SOME RESULTS ON THE REES ALGEBRAS AND ANALYTICALLY INDEPENDENT OF IDEALS

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Abstract. In this paper, we study analytically independent elements and the equations of the Rees algebra of an ideal. Also we define the structure of the fiber cones, where the elements are analytically independent.

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1. Introduction

The powers of an ideal has been extensively studied in order to define classical notions in commutative ring theory and algebraic geometry. For example, the Rees algebra $R(I) = \bigoplus_{n\geq 0} I^n$ and the Symmetric algebra $S(I)$, where R is a commutative ring with identity and I is an ideal of R. The applications of such algebras are determined the moving curve of ideals and its relation to adjoint curve [3].

If $I = (x_1, \ldots, x_n)$, then the Rees algebra of an ideals is defined as the quotient of polynomial ring in n-variables as follows: a graded epimorphism $\phi : R[X_1, \ldots, X_n] \to R(I)$ such that $X_i \to x_i$, where $x_i \in I^i$ whose kernel is the ideal Q of $R[X_1, \ldots, X_n]$ generated by the homogeneous polynomials $f(X_1, \ldots, X_n)$ such that $f(x_1, \ldots, x_n) = 0$. The generators of the $ker(\phi)$ is called equation of the Rees algebra. The least integer $N \geq 1$ such that $Q = Q(N)$ is called the relation type of I, where $Q(N)$ is the ideal generated by homogeneous polynomial $R[X_1, \ldots, X_n]$ of degree at most N. It is denoted by $rt(I)$. It can also defined by the universal property of the Symmetric algebra. Consider $R^n \to I$ induces an epimorphism $R[X_1, \ldots, X_n] = S(R^n) \to S(I)$. So that kernel is the ideal $Q(1)$ of $R[X_1, \ldots, X_n]$ generated by the linear forms $\sum_{i=1}^{n} b_i X_i$ such that $\sum_{i=1}^{n} b_i x_i = 0$, where $b_i \in R$. Hence $Q(1)$ is contained in Q and equality hold if $S(I)$ is isomorphic to $R(I)$. An ideal I is said be of linear type if $Q(1) = Q$. Therefore $rt(I)$ is independent of the set of generators of an ideal.

The connection between the Rees algebra $R(I)$, the reduction of ideals and the symmetric algebra $S(I)$ have an important role in algebraic geometry. From geometric point of view it would be interesting that $Proj(\alpha) : Proj(R(I)) \to Proj(S(I))$ is an isomorphism, where I is a an regular sequence, $\alpha : R(I) \to (S(I))$ [1] and reduction number shows that analytically independent element and minimal generating set of the Rees algebra $R(I)$ [8]. These results give to the study of relation between the maximal minor of the generic matrix and generator of ideal, almost complete intersection ideals, projective dimension, reduction number. In [2] author investigated the results when $S(I)$ and $R(I)$ are isomorphic if and only if normal cone and normal bundle to the closed subscheme $spec(R/I)$ in $spec(R)$ are isomorphic. If I is of linear type , then I is minimal reduction itself [11]. There are many algebraist to discuss the results see [1], [2], [3], [4], [6], [7], [8]. This paper is based on work Valla on Rees algebra of an ideal, analytically independent element and begins the study of equation of the Rees algebra.

2. Main Results

Definition 2.1. *For the Noetherian local ring (R, m), the fiber cone of I,* $F_I(R) = \dfrac{R(I)}{mR(I)} = \bigoplus_{n \geq 0} \dfrac{I^n}{mI^n}$.

Definition 2.2. *The elements $x_1, \ldots, x_n \in I$ are said to be analytically independent in I if for any homogeneous polynomial $f(X_1, \ldots, X_n) \in R[X_1, \ldots, X_n]$ of degree r, the condition $f(x_1, \ldots, x_n) \in mI^r$ implies that all the coefficients of $f(X_1, \ldots, X_n)$ are in m.*

Theorem 2.3. *Let (R, m) be a Noetherian local ring and I be an ideal of R. Suppose $x_1, \ldots, x_n$ are analytically independent in I. Then :*

(1) *The elements $x_1, \ldots, x_n$ are minimally generate $(x_1, \ldots, x_n)$.*
(2) *If $(y_1, \ldots, y_n) = (x_1, \ldots, x_n)$, then $y_1, \ldots, y_n$ are analytically independent.*
(3) *If $J = (x_1, \ldots, x_n)$, then $F_J(R)$ is isomorphic to a polynomial ring in n variable over R/m.*

