Nonlinear Implicit Fractional Dynamic Equation on Time Scale

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Abstract. In this paper we inquire into the result of existence and uniqueness theorem to the initial value non-linear implicit type dynamic equation by using the nabla Caputo fractional derivative operator. The existency is based on Scahuder's fixed point theorem together with Arzela-Ascoli theorem on left-dense (ld) continuous function in time scale domain.

Keywords: Fractional dynamic equation, nabla Caputo fractional derivative and nabla Riemann-Liouville fractional integro-differential equation, Schaduer's fixed point theorem, Arzela-Ascoli theorem on time scale.

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1. Introduction

The primary aim of this research paper is to introduce the concept of dynamic equations, which serve as a unified framework encompassing both differential and difference equations within a single domain known as the time scale, represented by $\mathbb{T}$. The inception of this topic can be attributed to Stefan Hilger's pioneering work in 1988, as part of his Ph.D. research under the guidance of Bernd Aulbach, as documented in [6, 7].

The notion of dynamic equations holds immense utility in the simultaneous modeling of phenomena exhibiting both continuous and discrete behaviors. One such example is the intriguing life span of the species "Periodical cicada," which defies easy classification into either continuous or discrete domains. To appropriately study the population dynamics of the Cicada species, a unified approach is indispensable, one that embraces both continuous and discrete elements.

Consider the case of the Floridian Brood cicada, which undergoes a peculiar life cycle of 13 years as a moth, followed by 28 days as a mature individual. To address this scenario effectively, we must approach the problem within the time

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scale domain $D = \bigcup_{J=0}^{\infty} \big[J(13y + 28d), J(13y + 28d) + 28d\big]$, where d denotes days, y represents years, and $J \in \mathbb{N} \cup 0$. Embracing the time scale calculus allows us to comprehensively analyze and comprehend the complexities inherent in such dynamic systems that straddle both continuous and discrete phenomena.

Recently the topic has attained too much attention due to its application in Engineering and Biological sciences. For further details of time scale we refer the reader to see [1, 5, 8, 10, 11, 12, 20, 21] and the references cited therein.

The investigation of fractional order calculus has demonstrated its ability to provide more accurate results than ordinary calculus when applied to real-world problems. As a result, we assert that the incorporation of fractional calculus into time scale calculus not only expands the concepts of ordinary calculus but also opens up promising avenues in the field of mathematical science. This fusion of fractional and time scale calculus was first explored in 2012 through the dissertation work of N. R. O. Bostos under the guidance of his Ph.D. supervisor, D. F. M. Torres.

The existing literature contains a wealth of research on fractional linear and non-linear dynamic equations, featuring various operators like Caputo, Riemann-Liouville, Caputo-Hadamard, among others. For further details, we encourage readers to explore references such as [2, 3, 13, 14, 15, 17, 18, 9, 19, 22, 23, 24] and the works cited therein.

As real-world scenarios often involve nonlinear differential equations, it becomes imperative to study the diverse range of nonlinear dynamic equations. Inspired by the aforementioned research, our study delves into the realm of implicit type fractional dynamic equations, characterized by the form:

$$\begin{cases} {}^{C}D^{\psi}0x(t) = \mathcal{H}\big(t, x(t), {}^{C}D^{\psi}0x(t)\big) \\ x(t)_{|t=0} = x_0, \forall x_0 \in \mathbb{R}. \end{cases} \tag{1.1}$$

Here, $t \in \mathcal{J}_{\mathcal{T}}$, with $\mathcal{J}_{\mathcal{T}} = [0, T] \cap \mathbb{T}_{\mathcal{K}}$, $T \in \mathbb{T}^{+}$, and $\mathcal{H} : \mathcal{J}_{\mathcal{T}} \times \mathbb{R} \times \mathbb{R}$ denotes a locally continuous function, the details of which will be discussed further in the study.

The rest of the paper is organized as follows: In section 2, we have highlighted some auxilary results related to fractional dynamic equation on time scales. In section 3, we have presented the existence and uniqueness of the dynamic equation (1.1). In section 4, we have given an example related to our main findings which make the manuscript easier to understand and the conclusion part of the paper is presented in section 5.

