Dynamics of a Prey and Two predators with Distributed Time delay

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ABSTRACT:
This chapter describes stability analysis of a Prey and Two predators Ecological model. Two predators are competing for same prey and they have alternative food resources other than prey. Distributed type delay is incorporated in the interaction of prey and second predator is taken for investigation. The system is described by a system of integro-differential equations and local stability is studied at its interior equilibrium points. The global stability is addressed by constructing a suitable Lyapunov function. The effect of Time delay on the dynamical behaviour of the system is studied using exponential delay kernels. The delay kernels with different strengths are identified in which prey population growth is significant is shown using Numerical simulation. The weight kernel dynamics is compared with the system when no delay arguments are present.
Keywords: Equilibrium points, Local stability, Global stability, Time delay.

I.INTRODUCTION

Mathematical methods are well known in the field of ecological classifications. Ecological stability drive intention of many mathematicians in to this field in recent era. The ecological interactions are broadly classified in to prey-predation, competition mutualism, ammensalism etc. Prey-predator models always draw the attention of many researchers The ecological models with mathematical treatment are initiated by Lokta [1] and volterra [2]. Later on, Kapur [3,4] discussed various models related to ecology. May [5], Freedman [6], Paul colinvaux [7] contributed a lot to this field. The modelling of ecological system is mainly by differential equations. Braun [8] and Simon's [9] explain the applications of differential equations in this area. Prey-predator interaction has immense importance. A lot research has been done on prey-predator models. Recently stochastic prey-predation interaction and prey-refuge and additional food by A. Das, G.P. Samanta [18,20]. Bapan Ghosh [19] studied the stability switching and hydra effect in a predator–prey population.

Three species models are also well versed in ecological systems Vidyanath e.t al [21] studied the dynamics of one -predator and two preys. Shiva Reddy et.al [22,23] the dynamics of the three species model with two predators and one prey as well as prey, predator and super predator models.
Much work is done in two species dynamics. Time delay are very common in ecological phenomenon. A time delay occurs in any ecological interaction. These delays cause a cascade effect in stability of the ecological system. A small delay can cause a big change in the system stability. Naturally the delay can be classified as discrete, continuous and distributed. The nature of the delay depends upon the past history the models can be well explained by using distributed type delays.

The distributed time lags are more appropriate to represent the ecological patterns where time delays are depending on past history. The stability aspects of distributed time lags are widely studied by Cushing, J.M [10], and Sreehari Rao [11], Gopala swamy. K [12]. Time delay in interactions in three species models with a prey, predator and competitor models are discussed by paparao [13-17]. In spite of this, we proposed a three species ecological model with a single prey with two predators with logistic growth type. The system dynamics is studied at interior equilibrium point. Numerical simulation is carried out for different delay kernel strengths in support of stability analysis.
The chapter divided in to five sections in which the delay dynamics of the model studied both locally and globally with suitable numerical simulation.

2. MATHEMATICAL MODEL:

The model consisting of a single prey (x) and two predators namely first predator(y), second predator (z). Here two predators are competing for the same food (x). A time delay is induced in the interaction of prey and second predator (Gestation period of the prey). Predators are of generalist type and can sustain in absence of prey population. The ecological system is considered with all three population are non-zero and the interaction coefficients are positive in nature. The model equations are formed using the following system of integro - differential equations.

$$\frac{dx}{dt} = a_1 \mathrm{x}(1 - \frac{\mathrm{x}}{\mathrm{L}_1}) - \alpha_{12}\mathrm{xy} - \alpha_{13}\mathrm{x}\int_{-\infty}^{t} k_1(t-u)\,\mathrm{z}(u)du$$
$$\frac{dy}{dt} = a_2 \mathrm{y}\left(1 - \frac{\mathrm{y}}{\mathrm{L}_2}\right) + \alpha_{21}\mathrm{xy} - \alpha_{23}\mathrm{yz} \qquad (2.1)$$
$$\frac{dz}{dt} = a_3 \mathrm{z}(1 - \frac{\mathrm{z}}{\mathrm{L}_3}) + \alpha_{31}\mathrm{z}\int_{-\infty}^{t} k_2(t-u)\,\mathrm{x}(u)du - \alpha_{32}\mathrm{yz}$$