Proof. (1) We have to show that $\{\overline{x_1}, \ldots, \overline{x_n}\}$ is a basis of vector space J/mJ over R/m, where $\overline{x_i} = x_i + mJ$, $J = (x_1, \ldots, x_n)$, $i = 1, \ldots, n$. Let $x \in J$ such that

$$x = \sum_{i=1}^{n} a_i \ x_i, where \ a_i \in R.$$

$$x + mJ = \sum_{i=1}^{n} a_i \ x_i + mJ.$$

$$\overline{x} = \sum_{i=1}^{n} \overline{a_i} \ \overline{x_i}.$$

Therefore, $\overline{x}$ generates J.
Claim: $\{\overline{x_1}, \ldots, \overline{x_n}\}$ is a linear independent set over R/m.

$$\sum_{i=1}^{n} \overline{a_i} \ \overline{x_i} = mJ.$$

$$\sum_{i=1}^{n} a_i \ x_i + mJ = mJ.$$

$$\sum_{i=1}^{n} a_i \ x_i \ \in mJ \ \subseteq mI.$$

Since $x_1, \ldots, x_n$ are analytically independent in I, the polynomial $f(X_1, \ldots, X_n) = a_1X_1 + \cdots + a_nX_n$ of degree one with coefficient of $f(X_1, \ldots, X_n)$ are in m. Therefore, $\overline{a_i} = a_i + m = \overline{0}$. So that $\{\overline{x_1}, \ldots, \overline{x_n}\}$ is a basis.

(2) Let $J = (x_1, \dots, x_n) \subseteq I$ and $f(x_1, \dots, x_n) \in mJ^r$ for polynomial $f(X_1, \dots, X_n) \in R[X_1, \dots, X_n]$ with $deg(f) = r$. Note that $f(x_1, \dots, x_n) \in mJ^r \subseteq mI^r$. Since $x_1, \dots, x_n$ are analytically independent in I, all the coefficient of polynomial $f(X_1, \dots, X_n)$ are in m. Therefore, $x_1, \dots, x_n$ are analytically independent in $J = (y_1, \dots, y_n)$ and $y_1, \dots, y_n$ are analytically independent element.

(3) Consider the R/m- algebra homomorphism $g : R/m[X_1, \dots, X_n] \to F_J(R)$ such that

$$g(\sum_{i_1+i_2+\cdots+i_n=0}^{r} \overline{a_{i_1 i_2 \dots i_n}} X_1^{i_1} \dots X_n^{i_n}) = \sum_{i_1+i_2+\cdots+i_n=0}^{r} \overline{a_{i_1 \dots i_n}} \overline{x_1^{i_1}}\, \overline{x_2^{i_2}} \dots \overline{x_n^{i_n}}.$$

Then g is onto. By using fundamental theorem of R/m- algebra homomorphism $\dfrac{R/m[X_1, \dots, X_n]}{ker(g)} \cong F_J(R)$, where

$$\begin{aligned} ker(g) = \{ &\sum_{i_1+i_2+\cdots+i_n=0}^{r} \overline{a_{i_1 i_2 \dots i_n}} X_1^{i_1} \dots X_n^{i_n} \mid \\ &g(\sum_{i_1+i_2+\cdots+i_n=0}^{r} \overline{a_{i_1 i_2 \dots i_n}} X_1^{i_1} \dots X_n^{i_n}) = 0\}. \end{aligned}$$

Since $x_1, \dots, x_n$ are analytically independent in J, the polynomial $f(X_1, \dots, X_n) \in R[X_1, \dots, X_n]$ with $deg(f) = r$ such that $f(x_1, \dots, x_n) \in mJ^r$ with all the coefficient of polynomial $f(X_1, \dots, X_n)$ are in m for $r \geq 1$. Therefore $\sum_{i_1+i_2+\cdots+i_n=0}^{r} \overline{a_{i_1 i_2 \dots i_n}} X_1^{i_1} \dots X_n^{i_n} = 0$ and $ker(g) = 0$. Hence $R/m[X_1, \dots, X_n] \cong F_J(R)$.

□

Proposition 2.4. *Let R be a Noetherian ring, $I \subset R$ be an ideal of R. Suppose $\mathcal{A}$ is a flat R-algebra. Then*

$$R(I) \bigotimes_R \mathcal{A} \cong R(I \bigotimes_R \mathcal{A})$$

Proof. Consider the short exact sequence of algebras

$$0 \longrightarrow Ker(g) \longrightarrow S(I) \longrightarrow R(I) \longrightarrow 0.$$

Since $\mathcal{A}$ is a flat R-algebra,

$$0 \longrightarrow Ker(g) \bigotimes_R \mathcal{A} \longrightarrow S(I) \bigotimes_R \mathcal{A} \longrightarrow R(I) \bigotimes_R \mathcal{A} \longrightarrow 0.$$