2. Preliminaries

Definition 2.1. [20] *Time scale $\mathbb{T}$ is a non empty subset of which is closed in $\mathbb{R}$. For connectedness of $\mathbb{T}$, there are two jump operators called backward and forward jump operator. Let $\rho : \mathbb{T} \to \mathbb{R}$, defined by*

$$\rho(t) = \{s \in \mathbb{T} : s < t\}, \text{ for } t \in \mathbb{T},$$

then $\rho(t)$ is a backward jump operator and

$$\sigma(t) : \{s \in \mathbb{T} : s > t\}$$

is a forward jump operator. Later $\rho(t)$ is said to be left scattered and left dense if $\rho(t) < t$ and $\rho(t) = t$, respectively. If d is the possible minimum right scattered, then we write $\mathbb{T}_{\mathcal{K}} = \mathbb{T} \setminus d$, else $\mathbb{T}_{\mathcal{K}} = \mathbb{T}$.

Since our manuscript is concerned with the nabla derivative, the results related to the delta derivative is omitted and so is also forward jump operator.

Definition 2.2. [12] *Let $v : \mathcal{J}_{\mathcal{T}} \to \mathbb{R}$. If at the left dense point in $\mathcal{J}_{\mathcal{T}}$, v is continuous and right sided limit exists at right dense points of $\mathcal{J}_{\mathcal{T}}$, then v is said to be ld continuous.*

The set $\xi(\mathcal{J}_{\mathcal{T}}, \mathbb{R})$ is used to denote all ld continuous functions from $\mathcal{J}_{\mathcal{T}}$ to $\mathbb{R}$.

Definition 2.3. [12] *Let $x \in \xi(\mathbb{T}, \mathbb{R})$, if $X^{\nabla}(t) = x(t)$, the nabla integral of x is presented by*

$$\int_{t_0}^{t} x(v)\nabla v = X(t) - X(t_0),$$

where $t_0 \in \mathbb{T}$.

Definition 2.4. [20] *Consider a ld continuous function $v : \mathbb{T}_{\mathcal{K}} \to \mathbb{R}$, the nabla derivative of v(t) of order $\psi \in (0, 1)$ is*

$$v^{(\psi)}(t) = \begin{cases} \frac{v(t)-v(\rho(t))}{\mu(t)^{\psi}}, & if \rho(t) < t \\ \lim_{s\to t} \frac{v(t)-v(s)}{(t-s)^{\psi}}, & if \rho(t) = t, \end{cases}$$

where μ denotes the backward graininess function. Also for $t \in \mathbb{T}$ value of $\mu(t) = t - \rho(t)$.

Definition 2.5. [20, Riemann-Liouville integro differential equation] *Let $x : \mathbb{T}_{\mathcal{K}} \to \mathbb{R}$ be a ld continuous function, then the Riemann-Liouville integro-differential equation of order $\psi \in (0, 1)$ is given by*

$${}^{RL}D_a^{-\psi} x(t) = \mathbb{I}_a^{\psi} x(t) = \frac{1}{\Gamma(\psi)} \int_a^t \big(t - \rho(v)\big)^{\psi-1} x(v)\nabla v.$$

For any two ld continuous function x(t) and y(t), the operator $\mathbb{I}^{\psi}$ *is linear, that is*

$$\mathbb{I}^{\psi}x(t)-\mathbb{I}^{\psi}y(t)=\mathbb{I}^{\psi}\big(x(t)-y(t)\big)$$

Definition 2.6. [20, Higher order nabla derivative] *Assume* $\mathcal{X}:\mathbb{T}_{\mathcal{K}}\to\mathbb{R}$ *is a ld continuous function on a time scale* $\mathbb{T}$. *The second order nabla derivative* $\mathcal{X}_{\nabla\nabla}=\mathcal{X}_{\nabla}^{(2)}$ *can be defined, provided* $\mathcal{X}_{\nabla}$ *is differentiable on* ${\mathbb{T}_{\mathcal{K}}}^{(2)}=\mathbb{T}_{\mathcal{K}\mathcal{K}}$ *with derivative* $\mathcal{X}_{\nabla}^{(2)}=(\mathcal{X}_{\nabla})_{\nabla}:{\mathbb{T}_{\mathcal{K}}}^{(2)}\to\mathbb{R}$. *Similarly, the* n^{th} *order nabla derivative we get* $\mathcal{X}_{\nabla}^{(n)}:{\mathbb{T}_{\mathcal{K}}}^{n}\to\mathbb{R}$, *it is attained by cut out n right scattered left end points from* $\mathbb{T}$.

