Model for the effects of Industrialization, Population, Primary - Secondary Toxicants on depletion of Forestry Resource

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ABSTRACT

A nonlinear mathematical model is developed and analyzed in this research to explore the impacts of industrialization, population, and primary-secondary toxicants on the depletion of forestry resources. It is assumed that primary toxicant is emitted into the environment with a constant prescribed rate as well as its growth is enhanced by increase in population density and industrialization. Further, a part of primary toxicant is transformed into secondary toxicant, which is more toxic, both affecting the resource and population simultaneously. The nature and uniqueness of equilibrium, as well as the requirements for the existence of their local and global equilibrium points, have all been proven by using the stability theory of differential equations. Numerical simulations are performed to analyze the dynamics of the system using a fourth order Runge-Kutta method and determine the critical parameters that are responsible for depletion of forestry resource.

Keywords: Resource-biomass, Population, Primary & Secondary Toxicants, Industrialization, Stability.
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1. INTRODUCTION

The Environmental problems in India are growing rapidly. The WHO estimates that about two million people die prematurely every year as a result of air pollution while many more suffer from breathing ailments, heart disease, lung infection and even cancer. Fine particles or microscopic dust from coal or wood fires and unfiltered diesel engine are rated as one of the most lethal forms of air pollution caused by industry, transport, household heating, cooking and ageing coal or oil-fired power stations.

Airborne pollutants can be classified broadly into two categories: primary and secondary. Primary pollutant are those that are emitted into the atmosphere by the source such as fossil fuels combustion from power plant, vehicle engine and industrial production, by combustion of biomass from agriculture and land clearing purpose, and by natural processes. Secondary pollutants are formed within the atmosphere when primary pollutant reacts with sunlight, oxygen and water and other chemical present in the air. The question to what extent primary and secondary air pollutants are relevant to atmospheric pollution and their effects on biological species and the quality of the environment can be answered in a straight forward manner: atmospheric processes, including oxidation procedures, particle formation and equilibria, determine the fate of primary emission and, in most cases, the secondary product of these processes are the more important ones concerning their effects on human health and the quality of the environment. So, the pollutants in both of their forms are serious threat for the survival of the resource biomass and exposed population and in order to regulate these pollutant wisely, we must assess the risk of the resource biomass and population exposed to pollutants. Therefore, it is important to study the effects of pollutants on resource dependent biological population by making use of mathematical models. So in this research an attempt is made to model the effect of these environmental pollutants on resource dependent biological population.

In recent years, Freedman and Shukla [1] studied the effects of toxicants on a biological population and predator-prey system. They showed that if the emission rate of the toxicants increases, the equilibrium level of population decreases, and the magnitude of which depends on the influx and washout rates of the toxicant. Chattopadhyay [2] proposed a model to study the effect of toxic substances on a two species competitive system. Shukla and Dubey [3] studied the effect of two toxicants on the growth and survival of biological species. The survival (growth and existence) of a resource biomass dependent species in a forest habitat, which is depleted due to industrialization pressure, has also been studied in [4, 5]. Shukla and Dubey [6] studied the depletion of a forestry resource in a habitat, which is caused by an increase in population density and pollutant emission into the environment. Dubey et al. [7] studied the depletion of forestry resource by population and population pressure augmented industrialization. They showed that if the growth of population is only partially dependent on resource, still the resource biomass is doomed to extinction due to large population pressure augmented industrialization. Dubey and Narayanan [8] studied the effects of industrialization, population and pollution on a renewable resource. Shukla et al. [9] studied the effects of primary and secondary toxicants on renewable resources. In his study, the direct emission of primary toxicant is considered, a part of which is transformed into secondary toxicant, but in real situation, level of toxicant increases into the environment by increase in density of population and industrialization. Further Misra P. et al. [10] studied a mathematical model to study the optimal harvest policy for toxicant effected forestry biomass . Constant introduction of toxicant into the environment and dynamic harvesting effort of biomass with tax as control instrument have been taken Lata. K et al., [11] investigated the impact of industrialization on forestry resources, assessing the effect of wood and non-wood based industries on the depletion of forestry biomass. It was discovered that as the level of pollutants from wood and non-wood based businesses rises, the metabolism of forestry resources suffers due to the uptake of these pollutants by the forestry resources. Mishra & Lata, [12] investigated the depletion and conservation of forestry biomass in the presence of industrialization by assuming that industries migrate owing to forestry biomass availability and their expansion rises due to forestry biomass availability. Further Verma V. & Singh V. [13] studied the impact of media campaign to conserve forestry resources and control population pressure. The study concluded that if we conserve forestry resources and promote public understanding of the value of trees, we can protect them.

In view of above considerations, in this paper, a nonlinear mathematical model is proposed and analyzed for the survival of resource dependent biological population in the presence of two toxicants (primary and secondary). It is assumed that density of primary toxicant is enhanced by population and industrialization in the environment and the secondary toxicant is formed from it into the environment which is more toxic. This situation is modeled by the system of five ordinary differential equations. Stability theory of nonlinear differential equations and fourth order Runge-Kutta method are used to analyze and predict the behavior of the model.

2. MATHEMATICAL MODEL

We consider an ecosystem where the resource biomass is being depleted due to the pressure of industrialization, population, primary-secondary toxicants in the environment. It is assumed that the dynamics of the resource biomass, population and industrialization are governed by logistic type equations. It is also assumed that the growth rate of resource biomass decreases with increase in density of population and industrialization while its carrying capacity decreases with increase in environmental concentration of primary-secondary toxicant. It is further assumed that growth rate of population increases as the density of resource biomass and industrialization increases. Also the growth rate of industrialization increases with increase in density of resource biomass and population. It is also considered that the emission of primary toxicant into the environment is industrialization and population dependent and a secondary toxicant which is transformed from the primary toxicant into the environment and is more toxic. It is assumed that the rate of transformation of secondary toxicant is proportional to the environmental concentration of the primary toxicant. In view of these arguments, the system is assumed to be governed by the following differential equations:

$$
\begin{aligned}
\frac{dB}{dt} &= r_B(N)B - \frac{r_{B0}B^2}{K_B(P_1,.P_2)} - \alpha IB,\\
\frac{dN}{dt} &= r_P(B)N - \frac{r_{P0}N^2}{M(P_1,P_2)} + \gamma_1 IN,\\
\frac{dP_1}{dt} &= Q(I,N) - \delta_0 P_1 - \alpha_1 BP_1 - \alpha_2 NP_1 - gP_1,\\
\frac{dP_2}{dt} &= \theta g P_1 - \delta_1 P_2 - \beta_1 BP_2 - \beta_2 NP_2,\\
\frac{dI}{dt} &= r_1 I\left(1 - \frac{I}{L}\right) + \beta IB + \gamma_2 IN.
\end{aligned}
\qquad (2.1)
$$

$$B(0) \geq 0,\ N(0) \geq 0,\ P_1(0) \geq 0,\ P_2(0) \geq 0,\ I(0) \geq 0.$$

In model (2.1), B is the density of resource biomass, N is the density of population, P_1 and P_2 are the densities of primary and secondary toxicants into the environment. I is the density of industrialization. α is the depletion rates coefficients of the resource biomass due to the industrialization and β is the corresponding growth rate coefficient of industrialization. The positive constant k is the transformation rate coefficient of primary toxicant into secondary toxicant in the environment. γ_1 and γ_2 are the growth rate coefficients of industrialization and population respectively due to their interaction. r_1 is the intrinsic growth rate coefficient of industrialization. α_1, α_2 and β_1, β_2 are the depletion rate coefficients of primary and secondary toxicants due to resource biomass and population respectively. δ_0 and δ_1 are the natural washout rate coefficients of the primary and secondary toxicants respectively from the environment .The constant $\theta \leq 1$, is a fraction, which represent the magnitude of transformation of primary toxicant into secondary toxicant.

