Dual Graph and Line Graph - Algebraic Property and Euler Characteristic Derived From External Edge Connectivity

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Abstract :

In this note we discuss adjacency matrix and incidence matrix of some rarely known Euler graphs: 1) Devil's Star graph, and 2) Mohmad scimitar graph which relate to historian age of middle east countries. As known by the properties of 'External Connectivity of each edge of a given graph predicting about Eulerity of corresponding their line graphs, we also depict some salient features. In addition to this, we have, in this note discussed Eulerity of dual graphs using the notion of external edge connectivity.

Keywords:
Mohmad scimitar Graph, Devil's Star Graph, Adjacency Matrix,Incidence Matrix, Line graph, Dual graph, External edge connectivity.

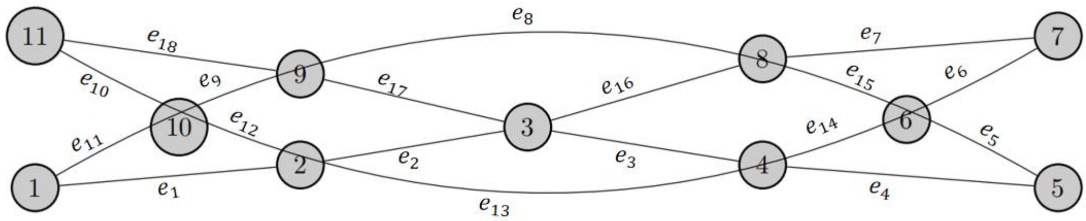
1) Introduction:

1.1) Euler Graph

I) A connected graph G is a Eulerian one if it has a Euler tour- a closed circuit that contains each edge traversed exactly once in one direction only.
This property is identified by checking evenness of degrees of each vertex.
II) Union of finite cycles:
A graph which has either one cycle of even degree vertices or a union of all cycles of even degree vertices is an Euler graph. Euler graph $G = C_1 \cup C_2 \cup \ldots\ldots. \cup C_n$ where each C_i is an even cycle.
i.e., each cycle has vertices of even degree.

1.2) Rarely known Euler graphs

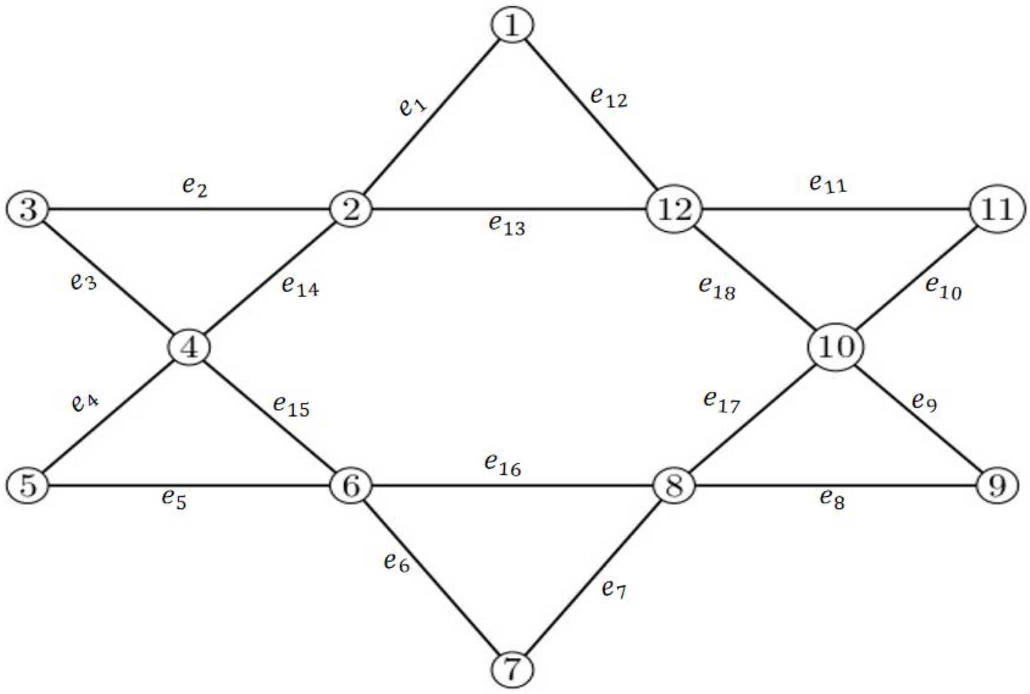
We have collection of some rare graphs which observe the Eulerian property.
I) Mohmad scimitar - the shape of the graph resembles to curved swords. The old-fashioned Arabian sword.