Where the parameters in the above model is described as follows

2.1: NOMENCLATURE:

S.No	Parameter	Description
1	x, y & z	Prey, first predator and a second predator populations respectively
2	$a_1\ a_2, a_3$	Growths rates of prey, first predator and second predator respectively
3	α_{12}	Interaction coefficient of prey and first predator (negative value)
4	α_{21}	Interaction coefficient of first predator and prey (positive value)
5	α_{23}	Interaction coefficient of first predator and second predator (negative value)
6	α_{32}	Interaction coefficient of second predator and first predator (negative value)
7	α_{13}	Interaction coefficient of prey and second predator (negative value)
8	α_{31}	Interaction coefficient of second predator and prey (positive value)
9	$k_1(t-u)$ & $k_2(t-u)$	Delay kernels of prey and second predator influence at time t.

Put $\frac{a_1}{L_1} = \alpha_{11}, \frac{a_2}{L_2} = \alpha_{22}, \frac{a_3}{L_3} = \alpha_{33}$ and t-u = w, we get the following system of equations

$$\frac{dx}{dt} = a_1\mathrm{x} - \alpha_{11}\mathrm{x}^2 - \alpha_{12}\mathrm{xy} - \alpha_{13}\mathrm{x}\int_0^\infty k_1(w)\,\mathrm{z(t-w)}dw$$
$$\frac{dy}{dt} = a_2\mathrm{y} - \alpha_{22}\mathrm{y}^2 + \alpha_{21}\mathrm{xy} - \alpha_{23}\mathrm{yz} \qquad (2.2)$$
$$\frac{dz}{dt} = a_3\mathrm{z} - \alpha_{33}\mathrm{z}^2 + \alpha_{31}\mathrm{z}\int_0^\infty k_2(w)\,\mathrm{x}(t-w)dw - \alpha_{32}\mathrm{yz}$$

Choose the kernels k_1 and k_2 such that

$$\int_0^\infty k_1(\mathrm{w})dw = 1, \int_0^\infty k_2(\mathrm{w})dw = 1, \int_0^\infty wk_1(w)dw < \infty, \& \int_0^\infty wk_2(\mathrm{w})dw < \infty \qquad (2.3)$$

Assume the solutions for the above model (2.3) as

$$x_1 = A_1e^{\lambda t}, \quad x_2 = A_2e^{\lambda t}, \quad x_3 = A_3e^{\lambda t}$$

and substituting in (2.3) we get the following system of equations

$$\frac{dx}{dt} = a_1\mathrm{x} - \alpha_{11}\mathrm{x}^2 - \alpha_{12}\mathrm{xy} - \alpha_{13}\mathrm{xz}\,w_1(\lambda)$$
$$\frac{dy}{dt} = a_2\mathrm{y} - \alpha_{22}\mathrm{y}^2 + \alpha_{21}\mathrm{xy} - \alpha_{23}\mathrm{yz} \qquad (2.4)$$
$$\frac{dz}{dt} = a_3\mathrm{z} - \alpha_{33}\mathrm{z}^2 + \alpha_{31}\mathrm{xz}w_2(\lambda) - \alpha_{32}\mathrm{yz}$$

Where

$$w_1(\lambda) = \int_0^\infty k_1(w)e^{-\lambda z}dz = L\{k_1(w)\},\ i.e., Laplace\ Transform\ of\ k_1(w)$$
$$w_2(\lambda) = \int_0^\infty k_2(w)e^{-\lambda z}dz = L\{k_2(w)\}\ i.e., Laplace\ Transform\ of\ k_2(w).$$