In model (2.1), the function $r_B(N)$ denotes the specific growth rate of resource biomass which decreases as N increases. Hence we take

$$r_B(0) = r_{B0} > 0, \quad r_B{}'(N) \leq 0 \quad \text{for } N \geq 0. \qquad (2.2)$$

The function $K_B(P_1,P_2)$ represent the maximum density of resource biomass which the environment can support in the presence of primary and secondary toxicants, and it also decreases as P_1 and P_2 increases. Hence we take

$$K_B(0,0) = K_{B0} > 0, \quad \frac{\partial K_B(P_1,P_2)}{\partial P_1} < 0, \quad \frac{\partial K_B(P_1,P_2)}{\partial P_2} < 0 \quad \text{for } P_1 \geq 0,\ P_2 \geq 0. \qquad (2.3)$$

The function $r_P(B)$ denotes the growth rate coefficient of the population and it increases as the resource biomass density increases. Hence we take

$$r_P(0) = r_{P0} > 0, \quad r_P{}'(B) \geq 0 \quad \text{for } B \geq 0. \qquad (2.4)$$

The function $M(P_1,P_2)$ represent the maximum density of population which the environment can support in the presence of primary and secondary toxicants, and it also decreases as P_1 and P_2 increases. Hence we take

$$M(0,0)=M_0>0, \quad \frac{\partial M(P_1,P_2)}{\partial P_1}<0, \quad \frac{\partial M(P_1,P_2)}{\partial P_2}<0 \tag{2.6}$$

for $P_1 \geq 0, P_2 \geq 0$.

The function $Q(I,N)$ is the rate of introduction of toxicant into the environment which increases as I and N increase. Hence we take

$$Q(0,0)=Q_0 \geq 0, \ \frac{\partial Q(I,N)}{\partial I} \geq 0, \ \frac{\partial Q(I,N)}{\partial N} \geq 0 \text{ for } I \geq 0, \ N \geq 0. \tag{2.7}$$

Before analyzing the model we state and prove the following lemma corresponding to the region of attraction for solution of model (2.1).

Lemma (2.1): The set $\Omega = \{(B,N,P_1,P_2,I): 0 \leq B \leq K_{B0}, 0 \leq N \leq N_m, 0 \leq P_1+P_2 \leq Q_m, 0 \leq I \leq L_a\}$ is the region of attraction for all solutions of model (2.1) initiating in the interior of positive orthant, where $Q_m = \frac{Q(L_a,N_m)}{\delta}$, $\delta = \min(\delta_0+g-\theta g, \delta_1)$.

3. EQUILIBRIUM ANALYSIS

The system (2.1) may have eight nonnegative equilibrium in the B,N,P_1,P_2,I space namely, $E_1\left(0,0,\frac{Q_0}{\delta_0+g},\frac{\theta Q_0 g}{\delta_1(\delta_0+g)},0\right)$, $E_2\left(0,0,\frac{Q_0}{\delta_0+g},\frac{\theta Q_0 g}{\delta_1(\delta_0+g)},L\right)$, $E_3(0,\tilde{N},\tilde{P}_1,\tilde{P}_2,0)$, $E_4\left(0,\tilde{\tilde{N}},\tilde{\tilde{P}}_1,\tilde{\tilde{P}}_2,\tilde{\tilde{I}}\right)$, $E_5(\hat{B},0,\hat{P}_1,\hat{P}_2,0)$, $E_6\left(\hat{\hat{B}},\hat{\hat{N}},\hat{\hat{P}}_1,\hat{\hat{P}}_2,0\right)$, $E_7(\breve{B},0,\breve{P}_1,\breve{P}_2,\breve{I})$, $E^*(B^*,N^*,P_1^*,P_2,I^*)$.

The existence of E_1 and E_2 is obvious. We prove the existence of other equilibrium points.

Existence of $E_3(0,\tilde{N},\tilde{P}_1,\tilde{P}_2,0)$:

In this case, $\tilde{N},\tilde{P}_1$ and $\tilde{P}_2$ are the positive solutions of the following equations:

$$N=M(P_1,P_2), \tag{3.1}$$

$$Q(0,N)-\delta_0 P_1-\alpha_2 NP_1-gP_1=0, \tag{3.2}$$

$$\theta gP_1-\delta_1 P_2-\beta_2 NP_2=0. \tag{3.3}$$

From equations (3.2) and (3.3), respectively we get

$$P_1=\frac{Q(0,N)}{\delta_1+\alpha_2 N+g}=f_1(N), \text{ say,} \tag{3.4}$$

and

$$P_2=\frac{\theta g f_1(N)}{\delta_1+\beta_2 N}=f_2(N), \text{ say.} \tag{3.5}$$

It is noted that from equation (3.4) and (3.5) that P_1 and P_2, are the functions of N only. To show the existence of E_3, we define a function $F_1(N)$ from equation (3.1), after using (3.4) and (3.5) as follows

$$F_1(N)=N-M(f_1(N),f_2(N)). \tag{3.6}$$

From equation (3.6), we note that

$$F_1(0)=-M(f_1(0),f_2(0))<0.$$

Also from (3.6), we note that

$$F_1(N_m)=N_m-M(f_1(N_m),f_2(N_m))>0,$$

under the condition, $N_m-M(f_1(N_m),f_2(N_m))>0$, (3.7)

Thus there exists a root $\tilde{N}$ in the interval $0<\tilde{N}<N_m$ given by

$$F_1(\tilde{N})=0. \tag{3.8}$$

Now, the sufficient condition for E_3 to be unique is $\frac{dF_1}{dN}>0$ at $\tilde{N}$, where

$$\frac{dF_1}{dN}=1-\left(\frac{\partial M}{\partial P_1}\frac{df_1}{dN}+\frac{\partial M}{\partial P_2}\frac{df_2}{dN}\right). \tag{3.9}$$

From (3.9), we note that $\frac{dF_1}{dN}>0$ at $\tilde{N}$, if $\frac{\partial M}{\partial P_1}\frac{df_1}{dN}+\frac{\partial M}{\partial P_2}\frac{df_2}{dN}<1$ with this value of $\tilde{N}$, value of $\tilde{P}_1$ and $\tilde{P}_2$ can be found from equation (3.4) and (3.5) and is positive since $\frac{\partial M}{\partial P_1}\frac{df_1}{dN}+\frac{\partial M}{\partial P_2}\frac{df_2}{dN}<1$.

Existence of $E_4\left(0,\tilde{\tilde{N}},\tilde{\tilde{P}}_1,\tilde{\tilde{P}}_2,\tilde{\tilde{I}}\right)$:

In this case $\tilde{\tilde{N}},\tilde{\tilde{P}}_1,\tilde{\tilde{P}}_2$ and $\tilde{\tilde{I}}$ are the solutions of the following equations:

$$r_{P0}-\frac{r_{P0}N}{M(P_1,P_2)}+\gamma_1 I=0, \tag{3.10}$$

$$Q(I,N)-\delta_0 P_1-\alpha_2 NP_1-gP_1=0, \tag{3.11}$$

$$\theta gP_1-\delta_1 P\ -\beta_2 NP_2=0, \tag{3.12}$$

$$I=\frac{L(r+\gamma_2 N)}{r}=g_1(N),\ \text{say} \tag{3.13}$$

Using the value of I, from equation (3.13) in equations (3.11) and (3.12) we obtain

$$P_1=\frac{Q(g_1(N),N)}{\delta_0+\alpha_2 N+g}=g_2(N),\ \text{say}, \tag{3.14}$$

$$P_2=\frac{\theta gQ_2(N)}{\delta_1+\beta_2 N}=g_3(N),\ \text{say}. \tag{3.15}$$

It is noted from equations (3.13), (3.14) and (3.15) that I, P_1 and P_2, are the functions of N, only. To show the existence of E_4, we define a function $F_2(N)$ from equation (3.10), after using (3.13), (3.14) and (3.15) as follows

$$F_2(N)=r_{P0}N-(r_{P0}+\gamma_1 g_1(N))M(g_2(N),g_3(N)). \tag{3.16}$$

From equation (3.16), we note that

$$F_2(0)=-(r_{P0}+\gamma_1 L)M\left(\frac{Q(L,0)}{\delta_0+g},\frac{\theta gQ(L,0)}{\delta_1(\delta_0+g)}\right)<0.$$

Also from (3.16), we note that

$$F_2(N_m)=r_{P0}N_m-(r_{P0}+\gamma_1 g_1(N_m))M(g_2(N_m),g_3(N_m))>0.$$

under the condition, $r_{P0}N_m>(r_{P0}+\gamma_1 g_1(N_m))M(g_2(N_m),g_3(N_m))$. (3.17)