Definition 2.7. [20, Caputo nabla derivative] *Let* $\mathcal{X}$ *be a ld continuous function, where* $\mathcal{X}_{\nabla}^{(n)}$ *exists on* ${\mathbb{T}_{\mathcal{K}}}^{n}$. *Then the Caputo nabla derivative of order* $\psi>0$ *is defined by*

$${}^{C}D_{a}^{(\psi)}\mathcal{X}(t)=\frac{1}{\Gamma(n-\psi)}\int_{a}^{t}(t-\mu(u))^{n-\psi-1}\mathcal{X}_{\nabla}^{(n)}(u)\nabla(u),$$

where $n=[\psi]+1$. *In general if* $\psi\in(0,1)$ *then,*

$${}^{C}D_{a}^{(\psi)}\mathcal{X}(t)=\frac{1}{\Gamma(1-\psi)}\int_{a}^{t}\big(t-\mu(u)\big)^{-\psi}\mathcal{X}_{\nabla}(u)\nabla u.$$

Definition 2.8. [11] *Let* $\mathcal{G}$ *is a non empty convex and closed subset of a Banach space* F *and let* $x:G\to F$ *is a continuous mapping such that* $x(G)$ *is relatively compact in* F. *Then the function* x *contain a fixed point in* $\mathcal{G}$.

Theorem 2.9. [11] *A function* $u:\mathcal{X}\to\mathcal{Y}$ *is a completely continuous mapping, if* $B\subseteq\mathcal{X}$, *such that* B *is bounded then* $u(\mathcal{X})$ *is a relatively compact in* $\mathcal{X}$.

Proposition 2.10. [12] *Let* $\xi\big(\mathcal{J}_{\mathcal{T}},\mathbb{R}\big)$ *is a set of a ld-continuous functions from the time scale domain* $\mathcal{J}_{\mathcal{T}}$ *to* $\mathbb{R}$.

The set $\xi\big(\mathcal{J}_{\mathcal{T}},\mathbb{R}\big)$ *formed a Banach space equipped by the norm*

$$||x||_{\xi}=\sup_{t\in\mathcal{J}_{\mathcal{T}}}|x(t)|.$$

The set $\xi^{1}\big(\mathcal{J}_{\mathcal{T}},\mathbb{R}\big)$ *is used to denote all ld-continuously nabla derivable function from* $\mathcal{J}_{\mathcal{T}}$ *to* $\mathbb{R}$.

Definition 2.11. *For* $x\in\xi^{1}\big(\mathcal{J}_{\mathcal{T}},\mathbb{R}\big)$, *x(t) is a solution of the equation (1.1), if* $x(t)$ *satisfies the dynamic equation everywhere for each* $t\in\mathcal{J}_{\mathcal{T}}$.

Definition 2.12. [12, Arzela-Ascoli theorem] *Let* $\mathcal{D}\subseteq\xi\big(\mathcal{J}_{\mathcal{T}},\mathbb{R}\big)$. *Then* $\mathcal{D}$ *is said to be relatively compact if and only if at the same time* $\mathcal{D}$ *is bounded and equicontinuous.*

3. Main Findings

One can relate the dynamic equations (1.1) with a population dynamics problem of start - stop phenomenon, such as insect population in an area that is smooth during the incubating season and die out in winter. If we involve toxic effect in the population, then the change of population in that particular area can be modeled with the situation, where x(t) is the insect population at a particular time t, ${}^{C}D_{0}^{\psi}x(t)$ is the rate of change of the population with respect to t and $x(0) = x_0$, is the population of insect at an initial time in that area.