III. EQUILIBRIUM STATES:

The possible equilibrium points are identified by equating $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0, \frac{dz}{dt} = 0$. we get eight equilibrium points out of which the interior equilibrium point given by

E: INTERIOR EQUILIBRIUM POINT

$$\bar{\boldsymbol{x}} = \frac{a_1(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + a_2(\alpha_{13}\alpha_{32}w_1(\lambda) - \alpha_{12}\alpha_{33}) + a_3(\alpha_{12}\alpha_{23} - \alpha_{13}\alpha_{22}w_1(\lambda))}{\alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{12}(\alpha_{21}\alpha_{33} - \alpha_{31}\alpha_{23}) + w_1(\lambda)\alpha_{13}(\alpha_{31}w_2(\lambda)\alpha_{22} - \alpha_{21}\alpha_{32})}$$
$$\bar{\boldsymbol{y}} = \frac{a_1\left(\alpha_{21}\alpha_{33} - \alpha_{31}\alpha_{23}w_2(\lambda)\right) + a_2\left(\alpha_{11}\alpha_{33} + \alpha_{13}\alpha_{31}w_1(\lambda)w_2(\lambda)\right) - a_3(\alpha_{11}\alpha_{23} + \alpha_{13}\alpha_{21}w_1(\lambda))}{\alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{12}(\alpha_{21}\alpha_{33} - \alpha_{31}\alpha_{23}) + w_1(\lambda)\alpha_{13}(\alpha_{31}w_2(\lambda)\alpha_{22} - \alpha_{21}\alpha_{32})}$$

$$\bar{z} = \frac{a_1(\alpha_{22}\alpha_{31}w_2(\lambda) - \alpha_{21}\alpha_{32}) - a_2(\alpha_{11}\alpha_{32} + \alpha_{12}\alpha_{31}w_2(\lambda)) + a_3(\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})}{\alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{12}(\alpha_{21}\alpha_{33} - \alpha_{31}\alpha_{23}) + w_1(\lambda)\alpha_{13}(\alpha_{31}w_2(\lambda)\alpha_{22} - \alpha_{21}\alpha_{32})}$$

. (3.1)

This equilibrium state exist only when, $\bar{x} > 0, \bar{y} \geq 0, \bar{z} > 0$ (3.2)

IV. LOCAL STABILITY ANALYSIS AT CO-EXISTING STATE:

Theorem 4.1 : The interior equilibrium point $E(\bar{x}, \bar{y}, \bar{z})$ is locally asymptotically stable

Proof: Let the variational matrix is given by

$$J = \begin{bmatrix} -\alpha_{11}\bar{x} & -\alpha_{12}\bar{x} & -\alpha_{13}\bar{x}k_1^*(\lambda) \\ \alpha_{21}\bar{y} & -\alpha_{22}\bar{y} & -\alpha_{23}\bar{y} \\ \alpha_{31}\bar{z}k_2^*(\lambda) & -\alpha_{32}\bar{z} & -\alpha_{33}\bar{z} \end{bmatrix} \quad (4.1)$$

With The characteristic equation $\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$ (4.2)

Where $b_1 = (\alpha_{11}\bar{x} + \alpha_{22}\bar{y} + \alpha_{33}\bar{z})$

$(b_2 = (\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})\bar{x}\bar{y} + (\alpha_{11}\alpha_{33} + \alpha_{13}\alpha_{31}k_1(\lambda)k_2(\lambda)\bar{x}\bar{z} + (\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32})\bar{y}\bar{z}$

$b_3 = \bar{x}\bar{y}\bar{z}\,(\alpha_{11}\alpha_{22}\alpha_{33} + \alpha_{12}\alpha_{21}\alpha_{33} + \alpha_{13}\alpha_{22}\alpha_{31}k_1(\lambda)k_2(\lambda) - \alpha_{11}\alpha_{23}\alpha_{32} - \alpha_{12}\alpha_{23}\alpha_{31}k_2(\lambda) - \alpha_{13}\alpha_{21}\alpha_{32}k_1(\lambda))$ (4.3)