Thus there exists a root $\tilde{\tilde{N}}$ in the interval $0<\tilde{\tilde{N}}<N_m$, given by

$$F_2\left(\tilde{\tilde{N}}\right)=0. \tag{3.18}$$

Now, the sufficient condition for E_4 to be unique is $\frac{dF_2}{dN}>0$ at $\tilde{\tilde{N}}$, where

$$\frac{dF_2}{dN}=r_{P0}-L\frac{\gamma_1\gamma_2}{r}M(g_2(N),g_3(N))-(r_{P0}+\gamma_1 g_1(N))\left(\frac{\partial M}{\partial P_1}\frac{dg_2}{dN}+\frac{\partial M}{\partial P_2}\frac{dg_3}{dN}\right). \tag{3.19}$$

From (3.19), we note that $\frac{dF_2}{dN}>0$ at $\tilde{\tilde{N}}$, if

$$r_{P0}-L\frac{\gamma_1\gamma_2}{r}M(g_2(N),g_3(N))-(r_{P0}+\gamma_1 g_1(N))\left(\frac{\partial M}{\partial P_1}\frac{dg_2}{dN}+\frac{\partial M}{\partial P_2}\frac{dg_3}{dN}\right)>0. \tag{3.20}$$

With this value of $\tilde{\tilde{N}}$, value of $\tilde{\tilde{I}},\tilde{\tilde{P}}_1$ and $\tilde{\tilde{P}}_2$, can be found from equation (3.13), (3.14) and (3.15) and is positive since condition (3.20) is satisfied.

Existence of $E_5\left(\hat{B},0,\hat{P}_1,\hat{P}_2,0\right)$:

In this case $\hat{B},\hat{P}_1,\hat{P}_2$ are the solutions of the following equations

$$B=K_B(P_1,P_2), \tag{3.21}$$

$$P_1=\frac{Q_0}{\delta_0+\alpha_1 B+g}=h_1(B),\ \text{say}, \tag{3.22}$$

$$P_2=\frac{\theta gh_1(B)}{\delta_1+\beta_1 B}=h_2(B),\ \text{say}, \tag{3.23}$$

It is noted from equations (3.22) and (3.23) that P_1 and P_2, are functions of B only. To show the existence of E_5, we define a function $F_3(B)$ from equation (3.21), after using (3.22) and (3.23) as follows

$$F_3(B)=B-K_B.(h_1(B),h_2(B)) \tag{3.24}$$

From equation (3.24), we note that

$$F_3(0) = -K_B\left(\frac{Q_0}{\delta_0 + g}, \frac{\theta g Q_0}{\delta_1(\delta_0 + g)}\right) < 0.$$

Also from (3.24), we note that

$$F_3(K_{B0}) = K_{B0} - K_B(h_1(K_{B0}), h_2(K_{B0})) > 0, \text{ under the conditions}$$

$$K_{B0} > K_B(h_1(K_{B0}), h_2(K_{B0})). \tag{3.25}$$

Thus there exists a root $\hat{B}$, in the interval $0 < \hat{B} < K_{B0}$, given by

$$F_3(\hat{B}) = 0. \tag{3.26}$$

Now, the sufficient condition for E_5 to be unique is $\frac{dF_3}{dB} > 0$ at $\hat{B}$, where

$$\frac{dF_3}{dB} = 1 - \left(\frac{\partial K_B}{\partial P_1}\frac{dh_1}{dB} + \frac{\partial K_B}{\partial P_2}\frac{dh_2}{dB}\right). \tag{3.27}$$

From (3.27), we note that $\frac{dF_3}{dB} > 0$ at $\hat{B}$, if $\left(\frac{\partial K_B}{\partial P_1}\frac{dh_1}{dB} + \frac{\partial K_B}{\partial P_2}\frac{dh_2}{dB}\right) < 1.$

With this value of $\hat{B}$, value of $\hat{P}_1$ and $\hat{P}_2$, can be found from equations (3.22) and (3.23) and is positive since $\left(\frac{\partial K_B}{\partial P_1}\frac{dh_1}{dB} + \frac{\partial K_B}{\partial P_2}\frac{dh_2}{dB}\right) < 1.$

Existence of $E_6\left(\hat{\hat{B}}, \hat{\hat{N}}, \hat{\hat{P}}_1, \hat{\hat{P}}_2, 0\right)$:

In this case, $\hat{\hat{B}}, \hat{\hat{N}}, \hat{\hat{P}}_1, \hat{\hat{P}}_2$ are the solutions of the following equations:

$$r_B(N) - \frac{r_{B0}B}{K_B(P_1, P_2)} = 0, \tag{3.28}$$

$$r_P(B) - \frac{r_{P0}N}{M(P_1, P_2)} = 0, \tag{3.29}$$

$$Q(0, N) - \delta_0 P_1 - \alpha_1 B P_1 - \alpha_2 N P_1 - g P_1 = 0, \tag{3.30}$$

$$\theta g P_1 - \delta_1 P_2 - \beta_1 B P_2 - \beta_2 N P_2 = 0. \tag{3.31}$$

From the equation (3.30), we have

$$P_1 = \frac{Q(0, N)}{\delta_0 + \alpha_1 B + \alpha_2 N + g} = d_1(B, N), \quad \text{say,} \tag{3.32}$$

With this value of P_1, and from the equation (3.31), we have

$$P_2 = \frac{\theta g}{(\delta_1 + \beta_1 B + \beta_2 N)} \frac{Q(0, N)}{(\delta_0 + \alpha_1 B + \alpha_2 N + g)} = d_2(B, N), \quad \text{say,} \tag{3.33}$$

Using values of P_1 and P_2 from (3.32) and (3.33) in equations (3.28) and (3.29) respectively, we get

$$(r_{B0} - r_{B1}N)(K_{B0} - K_{B1}d_1(B, N) - K_{B2}d_2(B, N)) - r_{B0}B = 0, \tag{3.34}$$

$$(r_{P0} + r_{P1}B)(M_0 - M_1 d_1(B, N) - M_2 d_2(B, N)) - r_{P0}N = 0, \tag{3.35}$$

From (3.34), we note that $\frac{dN}{dB} > 0$, if

$$r_{B0} + r_B(N)\left(K_{B1}\frac{\partial d_1}{\partial B} + K_{B2}\frac{\partial d_2}{\partial B}\right) < 0, \text{ and}$$

$$r_{B1}K_B(d_1(B, N), d_2(B, N)) + r_B(N)\left(K_{B1}\frac{\partial d_1}{\partial N} + K_{B2}\frac{\partial d_2}{\partial N}\right) > 0.$$

From (3.35), we note that $\frac{dN}{dB} < 0$, if

$$r_P(B)\left(M_1\frac{\partial d_1}{\partial B} + M_2\frac{\partial d_2}{\partial B}\right) > r_{P1}M(d_1(B, N), d_2(B, N)), \quad \text{and}$$

$$r_{P0} + r_P(B)\left(M_1\frac{\partial d_1}{\partial N} + M_2\frac{\partial d_2}{\partial N}\right) > 0.$$

Thus the two isoclines (3.34) and (3.35) intersects at $\hat{\hat{B}}$ and $\hat{\hat{N}}$ provided

$$r_{B0} + r_B(N)\left(K_{B1}\frac{\partial d_1}{\partial B} + K_{B2}\frac{\partial d_2}{\partial B}\right) < 0,$$

$$r_{B1}K_B(d_1(B,N), d_2(B,N)) + r_B(N)\left(K_{B1}\frac{\partial d_1}{\partial N} + K_{B2}\frac{\partial d_2}{\partial N}\right) > 0.$$

$$r_P(B)\left(M_1\frac{\partial d_1}{\partial B} + M_2\frac{\partial d_2}{\partial B}\right) > r_{P1}M(d_1(B,N), d_2(B,N)),$$

$$r_{P0} + r_P(B)\left(M_1\frac{\partial d_1}{\partial N} + M_2\frac{\partial d_2}{\partial N}\right) > 0.$$

Using these values of $\hat{\hat{B}}$ and $\hat{\hat{N}}$ we get $\hat{\hat{P}}_1$ and $\hat{\hat{P}}_2$ from (3.32) and (3.33), respectively as follows

$$P_1 = \frac{Q(0,N)}{\delta_0 + \alpha_1 B + \alpha_2 N + g}, \quad \text{and}$$

$$P_2 = \frac{\theta g}{(\delta_1 + \beta_1 B + \beta_2 N)}\frac{Q(0,N)}{(\delta_0 + \alpha_1 B + \alpha_2 N + g)}.$$