[Figure 1 : Mohmad scimitar G_1]

II) Devil's Star

It is also inherits all the characteristics of Euler graph.



[Figure 2: Devil's Star G_2]

It is on the national flag of the country Israel.it is a graph with two equilateral triangles, one with upside down.

1.3) Notion of External Connectivity of an Edge and related Euler graph :

Let $G = < V, E >$ be a given connected graph with n vertices and k number of edges; $(m, k \in N)$ External connectivity of an edge e_i of the set $E\ [\ i = 1, 2 ... k]$ Is the sum of number of edges incident on end vertices of the edge e_i.This is denoted as $E_c(e_i) = m_1 + m_2 = m$ where m_1 and m_2 are the number of edges from the end vertices of the said edge; $m_1 + m_2 \in N \cup \{0\}$

Theorem : For the given connected graph G [may not be a simple graph] if the external edge connectivity of every edge is even then its corresponding line graph is an Euler graph.

1.4) Line Graph:

Let $G = < V, E >$ be the given graph with vertex set $V = \{v_1, v_2, \ldots\ldots\ldots, v_n\}$ and the set of edges $E = \{e_1, e_2, \ldots\ldots, e_m\}$ — n vertices and m edges. Corresponding to each edge $e_j, j = 1$ to 4 there is a corresponding set $X = \{x_1, x_2, \ldots\ldots, x_n\}$ of vertices of the said line graph, denoted as L(G), such that if any two edges e_i and e_j of the graph G are incident on a vertex then in accordance to that there are vertices $x_i$and x_j which are adjacent to each other in the graph L(G).

i.e. (1) The number of edges of G equals number of vertices of the line graph $L(G)$.

(2) There is a strict one –one sequential correspondence between the set $E = \{e_1, e_2, \ldots\ldots, e_n\}$ of G and the set of vertices $X = \{x_1, x_2, \ldots\ldots, x_m\}$ of the graph $L(G)$; so that if any two edges e_i and e_j are incident on a vertex in G parallels that there is an edge connecting the vertices x_i and x_j of the line graph $L(G)$.

1.5) Notion of Dual Graph:

As we know that each graph has its reflex with respect to faces and edges. As known, some graphs are self duals and they stands for the basic concept that dual of the dual graph is the primal graph.

In our note the Eulerity of the dual graph is also discussed keeping the central idea of Eulerity and external edge connectivity.

2) Adjacency Matrix and Related Properties :

The Adjacency Matrix of a graph G with n vertices and no parallel edges is an n by n symmetric binary matrix $X = [x_{ij}]$ defined over the ring of integers such that

$$x_{ij} = \begin{cases} 1\ ;\ \text{if there is an edge between } i^{th} \text{ and } j^{th} \text{ vertices} \\ 0\ ;\ \text{if there is no edge between } i^{th} \text{ and } j^{th} \text{ vertices} \end{cases}$$

Some prime observations related to adjacency matrix (X) of a given graph are as follows :

i. The entries along the principal diagonal of X are all zero's if and only if if the graph has no self - loops. A self - loop at the i^{th} vertex corresponds to $x_{ij} = 1$.

ii. If the graph has no self - loops, the degree of a vertex equals the number of one's in the corresponding row or column of matrix X.

iii. If a graph is disconnected and consists two components G1 and G2. The adjacency matrix X(G) of graph G can be written in a block diagonal form as

$$X(G) = \begin{bmatrix} X(G_1) & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & X(G_2) \end{bmatrix}$$

2.1) Adjacency Matrix of G_1 :

	1	2	3	4	5	6	7	8	9	10	11	Row Total
1	0	1	0	0	0	0	0	0	0	1	0	2
2	1	0	1	1	0	0	0	0	0	1	0	4
3	0	1	0	1	0	0	0	1	1	0	0	4
4	0	1	1	0	1	1	0	0	0	0	0	4
5	0	0	0	1	0	1	0	0	0	0	0	2
6	0	0	0	1	1	0	1	1	0	0	0	4
7	0	0	0	0	0	1	0	1	0	0	0	2
8	0	0	1	0	0	1	1	0	1	0	0	4
9	0	0	1	0	0	0	0	1	0	1	1	4
10	1	1	0	0	0	0	0	0	1	0	1	4
11	0	0	0	0	0	0	0	0	1	1	0	2
Column Total	2	4	4	4	2	4	2	4	4	4	2	36

$$
= \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}
$$

2.2) Incidence Matrix of G_1 :