By Routh-Hurwitz criteria, the system is stable if $b_1 > 0$, $(b_1b_2 - b_3) > 0$ and $b_3(b_1b_2 - b_3) > 0$

Clearly $b_1 = (\alpha_{11}\bar{x} + \alpha_{22}\bar{y} + \alpha_{33}\bar{z}) > 0$

By algebraic calculations

$$\begin{aligned}(b_1b_2 - b_3) = {} & (\alpha_{11}{}^2\alpha_{22} + \alpha_{11}\alpha_{12}\alpha_{21})\bar{x}^2\bar{y} + (\alpha_{11}{}^2\alpha_{33} + \alpha_{11}\alpha_{13}\alpha_{31}w_1(\lambda)w_2(\lambda))\bar{x}^2\bar{z} \\ & +(\alpha_{22}{}^2\alpha_{33} - \alpha_{22}\alpha_{23}\alpha_{32})\bar{y}^2\bar{z} + (\alpha_{22}{}^2\alpha_{11} + \alpha_{22}\alpha_{12}\alpha_{21})\bar{y}^2\bar{x} \\ & +(\alpha_{11}\alpha_{33}{}^2 + \alpha_{33}\alpha_{13}\alpha_{31}w_1(\lambda)w_2(\lambda))\bar{z}^2\bar{x} + (\alpha_{22}\alpha_{33}{}^2 - \alpha_{33}\alpha_{23}\alpha_{32})\bar{z}^2\bar{y} \\ & +\bar{x}\bar{y}\bar{z}(2\alpha_{11}\alpha_{22}\alpha_{33} + \alpha_{12}\alpha_{23}\alpha_{31}w_2(\lambda) + \alpha_{13}\alpha_{21}\alpha_{32}w_1(\lambda))\end{aligned}$$

$(b_1b_2 - b_3) > 0$ (Majority of the terms are positive) (4.4)

Also $b_3(b_1b_2 - b_3) > 0$

Hence the interior equilibrium point $E(\bar{x}, \bar{y}, \bar{z})$ is locally asymptotically stable

V. GLOBAL STABILITY:

Theorem 5.1: The interior equilibrium point $E(\bar{x}, \bar{y}, \bar{z})$ is globally asymptotically stable

Proof: Consider the Lyapunov function be

$$V(\bar{x}, \bar{y}, \bar{z}) = (x - \bar{x}) - x\log\frac{x}{\bar{x}} + (y - \bar{y}) - y\log\frac{y}{\bar{y}} + (z - \bar{z}) - z\log\frac{z}{\bar{z}} \quad (5.1)$$

The time derivate of 'V' along the solutions of equations (2.1) is

$$\begin{aligned}V^1(t) = {} & [x - \bar{x}]\left(a_1 - \alpha_{11}x - \alpha_{12}y - \alpha_{13}\int_0^\infty k_1(w)z(t-w)dw\right) \\ & +[y - \bar{y}](a_2 - \alpha_{22}y + \alpha_{21}x - \alpha_{23}z) \\ & +[z - \bar{z}]\left(a_3 - \alpha_{32}z + \alpha_{31}\int_0^\infty k_2(w)x(t-w)dw - \alpha_{32}y\right)\end{aligned} \quad (5.2)$$

by proper choice of $a_1, a_2 \,\&\, a_3$

$$a_1 = \alpha_{11}\bar{x} + \alpha_{12}\bar{y} + \alpha_{13}\int_0^\infty k_2(w)z(t-w)dw$$

$$a_2 = -\alpha_{21}\bar{x} + \alpha_{22}\bar{y} + \alpha_{23}\bar{z}$$

$$\&\; a_3 = -\alpha_{31}\int_0^\infty k_2(w)x(t-w)dw + \alpha_{33}\bar{z} + \alpha_{32}\bar{y}$$