Existence of $E_7(\breve{B}, 0, \breve{P}_1, \breve{P}_2, \breve{I})$:

In this case $\breve{B}, \breve{P}_1, \breve{P}_2, \breve{I}$ are the solutions of the following equations

$$r_{B0} - \frac{r_{B0}B}{K_B(P_1, P_2)} - \alpha I = 0, \tag{3.36}$$

$$P_1 = \frac{Q(e_1(B), 0)}{\delta_0 + \alpha_1 B + g} = e_2(B), \quad \text{say}, \tag{3.37}$$

$$P_2 = \frac{\theta g e_2(B)}{\delta_1 + \beta_1 B} = e_3(B), \quad \text{say}, \tag{3.38}$$

$$I = L\left(1 + \frac{\beta B}{r_1}\right) = e_1(B), \quad \text{say}. \tag{3.39}$$

It is noted from equations (3.37), (3.38) and (3.39) that P_1, P_2 and I are functions of B only. To show the existence of E_7, we define a function $F_5(B)$ from equation (3.36), after using (3.37), (3.38) and (3.39) as follows

$$F_5(B) = r_{B0}B - (r_{B0} - \alpha e_1(B))K_B.(e_2(B), e_3(B)) \tag{3.40}$$

From equation (3.40), we note that

$$F_5(0) = -(r_{B0} - \alpha L)K_B\left(\frac{Q(L,0)}{\delta_0 + g}, \frac{\theta g Q(L,0)}{\delta_1(\delta_0 + g)}\right) < 0.$$

Also from (3.40), we note that

$$F_5(K_{B0}) = r_{B0}K_{B0} - (r_{B0} - \alpha e_1(K_{B0}))K_B(e_2(K_{B0}), e_3(K_{B0})) > 0.$$

under the conditions

$$r_{B0}K_{B0} > (r_{B0} - \alpha e_1(K_{B0}))K_B(e_2(K_{B0}), e_3(K_{B0})). \tag{3.41}$$

Thus there exists a root $\breve{B}$, in the interval $0 < \breve{B} < K_{B0}$, given by

$$F_3(\breve{B}) = 0. \tag{3.42}$$

Now, the sufficient condition for E_7 to be unique is $\frac{dF_5}{dB} > 0$ at $\breve{B}$, where

$$\frac{dF_5}{dB} = r_{B0} + \alpha e_1'(B)K_B(e_2(B), e_3(B)) - (r_{B0} - \alpha e_1(B))\left(\frac{\partial K_B}{\partial P_1}\frac{de_2}{dB} + \frac{\partial K_B}{\partial P_2}\frac{de_3}{dB}\right). \tag{3.43}$$

From (3.43), we note that $\frac{dF_5}{dB} > 0$ at $\breve{B}$, if

$$r_{B0} + \alpha e_1'(B)K_B(e_2(B), e_3(B)) > (r_{B0} - \alpha e_1(B))\left(\frac{\partial K_B}{\partial P_1}\frac{de_2}{dB} + \frac{\partial K_B}{\partial P_2}\frac{de_3}{dB}\right). \tag{3.44}$$

With this value of $\breve{B}$, value of $\breve{P}_1, \breve{P}_2$ and $\breve{I}$ can be found from equations (3.37), (3.38) and (3.39) and is positive since condition (3.44) is satisfied.

Existence of $E^*(B^*, N^*, P_1^*, P_2^*, I^*)$:

In this case, $B^*, N^*, P_1^*, P_2^*, I^*$ are the solutions of following equations:

$$r_B(N) - \frac{r_{B0} B}{K_B(P_1, P_2)} - \alpha I = 0, \tag{3.45}$$

$$r_P(B) - \frac{r_{P0} N}{M(P_1, P_2)} + \gamma_1 I = 0, \tag{3.46}$$

$$Q(I, N) - \delta_0 P_1 - \alpha_1 B P_1 - \alpha_2 N P_1 - g P_1 = 0, \tag{3.47}$$

$$\theta g P_1 - \delta_1 P_2 - \beta_1 B P_2 - \beta_2 N P_2 = 0, \tag{3.48}$$

$$r_1\left(1 - \frac{I}{L}\right) + \beta B + \gamma_2 N = 0. \tag{3.49}$$

From the equation (3.49), we have

$$I = \frac{L}{r_1}\left(r_1 + \beta B + \gamma_2 N\right) = s_1(B, N), \quad \text{say,} \tag{3.50}$$

With this value of I, and from the equation (3.47) and (3.48), we have

$$P_1 = \frac{Q\left(s_1(B, N), N\right)}{\left(\delta_0 + \alpha_1 B + \alpha_2 N + g\right)} = s_2(B, N), \quad \text{say,} \tag{3.51}$$

$$P_2 = \frac{\theta g s_2(B, N)}{\left(\delta_1 + \beta_1 B + \beta_2 N\right)} = s_3(B, N), \quad \text{say,} \tag{3.52}$$

Using values of I, P_1 and P_2 from (3.50), (3.51) and (3.52) in equations (3.45) and (3.46) respectively, we get

$$\left(r_{B0} - r_{B1} N - \alpha s_1(B, N)\right)\left(K_{B0} - K_{B1} s_2(B, N) - K_{B2} s_3(B, N)\right) - r_{B0} B = 0, \tag{3.53}$$

$$\left(r_{P0} + r_{P1} N + \gamma_1 s_1(B, N)\right)\left(M_0 - M_1 s_2(B, N) - M_2 s_3(B, N)\right) - r_{P0} N = 0, \tag{3.54}$$

From (3.53), we note that $\frac{dN}{dB} < 0$, if

$$\alpha \frac{\partial s_1}{\partial B} K_B\left(s_2(B, N), s_3(B, N)\right) + \left(r_B(N) - \alpha s_1(B, N)\right)\left(K_{B1}\frac{\partial s_2}{\partial B} + K_{B2}\frac{\partial s_3}{\partial B}\right) + r_{B0} > 0, \text{ and}$$

$$\left(r_{B1} + \alpha \frac{\partial s_1}{\partial N}\right) K_B\left(s_2(B, N), s_3(B, N)\right) + \left(r_B(N) - \alpha s_1(B, N)\right)\left(K_{B1}\frac{\partial s_2}{\partial N} + K_{B2}\frac{\partial s_3}{\partial N}\right) > 0,$$

From (3.54), we note that $\frac{dN}{dB} > 0$, if

$$\left(-r_{P1} - \gamma_1 \frac{\partial s_1}{\partial B}\right) M\left(s_2(B, N), s_3(B, N)\right) + \left(r_P(B) + \gamma_1 s_1(B, N)\right)\left(M_1 \frac{\partial s_2}{\partial B} + M_2 \frac{\partial s_3}{\partial B}\right) > 0, \quad \text{and}$$

$$\gamma_1 \frac{\partial s_1}{\partial N} M\left(s_2(B, N), s_3(B, N)\right) - \left(r_P(B) + \gamma_1 s_1(B, N)\right)\left(M_1 \frac{\partial s_2}{\partial N} + M_2 \frac{\partial s_3}{\partial N}\right) - r_{P0} > 0.$$

Thus the two isoclines (3.53) and (3.54) intersects at B^* and N^* provided

$$\alpha \frac{\partial s_1}{\partial B} K_B\left(s_2(B, N), s_3(B, N)\right) + \left(r_B(N) - \alpha s_1(B, N)\right)\left(K_{B1}\frac{\partial s_2}{\partial B} + K_{B2}\frac{\partial s_3}{\partial B}\right) + r_{B0} > 0,$$

$$\left(r_{B1} + \alpha \frac{\partial s_1}{\partial N}\right) K_B\left(s_2(B, N), s_3(B, N)\right) + \left(r_B(N) - \alpha s_1(B, N)\right)\left(K_{B1}\frac{\partial s_2}{\partial N} + K_{B2}\frac{\partial s_3}{\partial N}\right) > 0,$$

$$\left(-r_{P1} - \gamma_1 \frac{\partial s_1}{\partial B}\right) M\left(s_2(B, N), s_3(B, N)\right) + \left(r_P(B) + \gamma_1 s_1(B, N)\right)\left(M_1 \frac{\partial s_2}{\partial B} + M_2 \frac{\partial s_3}{\partial B}\right) > 0,$$

$$\gamma_1 \frac{\partial s_1}{\partial N} M\left(s_2(B, N), s_3(B, N)\right) - \left(r_P(B) + \gamma_1 s_1(B, N)\right)\left(M_1 \frac{\partial s_2}{\partial N} + M_2 \frac{\partial s_3}{\partial N}\right) - r_{P0} > 0.$$

Using these values of B^* and N^* we get P_1^*, P_2^* and I^* from (3.50), (3.51) and (3.52), respectively as follows

$$I = \frac{L}{r_1}\left(r_1 + \beta B + \gamma_2 N\right), \qquad P_1 = \frac{Q\left(s_1(B,N),N\right)}{\left(\delta_0 + \alpha_1 B + \alpha_2 N + g\right)}, \qquad P_2 = \frac{\theta g s_2(B,N)}{\left(\delta_1 + \beta_1 B + \beta_2 N\right)}.$$

4. STABILITY ANALYSIS

4.1 Local Stability

The local stability behavior of each equilibrium point can be studied by computing the corresponding variational matrix. From these matrices we note the following.