For the existency of the equation (1.1), we reduce the equation into a fixed point problem and we claim that the solution of (1.1) satisfies the Voltera integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\psi)} \int_0^t \big(t - \rho(v)\big)^{\psi-1} \mathcal{H}\big(t, x(v), {}^{C}D^{\psi}x(v)\big) \nabla v. \tag{3.1}$$

For showing our findings, we need to present the following results.

Proposition 3.1. [4] *Let $r, v \in \mathbb{T}$ such that $r < v$. Consider a ld continuous function g on $\mathcal{J}_{\mathcal{T}}$, then*

$$\int_r^v g(t)\nabla t = \int_{\rho(r)}^r g(t)\nabla t + \int_r^{\rho(v)} g(t)\nabla t + \int_{\rho(v)}^v g(t)\nabla t$$
$$= [r - \rho(r)]g(r) + [v - \rho(v)]g(v) + \int_r^{\rho(v)} g(t)\nabla t.$$

Theorem 3.2. *Let a ld continuous function g on a time scale interval $[r, v]_T$, and if the function $\mathcal{G}$ is extension of g in real interval $[r, v]$, then*

$$\mathcal{G}(s) = \begin{cases} g(s), & if s \in \mathbb{T} \\ g(t), & if t \in \big(\rho(t), t\big) \notin \mathbb{T}, \end{cases}$$

hence

$$\int_r^v g(t)\nabla t \leq \int_r^v \mathcal{G}(t)dt.$$

Proof. Let $a \in [r, v]$, such that $\rho(a) < a$. From the Proposition 3.1, we obtain

$$\int_{\rho(a)}^a g(t)\nabla t = [a - \rho(a)]g(a).$$

Since the function $g(t)$ is an increasing, so $\mathcal{G}(t)$, its extension is also increasing. So using the mean value theorem of integration, we get

$$[a - \rho(a)]\mathcal{G}(\rho(a)) \leq \int_{\rho(a)}^a \mathcal{G}(t)dt \leq [a - \rho(a)]\mathcal{G}(a)$$

$$[a - \rho(a)]g(\rho(a)) \leq \int_{\rho(a)}^a \mathcal{G}(t)dt \leq [a - \rho(a)]g(a)$$

and hence,

$$\int_{\rho(a)}^{a} g(t)\nabla t \le \int_{\rho(a)}^{a} \mathcal{G}(t)dt.$$

Now if the interval $[r, v]$ has only one left scattered point say a, then we acquire the following by using the Proposition 3.1 and the above results:

$$\begin{aligned}\int_{r}^{v} g(t)\nabla t &= \int_{r}^{\rho(a)} g(t)\nabla t + \int_{\rho(a)}^{a} g(t)\nabla t + \int_{a}^{v} g(t)\nabla t\\ &\le \int_{r}^{\rho(a)} \mathcal{G}(t)dt + \int_{\rho(a)}^{a} \mathcal{G}(t)dt + \int_{a}^{v} \mathcal{G}(t)dt.\end{aligned}$$

Similarly we can extend the above result for n left scattered point on $[r, v]$, and thus obtain

$$\int_{r}^{v} g(t)\nabla t \le \int_{r}^{v} \mathcal{G}(t)dt.$$

□

Lemma 3.3. *Any function $h \in \xi^1\big(\mathcal{J}_\mathcal{T}, \mathbb{R}\big)$ is a solution of (1.1), if $h \in \xi\big(\mathcal{J}_\mathcal{T}, \mathbb{R}\big)$ satisfy the following Voltera integral equation*

$$h(t) = x_0 + \frac{1}{\Gamma(\psi)} \int_{0}^{t} \big(t - \rho(v)\big)^{\psi-1} x(v)\nabla v, \tag{3.2}$$

where $x(t)$ is a solution of the integral equation (3.1).

Proof. Let ${}^{C}D^{\psi}x(t) = h(t)$ with $x(t)|_{t=0} = x_0$, then from the equation (3.1), we obtain

$$\begin{aligned}x(t) =& x_0 + \frac{1}{\Gamma(\psi)} \int_{0}^{t} \big(t - \rho(v)\big)^{\psi-1} h(v)\nabla v\\ =& x_0 + \mathbb{I}^{\psi}h(t).\end{aligned}$$

Moreover, the dynamic equation (1.1) is equivalent to

$$h(t) = \mathcal{H}\big(t, x_0 + \mathbb{I}^{\psi}h(t), h(t)\big). \tag{3.3}$$

Later, we create some hypotheses that will be used to support our principal result.

