_{j_{p+1} \ldots\ldots \overset{s'}{\overset{\frown}{t}} \ldots j_{p+q}} \;]$$

$$= \left(\frac{1}{(p!q!)}\right) \textstyle\sum_{n=1}^{p+q} \; A^t_{i_n} \; \varepsilon^{j_1 \ldots\ldots j_{p+q}}_{i_1 \ldots\ldots \overset{n}{\overset{\frown}{t}} \ldots\ldots i_{p+q}} \; u_{j_1 \ldots\ldots j_p} \; v_{j_{p+1} \ldots\ldots\ldots j_{p+q}}$$

$= \emptyset\ (u \wedge v)_{i_1 \dots\dots\dots\dots i_{p+q}}$

Thus, the operator $\emptyset$ is a derivation. From this, we have the following:

Corollary (2.1): In almost Kaehlerian space, the operator Υ is a skew-derivation.
Corollary (2.2): In almost Kaehlerian space, the relation

(2.4) $$d\,\Gamma + \Gamma d = d\,\Upsilon + \Upsilon d$$

holds.
Theorem (2.3): In almost Kaehlerian spacer, we have

(2.5) $$*\,\Upsilon\,* = -\,\boldsymbol{\vartheta} - i(\delta A)$$

where $i(\delta A)$ denotes the inner product with respect to a 1-form $\delta A (A = A_{ij})$
Proof: We have the definition of Υ, for a p-form u,

$$(\Upsilon u)_{i_0 \dots\dots i_p} = \sum_{n<m} (-1)^n\ T^t_{i_n i_m}\ u_{i_0 \dots\dots \hat{n} \dots\dots \overset{m}{\hat{t}} \dots\dots i_p},$$

Where, we write $T^t_{ij} = \nabla_i A^t_j - \nabla_j A^t_i$. Therefore we have

$$(*\,\Upsilon * u)_{i_2 \dots\dots i_p} = \frac{g}{(a-p+1)!p!} \sum_{1 \le r < s \le a-p+1} (-1)^{r-1}\ T^h_{j_r j_s}$$
$$.\,g^{t_1 j_1} \dots\dots\dots g^{t_{a-p+1} j_{a-p+1}} g^{h_1 k_1} \dots\dots\dots\dots\dots g^{h_p k_p}$$
$$.\,u_{k_1 \dots\dots k_p}\ \varepsilon_{h_1 \dots\dots h_p j_1 \dots \hat{r} \dots\dots \overset{s}{\hat{h}} \dots\dots j_{a-p+1}}\ \varepsilon_{t_1 \dots\dots t_{a-p+1} i_2 \dots\dots i_p}$$

$$= \frac{(-1)^{r(p-1)(a-p+1)}}{(a-p+1)(a-p)p!} \sum_{r<s} T^{t_r t_s}_{\tau}\ \varepsilon^{k_1 \dots\dots\dots k_p \tau}_{j_2 \dots\dots\dots j_p t_r t_s}\ u_{k_1 \dots\dots\dots k_p}$$

$$= -\,\nabla^l A^t_l\ u_{t\, i_2 \dots\dots i_p} - \sum_{n=2}^{p} (-1)^n\ \nabla^t\ A^h_{i_n}\ u_{th\, i_2 \dots\dots \hat{n} \dots i_p}$$
$$= [\ \mathrm{i}\ (\delta A) u - \boldsymbol{\vartheta} u\ ]_{i_2 \dots\dots\dots i_p}.$$

Similarly, we have proof of the following:

Theorem (2.4): In an almost Kaehlerian space, we have

(2.6) $$\boldsymbol{\partial} = (d - \sqrt{-1}\ \Gamma)/2\,,$$
(2.7) $$\mathfrak{D} = (\delta - \sqrt{-1}\ \mathrm{D})/2\,,$$
(2.8) $$\bar{\boldsymbol{\partial}} = (d + \sqrt{-1}\ \Gamma)/2\,,$$
(2.9) $$\bar{\mathfrak{D}} = (\delta + \sqrt{-1}\ \mathrm{D})/2\,,$$
(2.10) $$\boldsymbol{\partial} = [d - \sqrt{-1}\ (\Gamma - \Upsilon)]\,/\,2\,,$$
(2.11) $$\mathfrak{D} = [\delta - \sqrt{-1}\ \{\mathrm{D} - \boldsymbol{\vartheta} - \mathrm{i}\,(\delta A)\}\,]\,/\,2\,,$$

3. Structure on almost kaehlerian spaces:

Theorem (3.1): In an almost Kählerian space, the structure's integrability is both a necessary and sufficient condition when:

$$(\Gamma - \Upsilon)^2 = 0.$$

Proof. We have the intergability condition of the almost complex structure is defined by $\partial^2 = 0,$ given by [2] Kodaira and spencer (1957), Then by equation (2.10)

$$\partial^2 = \frac{1}{4}\,[-(\Gamma - \Upsilon)^2 + \sqrt{(-1)}\,(d\Gamma + \Gamma d - d\Upsilon - \Upsilon d\,)],$$

Considering that the imaginary components disappear due to the implication of *Corollary* (2.2), we derive the result:

$$\partial^2 = -\frac{1}{4}\,(\Gamma - \Upsilon)^2$$

Which is real operator.