1. E_1 is also a saddle point with stable manifold locally in the $P_1 - P_2$ plane and with unstable manifold locally in the $B - N - I$ space.
2. E_2 is a saddle point with stable manifold locally in the $P_1 - P_2 - I$ space and with unstable manifold locally in the $B - N$ plane.
3. E_3 is a saddle point with stable manifold locally in the $N - P_1 - P_2$ space and with unstable manifold locally in the $B - I$ plane.
4. E_4 is a saddle point with stable manifold locally in the $N - P_1 - P_2 - I$ space and with unstable manifold locally in the B direction.
5. E_5 is a saddle point with stable manifold locally in the $B - P_1 - P_2$ space and with unstable manifold locally in the $N - I$ plane.
6. E_6 is a saddle point with stable manifold locally in the $B - N - P_1 - P_2$ space and with unstable manifold locally in the I direction.
7. E_7 is a saddle point with stable manifold locally in the $B - P_1 - P_2 - I$ space and with unstable manifold locally in the N direction.

In the following theorem we show that E^* is locally asymptotically stable:

Theorem 1: If the following inequalities hold

$$r_{P1}N^* + \alpha_1 P_1^* + \beta_1 P_2^* + \beta I^* < \frac{r_{B0}B^*}{K_B(P_1^*, P_2^*)}, \tag{4.1}$$

$$r_{B1}B^* + Q_2 - \alpha_2 P_1^* + \beta_2 P_2^* + \gamma_2 I^* < \frac{r_{P0}N^*}{M(P_1^*, P_2^*)}, \tag{4.2}$$

$$\frac{K_{B1}}{K_B^2(P_1^*, P_2^*)} r_{B0} B^{*2} + \frac{M_1}{M^2(P_1^*, P_2^*)} r_{P0} N^{*2} + \theta g < \frac{Q(I^*, N^*)}{P_1^*}, \tag{4.3}$$

$$\frac{K_{B2}}{K_B^2(P_1^*, P_2^*)} r_{B0} B^{*2} + \frac{M_2}{M^2(P_1^*, P_2^*)} r_{P0} N^{*2} < \frac{\theta g P_1^*}{P_2^*}, \tag{4.4}$$

$$\alpha B^* + \gamma_1 N^* + Q_1 < \frac{r_1 I^*}{L}. \tag{4.5}$$

then E^* is locally asymptotically stable.

Proof: If inequalities (4.1) – (4.5) hold, then by Gerschgorin's theorem (Lancaster and Tismenetsky, 1985), all eigenvalues of $V(E^*)$ have negative real parts and interior equilibrium E^* is locally asymptotically stable.

4.2. GLOBAL STABILITY

Theorem 2: In addition to the assumption (2.2) – (2.7), let $r_B(N)$, $r_P(B)$, $K_B(P_1, P_2)$, $M(P_1, P_2)$ and $Q(I, N)$ satisfy the conditions

$$0 \le -r'_B(N) \le \rho_1, \quad 0 \le -r'_P(B) \le \rho_2, M_n \le M(P_1,P_2) \le M_0, \quad K_m \le K_B(P_1,P_2) \le K_{B0}, 0 \le \frac{\partial Q}{\partial I} \le \rho_3, 0 \le \frac{\partial Q}{\partial N} \le \rho_4,$$

$$0 \le -\frac{\partial K_B}{\partial P_1} \le k_1, 0 \le -\frac{\partial K_B}{\partial P_2} \le k_2, 0 \le -\frac{\partial M}{\partial P_1} \le m_1, 0 \le -\frac{\partial M}{\partial P_2} \le m_2. \tag{4.6}$$

in Ω for some positive constants $\rho_1, \rho_2, \rho_3, \rho_4, k_1, k_2, K_0, K_m, M_0, M_n, m_1, m_2$ Then if the following inequalities hol

$$(\rho_1 + \rho_2)^2 < \frac{1}{4} \frac{r_{B0}}{K_B(P_1^*, P_2^*)} \frac{r_{P0}}{M(P_1^*, P_2^*)}, \tag{4.7}$$

$$\left(\alpha_1 Q_m + r_{B0} K_{B0} \frac{k_1}{K_m^{\;2}}\right)^2 < \frac{1}{4} \frac{r_{B0}}{K_B(P_1^*, P_2^*)} (\delta_0 + g + \alpha_1 B^* + \alpha_2 N^*), \tag{4.8}$$

$$\left(\beta_1 Q_m + r_{B0} K_{B0} \frac{k_2}{K_m^{\;2}}\right)^2 < \frac{1}{3} \frac{r_{B0}}{K_B(P_1^*, P_2^*)} (\delta_1 + \beta_1 B^* + \beta_2 N^*), \tag{4.9}$$

$$(\beta + \alpha)^2 < \frac{1}{3} \frac{r_{B0}}{K_B(P_1^*, P_2^*)} \frac{r_1}{L}, \tag{4.10}$$

$$\left(\rho_4 + \alpha_2 Q_m + r_{P0} N_m \frac{m_1}{M_n^{\;2}}\right)^2 < \frac{1}{4} \frac{r_{P0}}{M(P_1^*, P_2^*)} (\delta_0 + g + \alpha_1 B^* \alpha_2 N^*), \tag{4.11}$$

$$\left(\beta_2 Q_m + r_{P0} N_m \frac{m_2}{M_n^{\;2}}\right)^2 < \frac{1}{4} \frac{r_{P0}}{M(P_1^*, P_2^*)} (\delta_1 + \beta_1 B^* \beta_2 N^*), \tag{4.12}$$

$$(\gamma_1 + \gamma_2)^2 < \frac{1}{3} \frac{r_{P0}}{M(P_1^*, P_2^*)} \frac{r_1}{L}, \tag{4.13}$$

$$(\theta g)^2 < \frac{1}{3} (\delta_1 + \beta_1 B^* + \beta_2 N^*)(\delta_0 + g + \alpha_1 B^* \alpha_2 N^*), \tag{4.14}$$

$$\rho_3^{\;2} < \frac{1}{3} \frac{r_1}{L} (\delta_0 + g + \alpha_1 B^* \alpha_2 N^*), 1 \tag{4.15}$$

E^* is globally asymptotically stable with respect to all solutions initiating in the positive orthant Ω.