(L1) The mapping $\mathcal{H} : \mathcal{J}_\mathcal{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is always ld continuous.

(L2) There exist $E_1 \in \xi(\mathcal{J}_\mathcal{T}, \mathbb{R})$ and two constants E_2 and E_3 such that $E_2 > 0$ and $0 < E_3 < 1$ satisfying

$$|\mathcal{H}(t, h_1, h_2)| \le |E_1(t)| + E_2|h_1| + E_3|h_2|,$$

for all $t \in \mathcal{J}_\mathcal{T}$ and $(h_1, h_2) \in \mathbb{R} \times \mathbb{R}$.

(L3) There exist two constants $F_1 > 0$ and $0 < F_2 < 1$ such that

$$|\mathcal{H}(t, h_1, h_2) - \mathcal{H}(t, r_1, r_2)| \le F_1|h_1 - r_1| + F_2|h_2 - r_2|,$$

for all $t \in \mathcal{J}_{\mathcal{T}}$ and $(r_1, r_2) \in \mathbb{R} \times \mathbb{R}$. □

Proposition 3.4. *For $t \in \mathcal{J}_{\mathcal{T}}$ and $x \in \xi\big(\mathcal{J}_{\mathcal{T}}, \mathbb{R}\big)$, the operator $\mathbb{I}^{\psi}x(t)$ is bounded with respect to the norm defined in the Proposition 2.10.*

Proof. Since, $(t - \rho(v))^{\psi-1}$ is an increasing monotonic function, so by using the Theorem 3.2 we get

$$\int_0^t (t - \rho(v))^{\psi-1} \nabla v \leq \int_0^t (t - v)^{\psi-1} dv. \tag{3.4}$$

Now from the Definition 2.6 we obtain

$$\begin{aligned} ||\mathbb{I}^{\psi}x||_{\xi} &= \sup_{t \in \mathcal{J}_{\mathcal{T}}} |\mathbb{I}^{\psi}x(t)| \\ &\leq \Big|\frac{1}{\Gamma(\psi)} \int_0^t (t - \rho(v))^{\psi_1} x(v) \nabla v\Big| \\ &< \frac{1}{\Gamma(\psi)} \int_0^t (t - s)^{\psi-1} dv \sup_{v \in \mathcal{J}_{\mathcal{T}}} |x(v)| \\ &\leq \frac{T^{\psi}}{\Gamma(\psi+1)} ||x||_{\xi}. \end{aligned}$$

□

Proposition 3.5. *For any $h(t), r(t) \in \xi\big(\mathcal{J}_{\mathcal{T}}, \mathbb{R}\big), t \in \mathcal{J}_{\mathcal{T}}$, then by using the Definition 2.6 we get*

$$\begin{aligned} ||\mathbb{I}^{\psi}h - \mathbb{I}^{\psi}r||_{\xi} &= \sup_{t \in \mathcal{J}_{\mathcal{T}}} |\mathbb{I}^{\psi}h(t) - \mathbb{I}^{\psi}r(t)| \\ &\leq |\mathbb{I}^{\psi}\big(h(t) - r(t)\big)| \\ &\leq \frac{T^{\psi}}{\Gamma(\psi+1)} ||h - r||_{\xi}. \end{aligned}$$

Thus the operator $\mathbb{I}^{\psi}$ is linear with reference to the norm indicated by the Proposition 2.10.

Theorem 3.6. *Consider that (L1), (L2) and (L3) hold true, and if*

$$\frac{F_1 T^{\psi}}{\Gamma(\psi+1)} + F_2 < 1,$$

then the equation (1.1) consist a solution.