Substitute the above in equation (5.2) we get

$$-\alpha_{11}(x - \bar{x})^2 - \alpha_{22}(y - \bar{y})^2 - \alpha_{33}(z - \bar{z})^2 - (\alpha_{32} + \alpha_{23})(y - \bar{y})(z - \bar{z}) + (\alpha_{21} - \alpha_{12})(y - \bar{y})(x - \bar{x})$$

(5.3)

Using the inequality $ab \leq \frac{a^2+b^2}{2}$

$$-\alpha_{11}(x-\bar{x})^2-\alpha_{22}(y-\bar{y})^2-\alpha_{33}(z-\bar{z})^2-\frac{(\alpha_{32}+\alpha_{23})}{2}[(y-\bar{y})^2+(z-\bar{z})^2]$$
$$+\frac{(\alpha_{21}-\alpha_{12})}{2}[(y-\bar{y})^2+(x-\bar{x})^2]$$
$$V'(t)\le-\mu[(x-\bar{x})^2+(y-\bar{y})^2+(z-\bar{z})^2]<0$$
Where $\mu=min\left(\alpha_{11}+\alpha_{22}+\alpha_{33}+\frac{1}{2}\alpha_{23}+\frac{1}{2}\alpha_{32}+\frac{1}{2}\alpha_{21}-\frac{1}{2}\alpha_{12}-\frac{1}{2}(\alpha_{31}+\alpha_{13})\right)$

$\frac{dV}{dt}<0$ Therefore, the system is globally stable at interior equilibrium $E(\bar{x},\bar{y},\bar{z})$

VI. NUMERICAL EXAMPLE:

GRAPHS DESCRIPTION:

Fig (A): Represents time series plot; Fig (B): The phase portrait

Example 6.1: Let a_1=2.5; a_2=1.5; a_3=2.5; α_{12}=0.05; α_{13}=0.05; α_{21}=0.05; α_{23}=0.05; α_{31}=0.05; α_{32}=0.05; x=15, y =15, z =15; L1=100; L2=100; L3=100. E (63, 89, 95)
The systems of equations (2.1) are simulated using MATLAB using ode45.The system of equations without delay is solved with the same package we get the following results illustrated by the graphs 6.1(A), 6.1(B) for the following parametric values:

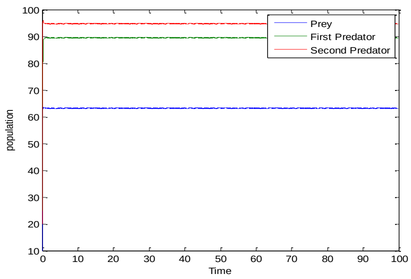


Fig:6. 1(A)

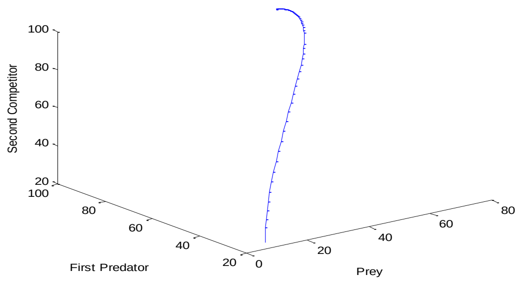


Fig: 6.1(B)

For the above-mentioned parametric values, the system is neutrally stable
Exponential function is given by $w_1(\lambda)=w_2(\lambda)=ae^{-aw}\ for\ a>0$
Then the Laplace transform of $w_1(\lambda)=w_2(\lambda)$ are defined as $w_1(\lambda)=w_2(\lambda)=\int_0^\infty e^{-\lambda t}a\,e^{-at}dt=\frac{a}{a+\lambda}$
The results are simulated for the above system of equations (2.3) Using MAT LAB simulation. With the parameters shown in Example 1 with different kernel values are plotted below.
λ=0. 001, a=0.5, E (80,99,80)