Proof: Consider the following positive definite function about E^*

$$V(B, N, P_1, P_2, I) = \left(B - B^* - B^* \ln \frac{B}{B^*}\right) + \left(N - N^* - N^* \ln \frac{N}{N^*}\right) + \frac{1}{2}(P_1 - P_1^*)^2 + \frac{1}{2}(P_2 - P_2^*)^2 + \left(I - I^* - I^* \ln \frac{I}{I^*}\right).$$

Differentiating V with respect to time t, we get

$$\frac{dV}{dt} = \left(\frac{B - B^*}{B}\right)\frac{dB}{dt} + \left(\frac{N - N^*}{N}\right)\frac{dN}{dt} + (P_1 - P_1^*)\frac{dP_1}{dt} + (P_2 - P_2^*)\frac{dP_2}{dt} + \left(\frac{I - I^*}{I}\right)\frac{dI}{dt}.$$

Substituting values of $\frac{dB}{dt}, \frac{dN}{dt}, \frac{dP_1}{dt}, \frac{dP_2}{dt}$ and $\frac{dW}{dt}$ from the system of equation (2.1) in the above equation and after doing some algebraic manipulations and considering functions,

$$\eta_B(N) = \begin{cases} \frac{r_B(N) - r_B(N^*)}{N - N^*}, & , N \neq N^*, \\ r_B'(N^*), & , N = N^* \end{cases} \tag{4.16}$$

$$\eta_P(B) = \begin{cases} \frac{r_P(B) - r_P(B^*)}{B - B^*}, & , B \neq B^*, \\ r_P{}'(B^*), & , B = B^* \end{cases} \tag{4.17}$$

$$\eta_{Q1}(I, N) = \begin{cases} \frac{Q(I, N) - Q(I^*, N)}{I - I^*}, & , I \neq I^*, \\ \frac{\partial Q(I^*, N)}{\partial I}, & , I = I^*, \end{cases} \tag{4.18}$$

$$\xi_{B1}(P_1, P_2) = \begin{cases} \dfrac{\frac{1}{K_B(P_1, P_2)} - \frac{1}{K_B(P_1^*, P_2)}}{P_1 - P_1^*}, & , P_1 \neq P_1^*, \\ -\dfrac{1}{K_B^{\;2}(P_1^*, P_2)} \dfrac{\partial K_B(P_1^*, P_2)}{\partial P_1}, & , P_1 = P_1^*, \end{cases} \tag{4.19}$$

$$\xi_{B2}(P_1^*, P_2) = \begin{cases} \dfrac{\frac{1}{K_B(P_1^*, P_2)} - \frac{1}{K_B(P_1^*, P_2^*)}}{P_2 - P_2^*}, & , P_2 \neq P_2^*, \\ -\dfrac{1}{K_B^{\;2}(P_1^*, P_2^*)} \dfrac{\partial K_B(P_1^*, P_2^*)}{\partial P_2}, & , P_2 = P_2^* \end{cases} \tag{4.20}$$

$$\tau_{P1}(P_1,P_2)=\begin{cases}\dfrac{\dfrac{1}{M(P_1,P_2)}-\dfrac{1}{M(P_1*,P_2)}}{P_1-P_1*}, & ,P_1\neq P_1*,\\ -\dfrac{1}{M^2(P_1*,P_2)}\dfrac{\partial M(P_1*,P_2*)}{\partial P_1}, & ,P_1=P_1*\end{cases} \quad (4.21)$$

$$\tau_{P2}(P_1*,P_2)=\begin{cases}\dfrac{\dfrac{1}{M(P_1*,P_2)}-\dfrac{1}{M(P_1*,P_2*)}}{P_2-P_2*}, & ,P_2\neq P_2*,\\ -\dfrac{1}{M^2(P_1*,P_2*)}\dfrac{\partial M(P_1*,P_2*)}{\partial P_2}, & ,P_2=P_2*\end{cases} \quad (4.22)$$

$$\eta_{Q2}(I*,N)=\begin{cases}\dfrac{Q(I*,N)-Q(I*,N*)}{N-N*}, & ,N\neq N*,\\ \dfrac{\partial Q(I*,N*)}{\partial N}, & ,N=N*\end{cases} \quad (4.23)$$

we get

$$\begin{aligned}\frac{dV}{dt}&=-\frac{1}{4}a_{11}(B-B*)^2+a_{12}(B-B*)(N-N*)-\frac{1}{4}a_{22}(N-N*)^2\\ &=-\frac{1}{4}a_{11}(B-B*)^2+a_{13}(B-B*)(P_1-P_1*)-\frac{1}{4}a_{33}(P_1-P_1*)^2\\ &=-\frac{1}{4}a_{11}(B-B*)^2+a_{14}(B-B*)(P_2-P_2*)-\frac{1}{3}a_{44}(P_2-P_2*)^2\\ &=-\frac{1}{4}a_{11}(B-B*)^2+a_{15}(B-B*)(I-I*)-\frac{1}{3}a_{55}(I-I*)^2\\ &=-\frac{1}{4}a_{22}(N-N*)^2+a_{23}(N-N*)(P_1-P_1*)-\frac{1}{4}a_{33}(P_1-P_1*)^2\\ &=-\frac{1}{4}a_{22}(N-N*)^2+a_{24}(N-N*)(P_2-P_2*)-\frac{1}{3}a_{44}(P_2-P_2*)^2\\ &=-\frac{1}{4}a_{22}(N-N*)^2+a_{25}(N-N*)(I-I*)-\frac{1}{3}a_{55}(I-I*)^2,\\ &=-\frac{1}{4}a_{33}(P_1-P_1*)+a_{34}(P_1-P_1*)(P_2-P_2*)-\frac{1}{3}a_{44}(P_2-P_2*)^2\\ &=-\frac{1}{4}a_{33}(P_1-P_1*)^2+a_{35}(P_1-P_1*)(I-I*)-\frac{1}{3}a_{55}(I-I*)^2.\end{aligned}$$

where

$a_{11}=\dfrac{r_{B0}}{K_B(P_1*,P_2*)}$, $a_{12}=\eta_B(N)+\eta_P(B)$, $a_{22}=\dfrac{r_{P0}}{M(P_1*,P_2*)}$, $a_{23}=-r_{P0}N\tau_{P1}(P_1,P_2)$, $a_{33}=\delta_0+g+\alpha_1B*+\alpha_2N*$, ,

$a_{13}=-\alpha_1P_1-r_{B0}B\xi_{B1}(P_1,P_2)$, $a_{14}=-\beta_1P_2-r_{B0}B\xi_{B2}(P_1*,P_2)$, $a_{34}=\theta g$, $a_{44}=\delta_1+\beta_1B*+\beta_2N*$, $a_{55}=\dfrac{r_1}{L}$, $a_{15}=-\alpha+\beta$,

$a_{24}=-r_{P0}N\tau_{P2}(P_1*,P_2)-\beta_2P_2$, $a_{25}=\gamma_1+\gamma_2$, $a_{35}=\eta_{Q1}(I,N)$

Then sufficient conditions for $\dfrac{dV}{dt}$ to be negative definite are that the following inequalities hold

$$a_{12}^2<\frac{1}{4}a_{11}a_{22},\quad a_{13}^2<\frac{1}{4}a_{11}a_{33},\quad a_{14}^2<\frac{1}{3}a_{11}a_{44},\quad a_{15}^2<\frac{1}{3}a_{11}a_{55},\quad a_{23}^2<\frac{1}{4}a_{22}a_{33},\quad a_{24}^2<\frac{1}{3}a_{22}a_{44},\quad a_{25}^2<\frac{1}{3}a_{22}a_{55}.$$

$$a_{34}^2<\frac{1}{3}a_{33}a_{44},\quad a_{35}^2<\frac{1}{3}a_{33}a_{55}. \quad (4.24)$$

Now, from (4.6) and mean value theorem, we note that

$$|\eta_B(N)| \le \rho_1, \quad |\eta_P(B)| \le \rho_2, \quad |\eta_{Q1}(I,N)| \le \rho_3, |\eta_{Q2}(I^*,N)| < \rho_4, |\tau_{P1}(P_1,P_2)| < \frac{m_1}{{M_n}^2},$$

$$|\tau_{P2}(P_1^*,P_2)| < \frac{m_2}{{M_n}^2}, \quad |\xi_{B1}(P_1,P_2)| \le \frac{k_1}{{K_m}^2}, \quad |\xi_{B2}(P_1^*,P_2)| \le \frac{k_2}{{K_m}^2}. \qquad (4.25)$$

Further, we note that the stability conditions (4.7)-(4.15) as stated in theorem 2, can be obtained by maximizing the left-hand side of inequalities (4.24). This completes the proof of theorem 2.

5 NUMERICAL SIMULATIONS AND DISCUSSION

To facilitate the interpretation of our mathematical findings by numerical simulation, we integrated system (2.1) using fourth order Runge-Kutta method. We take the following particular form of the functions involved in the model (2.1):

$$r_B(N) = r_{B0} - r_{B1}N, \quad r_P(B) = r_{P0} + r_{P1}B, \quad K_B(P_1,P_2) = K_{B0} - K_{B1}P_1 - K_{B2}P_2,$$
$$M(P_1,P_2) = M_0 - M_1P_1 - M_2P_2, \quad Q(I,N) = Q_0 + Q_1I + Q_2N. \qquad (5.1)$$

Now we choose the following set of values of parameters in model (2.1) and equation (5.1).