Proof. Let ${}^{C}D^{\psi}x(t) = h(t)$ for $t \in \mathcal{J}_{\mathcal{T}}$. Using the equation (3.3) we obtain

$$\begin{aligned} |h(t)| =& \big|\mathcal{H}\big(t, x_0 + \mathbb{I}^{\psi}h(t), h(t)\big)\big| \\ \leq& |E_1(t)| + E_2|x_0| + E_2|\mathbb{I}^{\psi}h(t)| + E_3|h(t)| \\ \leq& |E_1(t)| + E_2|x_0| + \frac{E_2T^{\psi}}{\Gamma(\psi+1)}|h(t)| + E_3|h(t)| \\ \leq& \frac{|E_1(t)| + E_2|x_0|}{1 - \big(\frac{E_2T^{\psi}}{\Gamma(\psi+1)} + E_3\big)} \\ \leq& \gamma, \end{aligned} \tag{3.5}$$

where

$$\gamma = \frac{|E_1(t)| + E_2|x_0|}{1 - \big(\frac{E_2T^{\psi}}{\Gamma(\psi+1)} + E_3\big)}.$$

Taking the norm of $\xi\big(\mathcal{J}_{\mathcal{T}}, \mathbb{R}\big)$ on both side of the equation (3.5) we get

$$\begin{aligned} ||h||_{\xi} \leq& \frac{||E_1||_{\xi} + E_2||x_0||_{\xi}}{1 - \big(\frac{E_2T^{\psi}}{\Gamma(\psi+1)} + E_3\big)} \\ \leq& \mathcal{L}, \end{aligned} \tag{3.6}$$

where

$$\mathcal{L} = \frac{||E_1||_{\xi} + E_2||x_0||_{\xi}}{1 - \big(\frac{E_2T^{\psi}}{\Gamma(\psi+1)} + E_3\big)}. \tag{3.7}$$

Now consider a set $\mathcal{M}_{\mathcal{L}} = \{h \in \xi\big(\mathcal{J}_{\mathcal{R}}, \mathbb{R}) : ||h||_{\xi} \leq \mathcal{L}\} \subseteq \xi\big(\mathcal{J}_{\mathcal{T}}, \mathbb{R}\big)$ and an operator $\mathcal{Z} : \mathcal{M}_{\mathcal{L}} \to \mathcal{M}_{\mathcal{L}}$ such that

$$\mathcal{Z}(h) = \mathcal{H}\big(t, x_0 + \mathbb{I}^{\psi}h(t), h(t)\big). \tag{3.8}$$

Moreover for $h, r \in \mathcal{M}_{\mathcal{L}}$ and using the Proposition 3.4 and 3.5 we get

$$\begin{aligned} ||\mathcal{Z}(h) - \mathcal{Z}(r)||_{\xi} =& \sup_{t\in\mathcal{J}_{\mathcal{T}}} |\mathcal{Z}(h(t)) - \mathcal{Z}(r(t))| \\ \leq& F_1|\mathcal{H}(t, x_0 + \mathbb{I}^{\psi}h(t), h(t)) - \mathcal{H}(t, x_0 + \mathbb{I}^{\psi}r(t), r(t))| \\ \leq& F_1\big|\mathbb{I}^{\psi}h(t) - \mathbb{I}^{\psi}r(t)\big| + F_2|h(t) - r(t)| \\ \leq& \frac{F_1T^{\psi}}{\Gamma(\psi+1)}|h(t) - r(t)| + F_2|h(t) - r(t)| \\ \leq& \frac{F_1T^{\psi}}{\Gamma(\psi+1)}||h - r||_{\xi} + F_2||h - r||_{\xi} \\ \leq& \Big(\frac{F_1T^{\psi}}{\Gamma(\psi+1)} + F_2\Big)||h - r||_{\xi}. \end{aligned}$$

Since $\frac{F_1T^{\psi}}{\Gamma(\psi+1)} + F_2 < 1$, which is a contraction mapping, this hinted the existence of the solution to the equation (1.1). □

For sufficient conditions of the existency of the solution of (1.1), we will use the conditions of the Definition 2.8.