	1	2	3	4	5	6	7	8	9	10	11	Row Total
e_1	1	1	0	0	0	0	0	0	0	0	0	2
e_2	0	1	1	0	0	0	0	0	0	0	0	2
e_3	0	0	1	1	0	0	0	0	0	0	0	2
e_4	0	0	0	1	1	0	0	0	0	0	0	2
e_5	0	0	0	0	1	1	0	0	0	0	0	2
e_6	0	0	0	0	0	1	1	0	0	0	0	2
e_7	0	0	0	0	0	0	1	1	0	0	0	2
e_8	0	0	0	0	0	0	0	1	1	0	0	2
e_9	0	0	0	0	0	0	0	0	1	1	0	2
e_{10}	0	0	0	0	0	0	0	0	0	1	1	2
e_{11}	1	0	0	0	0	0	0	0	0	1	0	2
e_{12}	0	1	0	0	0	0	0	0	0	1	0	2
e_{13}	0	1	0	1	0	0	0	0	0	0	0	2
e_{14}	0	0	0	1	0	1	0	0	0	0	0	2
e_{15}	0	0	0	0	0	1	0	1	0	0	0	2
e_{16}	0	0	1	0	0	0	0	1	0	0	0	2
e_{17}	0	0	1	0	0	0	0	0	1	0	0	2
e_{18}	0	0	0	0	0	0	0	0	1	0	1	2
Column Total	2	4	4	4	2	4	2	4	4	4	2	36

$$
= \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}
$$

3) Incidence Matrix and Related Properties:

The Incidence Matrix of a graph G with n vertices and no self - loops is an n by e matrix $A = \left[a_{ij}\right]$, whose e rows, correspond to the e edges and the n columns correspond to the n vertices, such that

$$a_{ij} = \begin{cases} 1\,; & \text{if } j^{th} \text{ edge } e_j \text{ is incident on the } i^{th} \text{ vertex } v_i \\ 0\,; & Otherwise \end{cases}$$

It is also called vertex – edge incidence matrix and is denoted by A (G)

The incidence matrix contains only two types of elements 0 and I clearly, this is a binary matrix or a (0, I) matrix.

Some prime observations related to incidence matrix A(G) of a given graph are as follows :

i. Since every edges is incident on exactly two vertices, each row of matrix A has exactly two one's.
ii. The number of one's in each column equals the degree of the corresponding vertex.
iii. A row with all zeros represents an isolated vertex.
iv. Parallel edges in a graph produce identical column in its incidence matrix.
V. If a graph is disconnected and consists two components G1 and G2. The incidence matrix A (G) of graph G can be written in a block diagonal form as

$$A(G) = \begin{bmatrix} A(G_1) & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & A(G_2) \end{bmatrix}$$

3.1) Adjacency Matrix of G_2

	1	2	3	4	5	6	7	8	9	10	11	12	Row Total
1	0	1	0	0	0	0	0	0	0	0	0	1	2
2	1	0	1	1	0	0	0	0	0	0	0	1	4
3	0	1	0	1	0	0	0	0	0	0	0	0	2
4	0	1	1	0	1	1	0	0	0	0	0	0	4
5	0	0	0	1	0	1	0	0	0	0	0	0	2
6	0	0	0	1	1	0	1	1	0	0	0	0	4
7	0	0	0	0	0	1	0	1	0	0	0	0	2
8	0	0	0	0	0	1	1	0	1	1	0	0	4
9	0	0	0	0	0	0	0	1	0	1	0	0	2
10	0	0	0	0	0	0	0	1	1	0	1	1	4
11	0	0	0	0	0	0	0	0	0	1	0	1	2
12	1	1	0	0	0	0	0	0	0	1	1	0	4
Column Total	2	4	2	4	2	4	2	4	2	4	2	4	36

$$
= \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}
$$

3.2) Incidence Matrix of G_2

	1	2	3	4	5	6	7	8	9	10	11	12	Row Total
e_1	1	1	0	0	0	0	0	0	0	0	0	0	2
e_2	0	1	1	0	0	0	0	0	0	0	0	0	2
e_3	0	0	1	1	0	0	0	0	0	0	0	0	2
e_4	0	0	0	1	1	0	0	0	0	0	0	0	2
e_5	0	0	0	0	1	1	0	0	0	0	0	0	2
e_6	0	0	0	0	0	1	1	0	0	0	0	0	2
e_7	0	0	0	0	0	0	1	1	0	0	0	0	2
e_8	0	0	0	0	0	0	0	1	1	0	0	0	2