Note that $Ker(g) \bigotimes_R \mathcal{A} = Ker(g \bigotimes ids)$ and $S(I) \bigotimes_R \mathcal{A} \cong S(I \bigotimes_R \mathcal{A})$. So that commutative diagram with exact rows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & Ker(g) \bigotimes_R \mathcal{A} & \longrightarrow & S(I) \bigotimes_R \mathcal{A} & \longrightarrow & R(I) \bigotimes_R \mathcal{A} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Ker(g \bigotimes ids) & \longrightarrow & S(I \bigotimes_R \mathcal{A}) & \longrightarrow & R_A(I \bigotimes_R \mathcal{A}) \end{array}$$

Hence $R(I) \bigotimes_R \mathcal{A} \cong R(I \bigotimes_R \mathcal{A})$ □

Proposition 2.5. *Let R be a ring, $Q = ker(\phi)$ and $Q_{(r)} = \{f \in ker(\phi) \mid deg(f) \leq r\}$, where $\phi : R[X_1, \dots, X_n] \longrightarrow R(I)$. Then*

$$Q_{(0)} \subseteq Q_{(1)} \subseteq Q_{(2)} \dots . Q_{(r)} \dots \text{ and } \bigcup_{r \geq 0} Q_{(r)} = ker(\phi).$$

.

Proof. Let $\phi : R[X_1, \dots, X_n] \longrightarrow R(I)$ such that

$$\phi(\sum_{i_1+i_2+\cdots+i_n=0}^{m} a_{i_1 i_2 \dots i_n} X_1^{i_1} \dots X_n^{i_n}) = \sum_{i_1+i_2+\cdots+i_n=0}^{m} a_{i_1 \dots i_n} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}.$$

(1) $$Q_{(0)} = \{a_{i_0 \dots . 0} \mid a_{i_0 \dots .. 0} \in R \mid deg(f) = 0\}.$$

(2) $$Q_{(1)} = \{f \in ker(\phi) \mid deg(f) \leq 1\}$$
$$= \{a_{i_0 \dots 0} + a_{i_1 0 \dots 0} X_1 + \dot{a}_{i_0 1 \dots 0} X_n,\ a_{i_0 \dots 0}\}.$$

(3) $$Q_{(2)} = \{f \in ker(\phi) \mid deg(f) \leq 2\}$$
$$= \{a_{0 \dots 0}, a_{0 \dots 0} + a_{i_i \dots 0} X_1 + a_{i_2 0 \dots 0} X_2 + a_{0 \dots i_n} X_n, \sum_{i_1+i_2+\cdots+i_n=2} a_{i_1 i_2 \dots i_n} X_1^{i_1} X_2^{i_2} \dots X_n^{i_n}\}.$$

(4)
$$Q_{(r)} = \{a_{0 \dots 0}, a_{0 \dots 0} + a_{i_i \dots 0} X_1 + a_{i_2 0 \dots 0} X_2 + a_{0 \dots i_n} X_n,$$
$$\sum_{i_1+i_2+\cdots+i_n=r-1} a_{i_1 i_2 \dots i_n} X_1^{i_1} X_2^{i_2} \dots X_n^{i_n}, \sum_{i_1+i_2+\cdots+i_n=r} a_{i_1 i_2 \dots i_n} X_1^{i_1} X_2^{i_2} \dots X_n^{i_n}\}.$$

By $(1), (2), (3), \dots (4), \dots$, we can observe that $Q_{(0)} \subseteq Q_{(1)} \subseteq Q_{(2)} \dots . Q_{(r)} \dots$. Since $ker\phi$ is a graded ring, $\bigcup_{r \geq 0} Q_{(r)} = ker(\phi)$. □

Theorem 2.6. *Let R be a Noetherian ring and $I = (x_1, \dots, x_n)$ be an ideal of R. Suppose $T_1, T_2, \dots, T_n$ are variables over R. Consider a map $\phi : R[T_1, \dots, T_n] \longrightarrow R(I)$ with $\phi(T_i) = x_i$. Let $Q(1)$ be the subideal of $ker(\phi)$ generated by all homogeneous elements of degree1. Let $R^m \xrightarrow{A} R^n \xrightarrow{\phi} I \longrightarrow 0$ be a presentation of I, where $A = [a_{ij}]_{m \times n}$ and $T = [T_1, \dots, T_n]_{1 \times n}$ matrix and L be the ideal generated by the entries of the matrix TA that vanish after subsituation $T_i \longrightarrow x_i$. Then $Q(1) = L$.*

Proof. Note that $Q(1) = \{a_1 T_1 + \dots + a_n T_n \mid a_1 x_1 \dots + a_n x_n = 0; x_i \in I\}$. Define

$$T\ A = [T_1, \dots, T_n]_{1 \times n} \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1m} \\ a_{21} & a_{22} \cdots & a_{2m} \\ \vdots & & \vdots \\ a_{n1} & a_{n2} \cdots & a_{nm} \end{bmatrix}_{n \times m}$$