Theorem 3.7. *Consider that (L1) → (L3) hold true, then for* $t \in \mathcal{J}_{\mathcal{T}}$ *the equation (1.1) contain a unique solution.*

Proof. The proof of the theorem is presented in the following steps:
Step 1: Let $\mathcal{Z} : \mathcal{M}_{\mathcal{L}} \to \mathcal{M}_{\mathcal{L}}$ and a sequence (h_n) such that $h_n \to h$, then for $t \in \mathcal{J}_{\mathcal{T}}$, we obtain

$$\begin{aligned} ||\mathcal{Z}h_n - \mathcal{Z}h||_\xi &= \sup_{t\in\mathcal{J}_{\mathcal{T}}} |\mathcal{Z}(h_n(t)) - \mathcal{Z}(h(t))| \\ &\leq |\mathcal{H}\big(t, x_0 + \mathbb{I}^\psi(h_n(t)), h_n(t)\big) - \mathcal{H}\big(t, x_0 + \mathcal{I}^\psi(h(t)), h(t)\big)| \\ &\leq F_1\big|\mathbb{I}^\psi h_n(t) - \mathbb{I}^\psi h(t)\big| + F_2|h_n(t) - h(t)| \\ &\leq \frac{F_1 T^\psi}{\Gamma(\psi+1)}|h_n(t) - h(t)| + F_2|h_n(t) - h(t)| \\ &\leq \frac{F_1 T^\psi}{\Gamma(\psi+1)}||h_n - h||_\xi + F_2||h_n - h||_\xi. \end{aligned}$$

Since $||h_n - h|| \to 0$ as $n \to \infty$, we get $||\mathcal{Z}(h_n) - \mathcal{Z}(h)||_\xi \to 0$, as $n \to \infty$. Thus the mapping $\mathcal{Z} : \mathcal{M}_{\mathcal{L}} \to \mathcal{M}_{\mathcal{L}}$ is continuous.
Step 2: For each $h \in \mathcal{M}_{\mathcal{L}}$, the mapping $\mathcal{Z}$ conveyed bounded sets of $\xi\big(\mathcal{J}_{\mathcal{T}}, \mathbb{R}\big)$ into bounded set of $\xi\big(\mathcal{J}_{\mathcal{T}}, \mathbb{R}\big)$.

$$\begin{aligned} ||\mathcal{Z}h||_\xi &= \sup_{t\in\mathcal{J}_{\mathcal{T}}} |\mathcal{Z}(h(t))| \\ &\leq |\mathcal{H}(t, x_0 + \mathbb{I}^\psi h(t), h(t)| \\ &\leq |E_1(t)| + E_2|x_0 + \mathbb{I}^\psi h(t)| + E_3|h(t)| \\ &\leq |E_1(t)| + E_2|x_0| + E_2|\mathbb{I}^\psi h(t)| + E_3|h(t)| \\ &< ||E_1||_\xi + E_2|x_0| + E_2\Big|\frac{1}{\Gamma(\psi)}\int_0^t (t-\rho(s))^{\psi-1} h(s)\nabla s\Big| + E_3||h||_\xi \\ &\leq ||E_1||_\xi + E_2||x_0||_\xi + \frac{E_2 T^\psi}{\Gamma(\psi+1)}||h||_\xi + E_3||h||_\xi \\ &\leq ||E_1||_\xi + E_2||x_0||_\xi + \Big(\frac{E_2 T^\psi}{\Gamma(\psi+1)} + E_3\Big)||h||_\xi \\ &\leq \mathcal{L}. \end{aligned}$$

Thus from the equation (3.7) we obtain the operator $\mathcal{Z}$ is a bounded.
Step 3: Let $\mathcal{Z} : \mathcal{M}_{\mathcal{L}} \to \mathcal{M}_{\mathcal{L}}$, and cosider $t_1, t_2 \in \mathcal{J}_{\mathcal{T}}$ such that $t_1 < t_2$, then for any

$h \in \mathcal{M}_{\mathcal{L}}$, the map $\mathcal{Z}$ conveyed bounded set into equicontinuous set.

$$
\begin{aligned}
||(\mathcal{Z}h)(t_2) - (\mathcal{Z}h)(t_1)||_{\xi} &= \sup_{t \in \mathcal{J}_{\mathcal{T}}} |\mathcal{Z}(h(t_2)) - \mathcal{Z}(h(t_1))| \\
&\leq |\mathcal{H}\big(t_2, x_0 + \mathbb{I}^{\psi} h(t_2)), h(t_2)\big) - \mathcal{H}\big(t_1, x_0 + \mathbb{I}^{\psi} h(t_1)), h(t_1)\big)| \\
&\leq \frac{F_1 T^{\psi}}{\Gamma(\psi+1)} |h(t_2) - h(t_1)| + F_2 |h(t_2) - h(t_1)|,
\end{aligned}
$$

when $t_2 \to t_1$, then $|h(t_2) - h(t_1)| \to 0$. So $||(\mathcal{Z}h)(t_2) - (\mathcal{Z}h)(t_1)||_{\xi} \to 0$.