e_9	0	0	0	0	0	0	0	0	1	1	0	0	2
e_{10}	0	0	0	0	0	0	0	0	0	1	1	0	2
e_{11}	0	0	0	0	0	0	0	0	0	0	1	1	2
e_{12}	1	0	0	0	0	0	0	0	0	0	0	1	2
e_{13}	0	1	0	0	0	0	0	0	0	0	0	1	2
e_{14}	0	1	0	1	0	0	0	0	0	0	0	0	2
e_{15}	0	0	0	1	0	1	0	0	0	0	0	0	2
e_{16}	0	0	0	0	0	1	0	1	0	0	0	0	2
e_{17}	0	0	0	0	0	0	0	1	0	1	0	0	2
e_{18}	0	0	0	0	0	0	0	0	0	1	0	1	2
Column Total	2	4	2	4	2	4	2	4	2	4	2	4	36

$$= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

4) Eulerity of Line of of G_1 and G_2:

As we know that, For the given connected graph *G* [may not be a simple graph] if the external edge connectivity of every edge is even then its corresponding line graph is an Euler graph.

4.1) External Connectivity of an Edge of G_1 :

$E_c(e_1) = 3 + 1 = 4$, $E_c(e_2) = 4, E_c(e_3) = 4, E_c(e_4) = 4, E_c(e_5) = 4, E_c(e_6) = 4,$ $E_c(e_7) = 4, E_c(e_8) = 4, E_c(e_9) = 4, E_c(e_{10}) = 4, E_c(e_{11}) = 4, E_c(e_{12}) =$ $4, E_c(e_{13}) = 4, E_c(e_{14}) = 4, E_c(e_{15}) = 4, E_c(e_{16}) = 4, E_c(e_{17}) = 4, E_c(e_{18}) = 4$

Line graph of graph G_1 is an Euler graph.

4.2) External Connectivity of an Edge of G_2 :

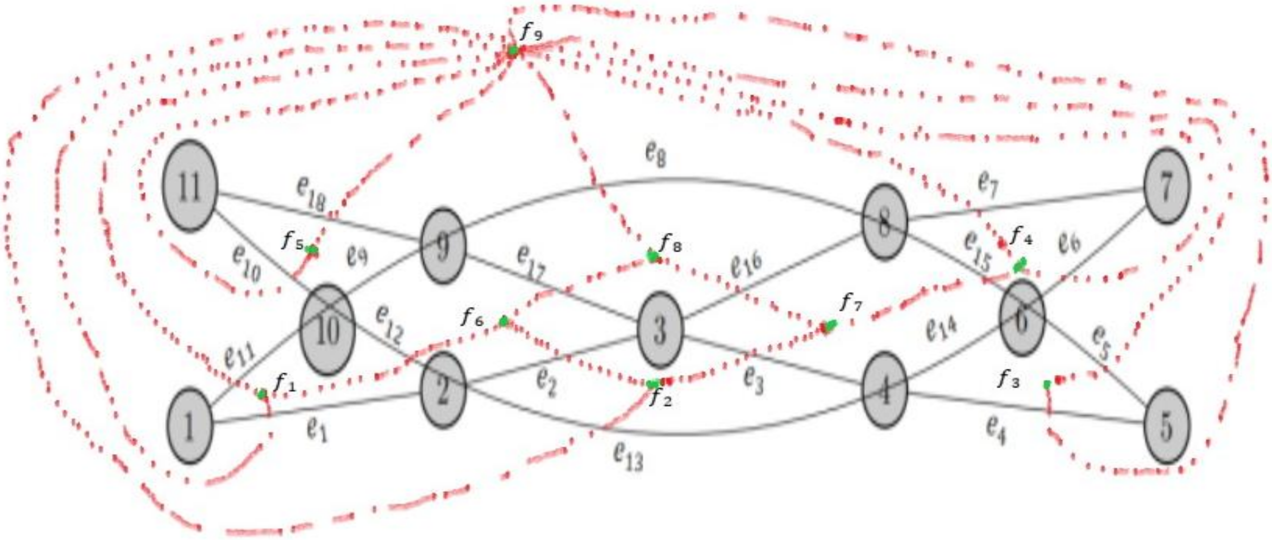
$E_c(e_1) = 3 + 1 = 4$, $E_c(e_2) = 4$, $E_c(e_3) = 4$, $E_c(e_4) = 4$, $E_c(e_5) = 4$, $E_c(e_6) = 4$, $E_c(e_7) = 4$, $E_c(e_8) = 4$, $E_c(e_9) = 4$, $E_c(e_{10}) = 4$, $E_c(e_{11}) = 4$, $E_c(e_{12}) = 4$, $E_c(e_{13}) = 4$, $E_c(e_{14}) = 4$, $E_c(e_{15}) = 4$, $E_c(e_{16}) = 4$, $E_c(e_{17}) = 4$, $E_c(e_{18}) = 4$