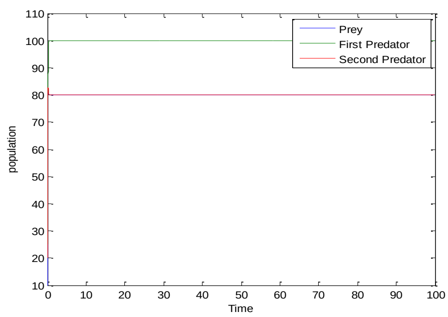


Fig: 6.2(A)

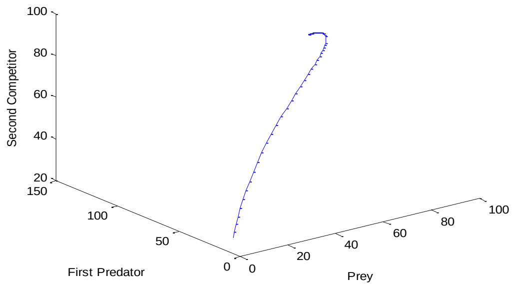
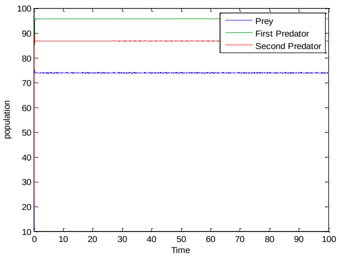


Fig: 6.2(B)

The system is neutrally stable and due to the delay parameter prey and first predator populations are increased from its initial growth and second predator population are decreases when compared with no delay arguments are induced.
λ=1.0, a= 1.5, E (74,96,87)



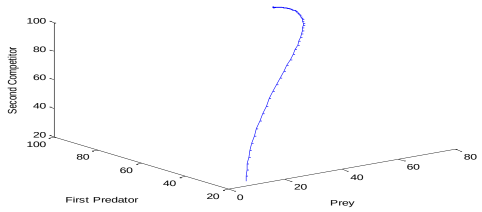
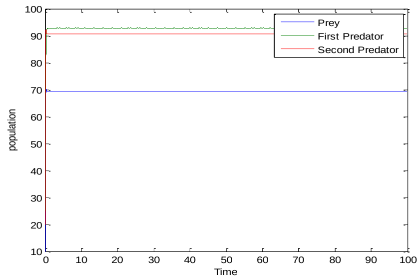


Fig: 6.3 (A) Fig :6.3(B)

The system is neutrally stable and due to the delay parameter prey and first predator populations are increased from its initial growth and second predator population are decreases when compared with no delay arguments are induced.

λ=1.0, a= 0.5, E (69,93,91)



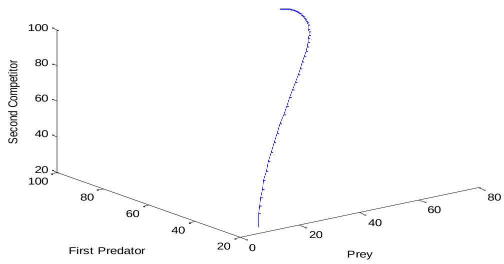


Fig: 6.4(A) Fig: 6.4(B)

The system is neutrally stable and due to the delay parameter prey and first predator populations are increased from its initial growth and second predator population are decreases when compared with no delay arguments are induced.

VII. CONCLUSION

We consider a model with a single prey and two predators for stability analysis with distributed type time delay. The co-existing state is identified and studied the local stability analysis at this point and shown that the system is asymptotically stable. The global stability is studied by Lyapunov's function. The dynamics of the system is studied using numerical simulation in support of stability analysis. We consider numerical example with delay and without delay agreements. The system is unstable if there is no delay impact. The impact of delay with different kernel strength is studied and observed that the systems are neutrally stable, for (i) λ=0. 001, a=0.5, (ii) λ=1.0, a= 1.5 (iii) λ=1.0, a= 0.5, there is a significant growth in the Prey and first predator populations and decay in second predator population. So, delay plays a significant role in system dynamics with suitable weight functions.

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