So as a consequences of Step 1 $\to$ Step 3 and using the Definition 2.12, we conclude that the mapping $\mathcal{Z}$ is continuous completely.

Step 4 (Priori bound): It has remain to show that the set given by

$$
\mathcal{N} = \{h \in \mathcal{M}_{\mathcal{L}} : h = \sigma \mathcal{Z}(h), 0 < \sigma < 1\}
$$

is a bounded.

Let $h(t) \in \mathcal{N}, t \in \mathcal{J}_{\mathcal{T}}$ and for $0 < \sigma < 1$ we have from the equations (3.3) and (3.5), that $\mathcal{N}$ is a bounded set.

Thus in view of the Definition 2.8, we must say that the operator $\mathcal{Z}$ consists a fixed point, which is a solution of the dynamic equation (1.1). □

4. Example

Example 4.1. Consider an initial value problem on $\mathbb{T} = [0,1] \cup [2,3]$

$$
\begin{cases}
{}^{C}D^{(\psi)} x(t) = \frac{e^{-2t}}{(e^{3t}+7)\left[1+|x(t)|+|{}^{C}D^{\psi} x(t)|\right]} \\
x(0) = 1
\end{cases}.
$$

For $t \in [0,3] \cap \mathbb{T}_{\mathcal{K}}$ and $\psi = \frac{1}{2}$, we set

$$
\mathcal{Z}(t,u,v) = \frac{e^{-2t}}{(e^{3t}+7)[1+u+v]}.
$$

Right hand side of the problem satisfies all the conditions of the Definition 2.8. So as to solve the Example 4.1 completely, the condition of the Theorem 3.6 is yet to be solved. Now,

$$
\begin{aligned}
|\mathcal{Z}(t,u_2,v_2) - \mathcal{Z}(t,u_1,v_1)| =& \frac{e^{-2t}}{(e^{3t}+7)} \Big[\frac{1}{1+u_2,v_2} - \frac{1}{1+u_1,v_1}\Big] \\
=& \frac{e^{-2t}}{(e^{3t}+7)} \Big[\frac{(u_2-u_1)+(v_2-v_1)}{(1+u_2+v_2)(1+u_1+v_1)}\Big] \\
\leq& \frac{e^{-2t}}{(e^{3t}+7)} \big(|u_2-u_1| + |v_2-v_1|\big) \\
\leq& \frac{1}{8}|u_2-u_1| + \frac{1}{8}|v_2-v_1|.
\end{aligned}
$$

From the conditions (L3) of Lemma 3.3, we get $F_1 = \frac{1}{8}$ and $F_2 = \frac{1}{8}$. Now

$$\begin{aligned}\frac{F_1 T^{\psi}}{\Gamma(\psi+1)} + F_2 &= \frac{\frac{1}{8}T^{\frac{1}{2}}}{\Gamma(\frac{1}{2}+1)} + \frac{1}{8} \\ &\leq \frac{3^{\frac{1}{2}}}{8 \times 0.887} + \frac{1}{8} \\ &< 1,\end{aligned}$$

satisfies the Banach contraction principle. Hence the equation (1.1) has a solution which is unique.

5. Conclusion

In this manuscript we discussed the implicit type nonlinear fractional dynamic equation involving Caputo nabla fractional derivative, using the Schauder's fixed point theorem which is a new result in a time scale context. Due to the physical interpretation of Caputo derivative, the result of this paper can be used in real world problems which can be easily discuss in time scale domain. Moreover the characteristics of the solutions such as dependency, stability analysis are our future scope of study. Recently, the topic is getting too much heed in Engineering fields due to its application in fractional two dimensional time scale [16].

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