$r_{B0} = 11,\ r_{B1} = 0.2,\ K_{B0} = 12.2,\ K_{B1} = 0.1,\ K_{B2} = 0.3,\ \alpha = 0.01,\ r_{P0} = 20,\ r_{P1} = 0.1, M_0 = 10,\ M_1 = 0.1, M_2 = 0.2,\ \gamma_1 = 0.02, Q_0 = 20,\ Q_1 = 0.3,$
$Q_2 = 0.2,\ \delta_0 = 14, \alpha_1 = 0.001, \alpha_2 = 0.08, \mathrm{g} = 5,\ \theta = 0.5,\ \delta_1 = 17,\ \beta_1 = 0.6,\ \beta_2 = 0.1,\ r_1 = 9, l = 5,\ \beta = 0.1, \gamma_2 = 0.2, K_m = 0.001$
$k_1 = 0.2, k_2 = 0.01,\ m_1 = 0.02, m_2 = 0.01,\ M_n = 1.3, \rho_1 = 0.2,\ \rho_2 = 0.1,\ \rho_3 = 1,\ \rho_4 = 0.1,$ (5.2)

With the above values of parameters, we note that condition for the existence of E^* are satisfied, and E^* is given by

$$B^* = 9.6912, \quad N^* = 10.3966, \quad P_1^* = 1.2140, \quad P_2^* = 0.1272, \quad I^* = 6.6936. \qquad (5.3)$$

It is further noted that all conditions of local stability (4.1) – (4.5), global stability (4.7) – (4.15) are satisfied for the set of values of parameters given in (5.2).

In fig. 1, the primary and secondary toxicants against time are plotted. It shows that as direct emission of toxicant i.e. Q_0, increases both primary and secondary toxicants into the environment increases rapidly. Also it has been taken in the model that emission of primary toxicant is industrialization and population dependent so its growth rate increases with increase in parameters Q_1 and Q_2, respectively, which ultimately result in increase of secondary toxicant into the environment. This can be seen in figs. 2-3. Fig. 4, shows the dynamics of resource-biomass for different values of α, w.r.t time t. This shows that density of resource-biomass decreases as α, increases. It is also noted that the resource-biomass density initially increases w.r.t time t and after certain time it settle down to its steady state. Figs. 5-7, show the effect of θ for $g = 12$ on the dynamics of resource-biomass, population and secondary toxicant w.r.t time t. From fig. 7, it is obvious that as θ, increases secondary toxicant into the environment increases rapidly. From figs 5-6, we can infer that as the level of secondary toxicant increases into the environment, densities of resource-biomass and population decreases.

Fig. 8, shows the dynamics of secondary toxicant for different values of g, with respect to time t. It is found that as g, rate of transformation of primary toxicant to secondary toxicant, increases density of secondary toxicant increases into the environment. Also table is formed for different values of g and $\theta = 1$, which shows resource-biomass, population, primary toxicant and industrialization decreases while secondary toxicant increases. From the table we can infer that resource-biomass, population may driven to extinction if rate of formation of secondary toxicant is large.

g	Resource-Biomass(B)	POPULATION (N)	Primary Toxicant (P_1)	Secondary Toxicant (P_2)	Industrialization (I)
0.5	9.6892	10.3788	1.5700	0.0329	6.6915
1	9.6861	10.3773	1.5204	0.0638	6.6912
5	9.6666	10.3684	1.2138	0.2547	6.6891
10	9.6511	10.3613	0.9694	0.4069	6.6874
15	9.6407	10.3566	0.8069	0.5082	6.6863

From figs. 9-10, we note that density of industrialization increases as β and γ_2, increases. Fig. 11, shows that density of population increases as γ_1, increases with time. Figs. 12-13, show the effects of K_{B1} and K_{B2}, on the dynamics of resource-biomass. In both cases the density of resource-biomass increases initially then decreases for some time and finally obtain its equilibrium level. These figs also show that primary pollutant has an adverse effect on the resource-biomass carrying capacity for a larger period than secondary toxicant. Similar behavior can be seen in figs. 14-15, which is plotted between population and time for different values of M_1 and M_2, respectively.

6. CONCLUSION

In this paper, a nonlinear mathematical model to study the effects of industrialization, population, primary–secondary toxicants on depletion of forestry resource is proposed and analyzed. It is assumed that primary toxicant is emitted into the environment with a constant prescribed rate as well as its growth is enhanced by increase in density of population and industrialization. Further, a part of primary toxicant is transformed into secondary toxicant, which is more toxic, both affecting the resource and population simultaneously. Criteria for local stability, instability and global stability are obtained by using stability theory of differential equation. It is found that if the densities of industrialization and population increases, then the density of primary toxicant into the environment become very large due to which the densities of resource biomass and population decreases & it settle down at its equilibrium level whose magnitude is lower than its original carrying capacity. It is also found that due to high level of primary toxicant into the environment which led in large transformation of secondary toxicant, which is more toxic, decreases the densities of resource biomass and population more than the case of single toxicant. Further, it is noted that if these factor increases unabatedly, then resource biomass and population may be driven to extinction.

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Figures

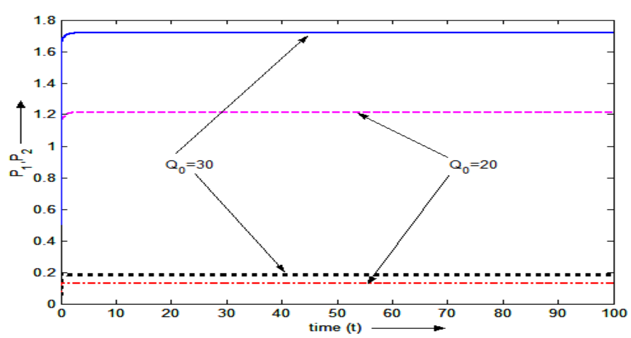

Fig.1, Variation of Primary and Secondary toxicants with time for different values of Q_0 and other values of parameters are same as in (5.2).

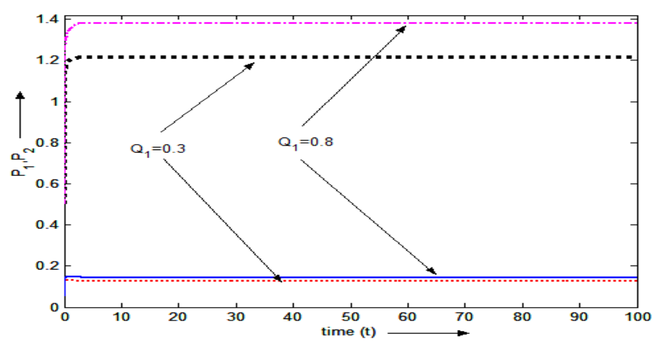

Fig.2, Variation of Primary and Secondary toxicants with time for different values of Q_1 and other values of parameters are same as in (5.2)

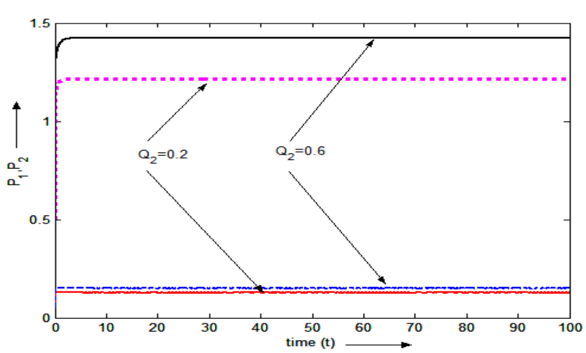

Fig.3, Variation of Primary and Secondary toxicants values of with time for different values of Q_2 and other values of parameters are same as in (5.2)

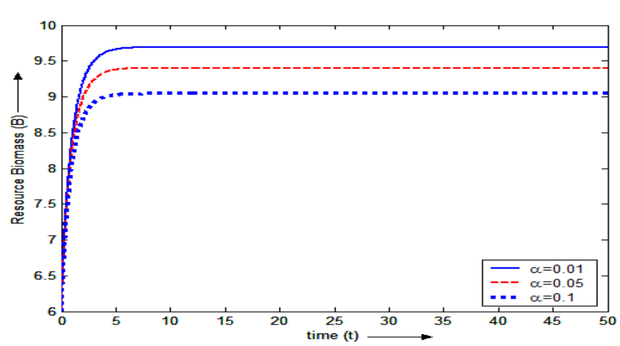

Fig. 4, Variation of resource-biomass with time for different α and other values of parameters are same as in (5.2)