$TA = [a_{11}T_1 + a_{21}T_2 + \cdots + a_{n1}T_n,\ a_{12}T_1 + a_{22}T_1 + \cdots + a_{n2}T_n,\ a_{1m}T_1 + a_{2m}T_2 + \cdots + a_{nm}T_n]$. This implies that L is ideal of $R[T_1, T_2, ..., T_n]$ defined by $L =< a_{11}T_1 + a_{21}T_2 + \cdots + a_{n1}T_n, a_{12}T_1 + a_{22}T_1 + \cdots + a_{n2}T_n, \dots a_{1m}T_1 + a_{2m}T_2 + \cdots + a_{nm}T_n >$.
Claim : $L = Q(1)$.
Let $x \in L$ such that $x = y_1(a_{11}T_1 + a_{21}T_2 + \cdots + a_{n1}T_n) + y_2(a_{12}T_1 + a_{22}T_1 + \cdots + a_{n2}T_n) + \cdots + y_n(a_{1m}T_1 + a_{2m}T_2 + \cdots + a_{nm}T_n)$.
Therfore $x = (y_1 a_{11} + a_{12} y_2 + \cdots + a_{1m} y_m) T_1 + (y_1 a_{21} + y_2 a_{22} + \cdots + y_m a_{2m}) T_2 + \cdots + (y_1 a_{n1} + y_2 a_{n2} + \cdots + y_m a_{nm}) T_n$. Take $a_i = \sum_{i=0}^{m} a_{ij}\ y_j$. Since $a_{ij} \in R$,

$a_{ij}\ y_j \in R$. Then $x = a_1T_1 + a_2T_2 \cdots + a_nT_n$. By assumption of L,
$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ (1).
This implies that $x \in Q(1)$.
Conversely, $A = [a_{ij}]_{n\times m}$. Let $x \in Q(1)$. Then $x = a_1T_1 + \cdots + a_nT_n$. Since $a_1x_1 + \cdots + a_nx_n = 0$, $(a_1, \ldots, a_n) \in ker(\phi) = Im(A)$,

where $Im(A) = [z_1z_2 \ldots z_m]_{1\times m} \begin{bmatrix} a_{11} & a_{12}\cdots & a_{n1} \\ a_{12} & a_{22}\cdots & a_{n2} \\ \vdots & & \vdots \\ a_{1m} & a_{2m}\cdots & a_{nm} \end{bmatrix}_{m\times n}$

$= [z_1a_{11} + z_2a_{12} + \cdots + z_ma_{1m}\ z_1a_{21} + z_2a_{22} + \cdots + z_ma_{2m}\ \ldots\ z_1a_{n1} + z_2a_{n2} + \cdots + z_ma_{nm}]_{1\times n}$.
So that $a_1 = z_1a_{11} + z_2a_{12} + \cdots + z_ma_{1m}$.
$a_2 = z_1a_{21} + z_2a_{22} + \cdots + z_ma_{2m}$
$\vdots$
$a_n = z_1a_{n1} + z_2a_{n2} + \cdots + z_ma_{nm}$
By (1), We can write $[z_1a_{11} + z_2a_{12} + \cdots + z_ma_{1m}]x_1 + [z_1a_{21} + z_2a_{22} + \cdots + z_ma_{2m}]x_2 + \cdots + [z_1a_{n1} + z_2a_{n2} + \cdots + z_ma_{nm}]x_n = 0$.
This implies that $z_1(a_1x_{11} + a_{21}x_2 + \cdots + a_{n1}x_n) + z_2(a_{12}x_1 + a_{22}x_2 + \cdots + a_{n2}) + \cdots + z_m(a_{1m}x_1 + \cdots + a_{nm}x_n) = 0$. Therefore $x \in L$. □

Example 2.7. *Consider the ring $R = k[X, Y, Z]$ and ideal $I = (X\ Y, Y\ Z, X\ Z)$ of R, where k is a field. Then the Rees algebra of I,*

$$R(I) \cong \frac{k[X_1, X_2, X_3, x, y, z]}{< X\ X_2 - Y\ X_3, Z\ X_1 - Y\ X_3 >},\ rt(I) = 1.$$

Proof. By using singular software, the Rees algebra of $R(I)$:
LIB"reesclos.lib";
ring$R = 0, (X, Y, Z)$, dp;
ideal $I = X\ Y, Y\ Z, X\ Z$;
list $L = ReesAlgebra(I)$;
def Rees $= L[1]$;
setring Rees;
Rees;
ker;
$ker[1] = X\ X_2 - Y\ X_3$,
$ker[2] = Z\ X_1 - Y\ X_3$

□

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