Line graph of graph G_2 is an Euler graph.

5) Dual Graph and its salient features for the graph G_1 and G_2:

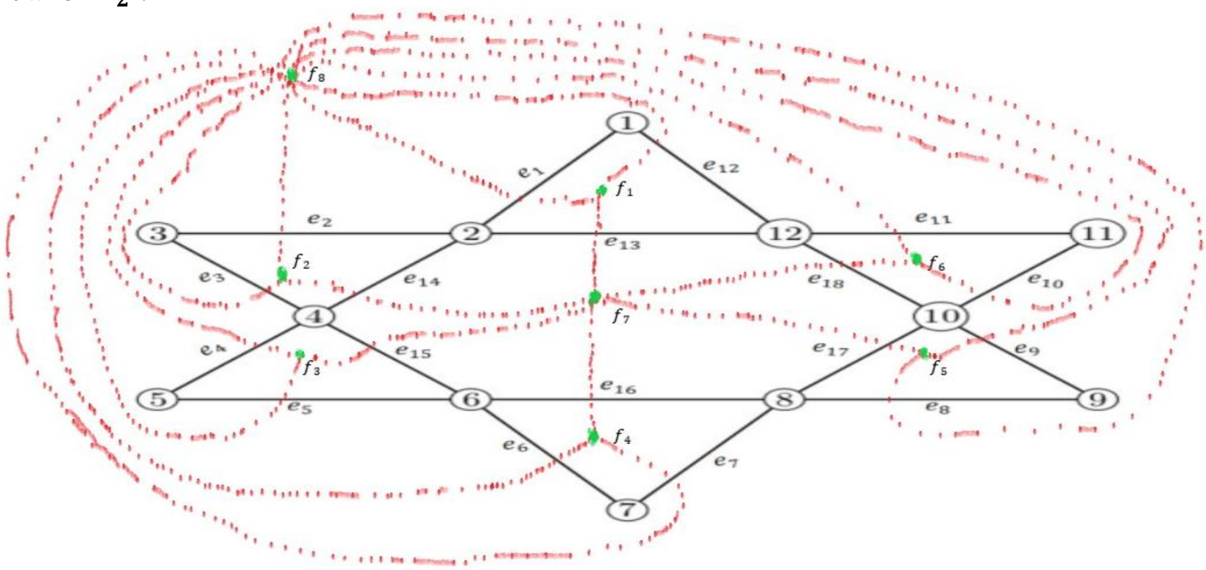
Salient features of dual graphs and its characteristic as follows:

5.1) Dual of G_1 :



For the further discussion we will refer the graph G_1 as Mohmad scimitar Graph.

5.2) Dual of G_2 :



For the further discussion we will refer the graph G_1 as Devil's Star Graph.

Here, G_1 and G_2 are Euler but its dual graphs are not Euler.

6) Results :

- External connectivity of an edge of each edge of these two graphs is even that's why line graph of both graph is also Euler.
- We have seen that Sum of the row total and column total are same for adjacency matrix for both graphs.
- In adjacency matrix $A(G)$, $a_{ij} = a_{ji}$; for each i and j.
  i.e. adjacency matrix $A(G)$ is symmetric
- All the entries of the main diagonal of $A(G)$ are 0.

- In incidence matrix the sum of the elements in the j^{th} column gives the degree of the vertex v_i and the sum of the elements in each row is 2.

Conclusion:

As known,there are some dominating characteristics related to vertices and cycles of an Euler graph which alternatively stand as defining properties. One additional notion introduced - 'External Edge Connectivity' of each edge, further simplify identity of Eulerity of line graph and dual graph of a given graph. Application of this new notion is tested and generalized to be applicable on any graph.

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