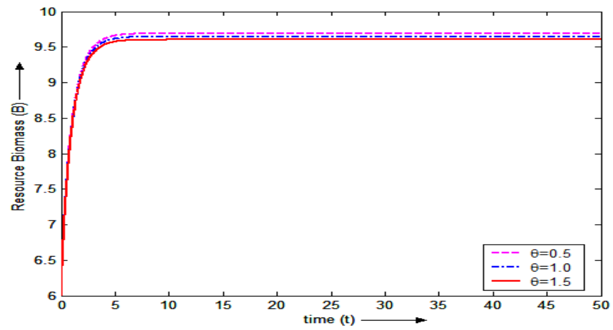

Fig. 5, Variation of resource-biomass with for different values of θ and other values of parameters are same as in (5.2)

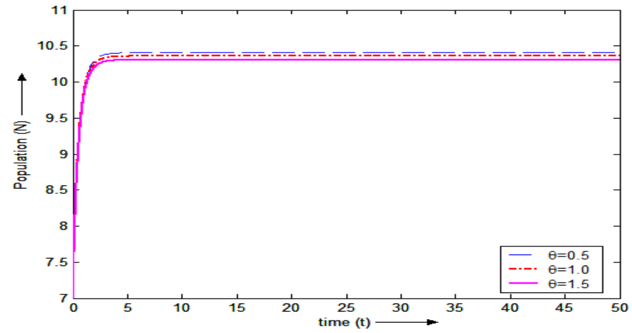

Fig. 6, Variation of population with time time for different values of θ and other values of parameters are same as in (5.2)

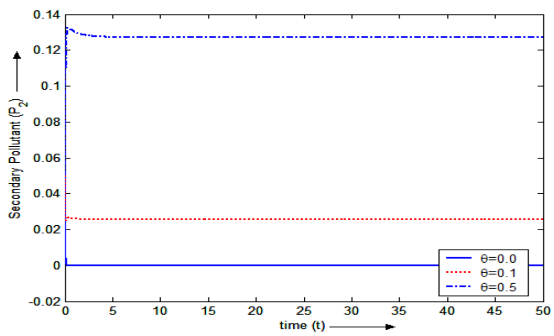

Fig. 7, Variation of secondary toxicant with time for different values of θ and other values Of parameters are same as in (5.2)

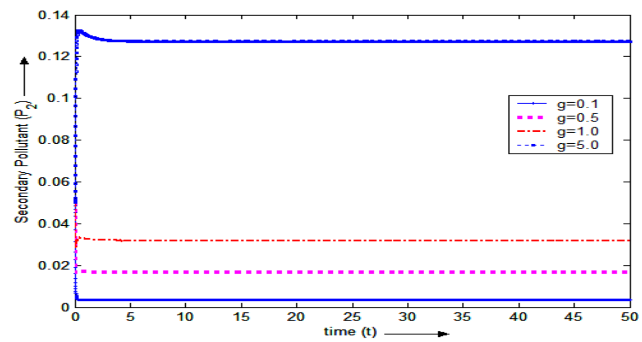

Fig. 8, Variation of secondary toxicant with time for different values of g and other values are same.

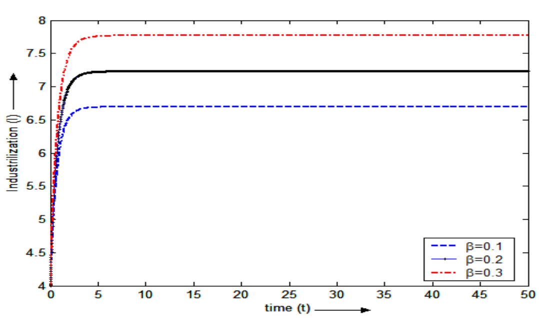

Fig. 9, Variation of Industrialization with time for different values of β and other values of parameters are same as in (5.2)

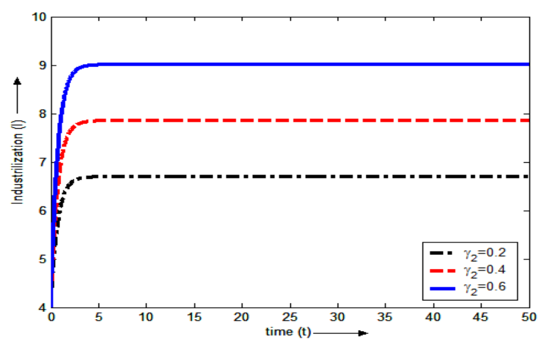

Fig. 10, Variation of Industrialization with time for different values of γ_2 and other values of parameters are same as in (5.2)

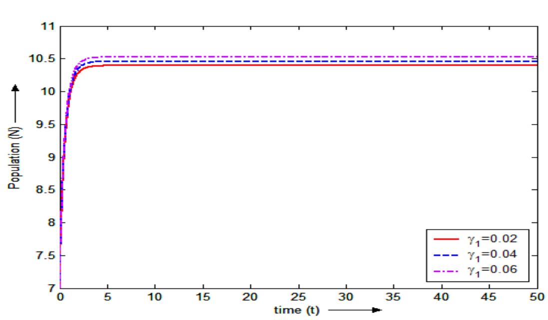

Fig. 11, Variation of Population with time for different values of γ_1 and other values of parameters are same as in (5.2)

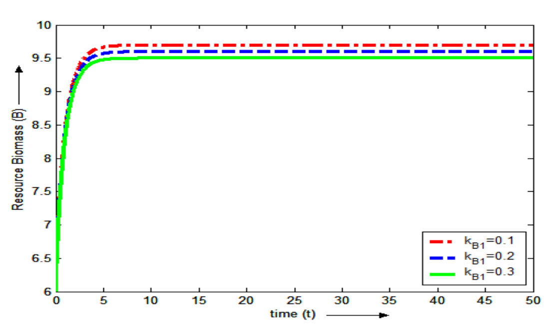

Fig. 12, Variation of resource-biomass with time for different values of K_{B1} and other values of parameters are same as in (5.2)

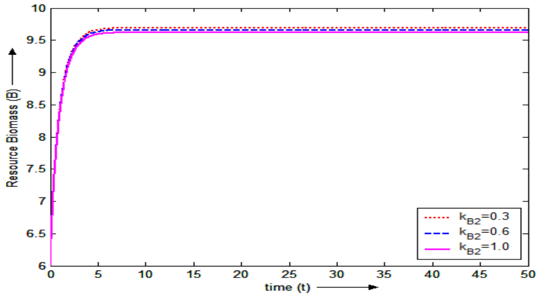

Fig. 13, Variation of resource-biomass with time for different values of K_{B2} and other values of parameters are same as in (5.2)

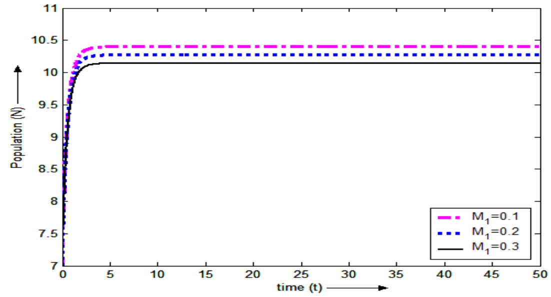

Fig. 14, Variation of population with time for different values of M_1 and other values of parameters are same as in (5.2)

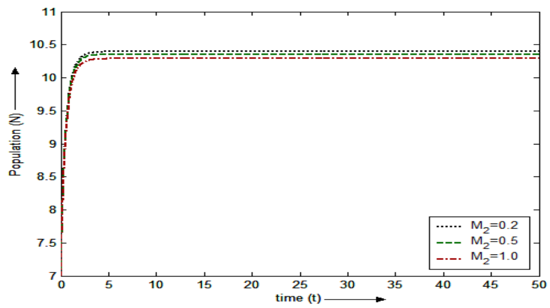

Fig. 15, Variation of population with time for different values of M_2 and other values of parameters are same as in (5.2)