The operator Γ which delineates a Kählerian structure through an almost Hermitian structure, demonstrates Kählerian characteristics only when the operator Γ^2 ceases to have an effect. As Γ functions as a skew- derivation, its second operation, Γ^2, acts as a derivation. Consequently, when Γ^2nullifies its impact on forms of degrees 0 and 1, its influence dissipates across forms of all degrees. Taking into consideration a 0-form f and a 1-form $u = (u_i),$ the following relationship holds:

$$(\Gamma^2 f\,)_{ij} = \left(A_i^t \nabla_t A_j^h - A_j^t \nabla_t A_i^h \right) \nabla_h f\ ,$$

$$(\Gamma^2 u\,)_{ijk} = \bigcup_{i,j,k} \left(A_i^t \nabla_t A_j^h - A_j^t \nabla_t A_i^h \right) \nabla_h u_k + \bigcup_{i,j,k} \left(A_i^t A_j^h R_{thk}^l \right) u_l$$

Where $\bigcup_{i,j,k}$ indicates that the terms are summed cyclically with respect to I, i, j, k. Consequently, the condition $\Gamma^2 = 0$ can be expressed equivalently through the following relationships:

(3.1) $$\left(A_i^t \nabla_t A_j^h - A_j^t \nabla_t A_i^h \right) = 0,$$

(3.2) $$\bigcup_{i,j,k} \left(A_i^t A_j^h R_{thk}^l \right) = 0.$$

Theorem (3.2): In an almost Kaehlerian space, the operator Γ^2 consistently equals zero.

Proof: Since the complex structure A_i^j is a covariant constant in an Kaehlerian space, we have from (3.1)

$$A_i^t R_{tjk}^\omega = A_j^t R_{tik}^\omega\ ,$$

and therefore $$A_i^t A_j^h R_{thk}^\omega = R_{ijk}^\omega,$$

which gives (3.2) holds.

Theorem (3.3): In an almost Kaehlerian space, when $\Gamma^2 = 0,$ if signifies that the structure is almost semi-Kaehlerian.

Proof: We have, Transvecting (3.1) with A_l^i , then

$$\nabla_l A_j^h + A_l^i A_j^t \nabla_t A_i^h = 0.$$

Contracting ***l* and h** and noting $A^{ih}\, \nabla_t\, A_{ih} = 0.$ prove the theorem.

Theorem (3.4): If $\Gamma^2 = 0$ in an almost Kaehlerian space, then we have

(3.3) $$\underset{i,j,k}{\cup} \left(A_i^t \ R_{jkt}^{\omega} \right) = 0.$$

(3.4) $$\frac{1}{2} A^{th} R_{thi}^{j} + A_i^t \ R_t^j = 0,$$

(3.5) $$A_i^t R_{tj} + A_j^t \ R_{ti}$$

Proof: Here, from equation (3.2), we get

(3.6) $$A_i^t A_j^h A_k^l R_{thl}^{\omega} = A_i^t R_{ktj}^{\omega} - A_j^t R_{kti}^{\omega}$$

Taking the sum of terms of (3.6) cyclically with respect to the indices **i, j, k**, we have

$$\underset{i,j,k}{\cup} A_i^t A_j^h A_k^l R_{thl}^{\omega} = \underset{i,j,k}{\cup} A_i^t R_{jkt}^{\omega} = 0.$$

gives (3.3). Contraction of **i and *ω*** in (3.3) yields

(3.7) $$A^{it} R_{itjk} + A_j^t R_{tk} - A_k^t R_{tj} = 0.$$

And, from equation (3.6) we get

$$- A_k^t A_i^h A_j^l R_{thl}^{\omega} = A_i^t R_{ktj}^{\omega} - A_k^t R_{itj}^{\omega} - A_j^t R_{kit}^{\omega},$$

Which can reduced to (3.4) by contracting with g^{ij}. Also, from (3.7) and (3.4), then we get the relation (3.5).

Theorem (3.5): If $\Gamma^2 = 0$ in an almost Kaehlerian space, then we have

(3.8) $$\nabla^i A^{jk} \nabla_j A_{i\omega} = 0.$$

Proof: We have, Differentiating (3.1) by ∇_i, then

$$A^{it} \nabla_i \nabla_t A_j^h = \nabla^i A_j^t \nabla_t A_i^h + A_j^t \left(R_{itl}^{i} A^{lh} + R_{itl}^{h} A^{il} \right).$$

From (3.4) and (3.5) and noting above equation, we get

$$\left(\frac{1}{2}\right) A^{it} \left(\nabla_i \nabla_t A_j^h - \nabla_t \nabla_i A_j^h \right)$$
$$= \left(-\frac{1}{2}\right) A^{it} R_{itj}^{l} A_l^h + \left(\frac{1}{2}\right) A^{it} R_{itl}^{h} A_j^l = -R_j^h + R_j^h = 0.$$

Here, the second and third terms on the right- hand side are reduced to $-R_j^h$ and R_j^h, respectively and thus we have (3.8).

Theorem (3.6): If the operator Γ^2 vanishes everywhere in an almost Kaehlerian space, it implies that the space is Kaehlerian.

Proof: Here, firstly we prove that

(3.9) $$A^{jk} \nabla^t \nabla_t A_{jk} = 0.$$

Then, by virtue of (3.1) we find

$$\nabla_i A_j^h = A_j^h A_l^t \nabla_t A_i^l$$

the above equation and (3.8) and (3.5) gives

$$\nabla^i \nabla_i A_j^h = A_j^h \nabla^i A_l^t \nabla_t A_i^l$$

Now contracting above equation with A_h^j and noting *theorem* (3.5), we obtain (3.9). From equation (3.9) follows immediately

$$\nabla^i A^{jk} \nabla_i A_{jk} = \left(\frac{1}{2}\right) \nabla^i \nabla_i \left(A^{jk} A_{jk}\right) - A^{jk} \nabla^i \nabla_i A_{jk} = 0.$$

Which means $\nabla_i A_{jk} = 0.$ proving the structure to be Kaehlerian.

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